Once upon a time...

“Life is worthless without love!”
- told Snow White the Seven Dwarfs
Once upon a time...

...and fell asleep
Once upon a time...

...for a long-long while.
Who will wake Snow White up?

- Prince (P)
Who will wake Snow White up?

- Prince (P)
- Prince Charming (C)
Who will wake Snow White up?

- Prince (P)
- Prince Charming (C)
- Little Prince (L)
Who will wake Snow White up?

- Prince (P)
- Prince Charming (C)
- Little Prince (L)
Who will wake Snow White up?

- Prince (P)
- Prince Charming (C)
- Little Prince (L)
- Batman (B)
The dwarfs have to choose

So they vote:

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P, C, L are tied (2 points)
The dwarfs have to choose

So they vote:

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C is elected (ties broken alphabetically)
The dwarfs have to choose

So they vote:

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C is elected (ties broken lexicographically)
But wait a minute...

The voters have the following preferences regarding the outcome:

1: $P \succ B \succ L \succ C$
2: $P \succ B \succ C \succ L$
3: $C \succ L \succ P \succ B$
4: $C \succ B \succ P \succ L$
5: $L \succ \ldots$
6: $L \succ \ldots$
7: $B \succ \ldots$
The dwarfs have incentives to strategise

So they may change their mind:

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C, L, B are tied (2 points)
The dwarfs have incentives to strategise

So they may change their mind:

\[
\begin{array}{c|c|c|c|c}
P & C & L & B \\
2 & 3 & 4 & 5 & 6 & 1 & 7 \\
\end{array}
\]

B is elected (lexicographic tie-breaking)
... and change the outcome!
It’s not yet the end...

The voters have the following preferences regarding the outcome:

1: $P \succ B \succ L \succ C$
2: $P \succ B \succ C \succ L$
3: $C \succ L \succ P \succ B$
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L is elected (unique winner)
It’s not yet the end...

The voters have the following preferences regarding the outcome:

1: \( P \succ B \succ L \succ C \)
2: \( P \succ B \succ C \succ L \)
3: \( C \succ L \succ P \succ B \)
4: \( C \succ B \succ P \succ L \)
5: \( L \succ \ldots \)
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7: \( B \succ \ldots \)
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B is elected (lexicographic tie-breaking)
No more objections!

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1: $P \succ B \succ L \succ C$
2: $P \succ B \succ C \succ L$
3: $C \succ L \succ P \succ B$
4: $C \succ B \succ P \succ L$
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6: $L \succ \ldots$
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Happy end!
What are we after?

- Agents have to agree on a joint plan of action or allocation of resources.
- Their individual preferences over available alternatives may vary, so they vote.
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Voting setting

- \( V = \{1, \ldots, n\} \) – set of voters (or agents)
- \( C = \{c_1, \ldots, c_m\} \) – set of candidates (or alternatives)
- \( \mathcal{L}(C) \) – set of all strict linear orders (transitive, antisymmetric and total relations) on \( C \)
- \( \succeq_i \in \mathcal{L}(C) \) – agent \( i \)’s private preference order over the candidates, for each \( i \in V \)
Voting profile

- $P_i \in \mathcal{L}(C)$ – vote of voter $i$ (may or may not coincide with $\succ_i$)
- $P = (P_1, \ldots, P_n) \in \mathcal{L}(C)^n$ – voting profile
  - $P = (P_i, P_{-i})$ where $P_{-i}$ – set of partial votes of a subset $V \setminus \{i\}$ of all the agents but $i$
- $P = (\succ_1, \ldots, \succ_n)$ – truthful profile
**Voting rule**

- $F : \mathcal{L}(C)^n \to 2^C \setminus \{\emptyset\} \quad \text{– voting rule}$
  - determines *the winners* of the election
Voting rule

- $F : \mathcal{L}(C)^n \rightarrow C$ – *resolute* voting rule
  - returns a single winner
  - paired with a *tie-breaking rule*
    - deterministic (e.g., lexicographic)
    - randomised
CONVERGENCE TO EQUILIBRIA IN PLURALITY VOTING

Model

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CONVERGENCE TO EQUILIBRIA IN PLURALITY VOTING

Model

Plurality

- Each voter reports his top candidate:
  - $P_i \in C$
- Voters may have different weights: $w_i \in \mathbb{N}$, $\forall i \in V$. 
Plurality

