Motivation for the theory of Perfectoid Spaces

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1 Introduction

Perfectoid spaces are a in one sense a tool. It is just a relatively new method to translate problems from ch. 0 to ch.p, where one may expect to get a solution... Before starting to deal with the very definition of such a spaces we would like to give some motivation on the study. The motivations are mainly connected with Galois representations. We will give in this paragraph a primary motivations, at the end we will open the landscape. This in order to answer to the following question: why to study Galois representations?

First of all we would like to say what we mean about Galois representations. Some notation. We indicate as usual $\mathbb{Q}$ the field of rational numbers, by $\overline{\mathbb{Q}}$ the algebraic closure and by $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ the Absolute Galois Group. It is naturally a topological group. A basis of open neighborhood of the identity automorphism is given by the subgroups of finite index i.e. $Gal(\overline{\mathbb{Q}}/K)$, where K is a finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. In fact it is a profinite group (at level of topology); it is given by

$$\lim_{\leftarrow} Gal(K/\mathbb{Q})$$

where the limit is taken all over the normal finite extensions K as before (seen as discrete sets). We want to study such a group: one may consider other structures than only the abstract (topological) group structure. Remember that $\mathbb{Q}$ is the fraction field of $\mathbb{Z}$ and then one may think at the primes in $\mathbb{Z}$. In fact, we can define for a prime number p, an absolute value $\mid - \mid_p$ such that in $\alpha \in \mathbb{Q}$ is given by $\mid \alpha \mid_p = p^s$, where $s \in \mathbb{Z}$ and $\alpha = \frac{a}{b} p^s$, where we ask that $a, b \in \mathbb{Z}$ are coprime with p.

We may complete $\mathbb{Q}$ with respect to such an absolute value and we get the field of p-adic numbers $\mathbb{Q}_p$: totally disconnected and locally compact topological field. Naturally we can take its algebraic closure $\overline{\mathbb{Q}}_p$ and to get $G_{\mathbb{Q}_p}$. Naturally we have $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (not unique), and to any such an embedding we have a closed embedding of groups $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$. Even if we don't have a unique inclusions they form a class for the conjugation.

Remark 1.1. The prrof of this fact gives some insights on the techniques one has to use. In fact call $j : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (choice). Of course if you use a different one, you will have to conjugate the final result by $G_{\mathbb{Q}_p}$. In order to define $\iota : G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$, if g is an element of the local (p-adic) Galois group $G_{\mathbb{Q}_p}$ we can consider $g \circ j$ and j : they are $\mathbb{Q}$ mono from $\overline{\mathbb{Q}}$ to $\overline{\mathbb{Q}}_p$. But $\overline{\mathbb{Q}}$ is normal over Q hence these two embeddings have the same image (all the elements of $\overline{\mathbb{Q}}_p$ which are sep. algebraic over $\mathbb{Q}$). Hence we can write $\iota(g) = j^{-1