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- The score of a candidate \( c \) is the total weight of agents voting for him:

\[
s(c) = \sum_{i \in V : P_i = c} w_i
\]
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Game-theoretical interpretation

The (Plurality) voting game is a normal form game $\langle V, C, F, \succ \rangle$ where:

- $V$ – set of agents = set of voters
- $C$ – set of strategies = set of candidates
- $F$ – voting rule (paired with a tie-breaking rule)
- $\succ$ – profile of voters’ preferences over the candidates
Game-theoretical interpretation

The (Plurality) voting game is a normal form game $\langle K, C, F, \succ \rangle$ where:

- $K \subseteq V$ – set of agents = set of strategic voters
- $B = V \setminus K$ – sincere (non-strategic) voters
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Voting as a normal form game

3 candidates with initial scores:

- $s_B(a) = 7$
- $s_B(b) = 9$
- $s_B(c) = 3$
Voting as a normal form game

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Voting as a normal form game

Agents’ preferences:

- 1: \( a \succ b \succ c \)
- 2: \( c \succ a \succ b \)

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} \\
\text{a} & (14, 9.3) & (10, 13, 3) & (10, 9, 7) \\
b & (11, 12, 3) & (7, 16, 3) & (7, 12, 7) \\
c & (11, 9.6) & (7, 13, 6) & (7, 9, 10) \\
\end{array}
\]
**Convergence to Equilibria in Plurality Voting**

**Model**

Voting as a normal form game

Agents’ preferences:

- 1: \(a \succ b \succ c\)
- 2: \(c \succ a \succ b\)

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Voting in turns (a.k.a. “iterative voting”) 

- Agents start from some initial profile (e.g., truthful).
- They change their votes in turns.
- At each step, a single agent makes a move.
- The game ends when there are no more objections.

- Implemented in polls via Doodle or Facebook.
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Improvement moves

Agents make rational moves to improve their state, when

▶ they do not know the preferences of the others,
▶ and cannot coordinate their actions.

⇒ The agents apply *myopic* (or, *local*) moves.
**Improvement moves**

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**Improvement moves**

3: \( C \succ L \succ P \succ B \)

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CONVERGENCE TO EQUILIBRIA IN PLURALITY VOTING

Model

Improvement moves

3: \[ C \succ L \succ P \succ B \]

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\[ B \overset{3}{\rightarrow} L \] is an *improvement move* (or *better reply*) of agent 3
**Convergence to Equilibria in Plurality Voting Model**

**Improvement moves**

3: \( C \succ L \succ P \succ B \)

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\( B \rightarrow^3 P \) is a best reply of agent 3
Improvement moves

3: \[ C \succ L \succ P \succ B \]

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\( B \overset{3}{\rightarrow} C \) is a restricted best reply (which is unique) for agent 3
Convergence to Equilibria in Plurality Voting

Model

Variations of the game

Voting setting:

- Voting rule
  - Plurality
Variations of the game

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  - Deterministic
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Dynamics:

- Initial state
  - Truthful
  - Arbitrary
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Dynamics:

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- Improvement moves
  - Better replies
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Dynamics:

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Our results

We show how the convergence depends on all of these game/dynamic attributes.
Deterministic tie-breaking

Theorem

*If all agents have weight 1 and use restricted best replies, the game converges to a Nash equilibrium from any state.*
Proof sketch

(by Reyhani & Wilson 2012)

- $o_t$ – outcome at step $t$
- Restricted best replies at any step $t$ are of 2 types:
  - **type 1:** $a \rightarrow b$ where $a \neq o_{t-1}$ and $b = o_t$
  - **type 2:** $a \rightarrow b$ where $a = o_{t-1}$ and $b = o_t$
- We will show that there are
  - $\leq m$ moves of type 1 in total, and
  - $\leq m - 1$ moves of type 2 for each voter.
Proof

\[ PW_t = \{ c \mid \exists i \in K : o_t \rightarrow^i c \Rightarrow c = o_{t+1} \} \] - potential winners at step \( t \)

Lemma

For \( t < t' \) we have \( PW_{t'} \subseteq PW_t \).
Proof of the lemma

- Let $a \rightarrow b$ at step $t$. Then, $b = o_t$.
- Let $c \in PW_t$.
- Consider the scores of $b, c, y \ \forall y \in C \setminus \{a, b\}$:

\[
\begin{align*}
  s_{t-1}(c) + 1 &= s_t(c) + 1 \geq s_t(b) - 1 = s_{t-1}(b) \\
  s_{t-1}(c) + 1 &= s_t(c) + 1 \geq s_t(y) = s_{t-1}(y)
\end{align*}
\]

where $c \geq c'$ if $s(c) > s(c')$ or $s(c) = s(c')$ and $c$ has a lower index.
Proof of the lemma (contd.)

- If $a \rightarrow b$ at step $t$ is of type 2, then followed by $b \rightarrow c$ at step $t + 1$ results in the same scores as $a \rightarrow c$ at step $t$. Hence, $c \in PW_{t-1}$.
- Otherwise, let $a' = o_t$ and note $a' \neq a, b$.
- We have:

$$s_{t-1}(c) + 1 \geq s_{t-1}(a')$$

$$s_{t-1}(a') \geq s_{t-1}(y) \quad \forall y \in C$$

$$\Rightarrow \quad s_{t-1}(c) + 1 \geq s_{t-1}(a') \geq s_{t-1}(a)$$

Hence, $c \in PW_{t-1}$. 

$\square$
Proof of the theorem (contd.)

- If $a \rightarrow b$ at step $t$ is of type 1 then $a \notin PW_t$:
  - If $a \in PW_t$ then $b \rightarrow a$ makes $a$ a winner, a contradiction to $a \rightarrow b$ being of type 1.
- By the lemma, $a \notin PW_{t'}$ for all $t' > t$
  $\Rightarrow$ the number of type 1 moves is bounded by $m$.

- At every improvement step $a \overset{i}{\rightarrow} b$ of type 2, it must hold that $b \succ_i a$
  $\Rightarrow$ each voter can make at most $m - 1$ steps of type 2.
Proof of the theorem (contd.)

- If $a \rightarrow b$ at step $t$ is of type 1 then $a \notin PW_t$:
  - If $a \in PW_t$ then $b \rightarrow a$ makes $a$ a winner, a contradiction to $a \rightarrow b$ being of type 1.

- By the lemma, $a \notin PW_{t'}$ for all $t' > t$ 
  $\Rightarrow$ the number of type 1 moves is bounded by $m$.

- At every improvement step $a \xrightarrow{i} b$ of type 2, it must hold that $b \succ_i a$
  $\Rightarrow$ each voter can make at most $m - 1$ steps of type 2.
(Not restricted) best replies

3 candidates with initial scores \((1, 0, 0)\)

2 voters with preferences

1: \(a \succ b \succ c\)
2: \(c \succ b \succ a\)

<table>
<thead>
<tr>
<th>voter 1</th>
<th>voter 2</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td>((1,2,0))</td>
<td>((1,1,1))</td>
</tr>
<tr>
<td>c</td>
<td></td>
<td>((1,1,1))</td>
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<td></td>
</tr>
</tbody>
</table>
Better replies

4 candidates with initial scores \((2, 2, 2, 0)\)

3 voters with preferences

- 1, 3: \(d \succ a \succ b \succ c\)
- 2: \(c \succ b \succ a \succ d\)

\[
dcd(2, 2, 3, 2) \xrightarrow{1} bcd(2, 3, 3, 1) \xrightarrow{3} bca(3, 3, 3, 0)
\]
Better replies

4 candidates with initial scores \((2, 2, 2, 0)\)

3 voters with preferences

- 1, 3 : \(d \succ a \succ b \succ c\)
- 2 : \(c \succ b \succ a \succ d\)

\[
dcd(2, 2, 3, 2) \rightarrow bcd(2, 3, 3, 1) \rightarrow bca(3, 3, 3, 0)
\]

\[
2 \rightarrow bba(3, 4, 2, 0) \rightarrow cba(3, 3, 3, 0) \rightarrow cca(3, 2, 4, 0)
\]

Cycle from the truthful state!
Better replies

4 candidates with initial scores \((2, 2, 2, 0)\)

3 voters with preferences

\begin{itemize}
\item 1, 3 : \text{d} \succ \text{a} \succ \text{b} \succ \text{c}
\item 2 : \text{c} \succ \text{b} \succ \text{a} \succ \text{d}
\end{itemize}

\[
dcd(2, 2, 3, 2) \xrightarrow{1} bcd(2, 3, 3, 1) \xrightarrow{3} bca(3, 3, 3, 0)
\]

\[
\uparrow_1
\]

\[
\xrightarrow{2} bba(3, 4, 2, 0) \xrightarrow{1} cba(3, 3, 3, 0) \xrightarrow{2} cca(3, 2, 4, 0)
\]

Cycle from the truthful state!
Weighted voters

- No convergence for 3+ voters, even when they start from the truthful state and use restricted best replies.
- Convergence for 2 voters, if they both use restricted best replies or start from the truthful state.
Randomised tie-breaking

- $\succeq_i$ does not induce a complete order over the outcomes, which are sets of candidates.
- We augment agents’ preferences with cardinal utilities:
  - $u_i(c) \in \mathbb{R}$ – utility of candidate $c$ to voter $i$,
  - for multiple winners, $u_i(W) = \sum_{c \in W} \frac{u_i(c)}{|W|}$.
- A utility function $u$ is consistent with a preference relation $\succeq_i$ if
  \[
u(c) > u(c') \iff c \succeq_i c'\]
To prove convergence, we must show it is guaranteed for *any* utility function which is consistent with the given preference order.

To disprove, it is sufficient to show a cycle for a *specific* assignment of utilities: *weak* counterexample.

If the counterexample holds for any profile of utility scales, it is *strong*. 

**Randomised tie-breaking**
### Weighted voters

3 candidates with initial scores $(0, 1, 3)$

2 voters weighted voters with preferences

- $1: a \succ b \succ c$
- $2: b \succ c \succ a$

<table>
<thead>
<tr>
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<th>$w_2 = 3$</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tr>
<td>a</td>
<td>(8,1,3)</td>
<td>(5,4,3)</td>
<td>(5,1,6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>(3,6,3)</td>
<td>(0,9,3)</td>
<td>(0,6,6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>(3,1,8)</td>
<td>(0,4,8)</td>
<td>(0,1,11)</td>
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Weighted voters

3 candidates with initial scores (0, 1, 3)

2 voters weighted voters with preferences

- 1: \( a \succ b \succ \{b, c\} \succ c \)
- 2: \( b \succ \{b, c\} \succ c \succ a \)

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No Nash equilibrium!
Weighted voters

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2 voters weighted voters with preferences

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No Nash equilibrium!
Unweighted voters

Theorem

*If all agents have weight 1 and use restricted best replies, the game converges to a Nash equilibrium from the truthful state.*
Proof (skipped)

We show that in each step, an agent votes for a less preferred candidate.

- Clearly holds for the first step. Proceed by induction.

Hence, each voter can make only $m - 1$ steps.
Less restricted dynamics

- **Arbitrary state:**
  - weak counterexample with 3 unweighted agents, even if they use restricted best replies

- **Better replies:**
  - strong counterexample with 3 unweighted agents
  - weak counterexample with 2 agents, even if they start from the truthful state
Summary

**Deterministic Tie breaking**

<table>
<thead>
<tr>
<th>Dynamics Initial state</th>
<th>R. best reply</th>
<th>Best reply</th>
<th>Any better reply</th>
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<tr>
<td></td>
<td>Truth</td>
<td>Any</td>
<td>Truth</td>
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<td>Weighted ((k &gt; 2))</td>
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<td>X</td>
</tr>
<tr>
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<td>V</td>
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</tr>
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<td>√</td>
<td>√</td>
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<tr>
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### Randomized Tie breaking

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Other remarks

- Truth biased agents
  - Vangelis’s talk tomorrow
Other remarks

- Truth biased agents
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- Quality of outcomes
Other remarks

- Truth biased agents
  - Vangelis’s talk tomorrow

- Quality of outcomes
  - Simina’s talk tomorrow
Other remarks

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Future work
Past future work
Past future work

- Rules other than Plurality
- Restricted Iterative Processes
Past future work

- Rules other than Plurality
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- Iterative processes as a single-round game
Past future work

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- Iterative processes as a single-round game
  - Today’s talks
Past future work

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Future work

- Rules other than Plurality
- Restricted Iterative Processes
- Iterative processes as a single-round game
- Weak acyclicity?
- Dynamics leading to desirable outcomes?
CONVERGENCE TO EQUILIBRIA IN PLURALITY VOTING

Happy end!

THE END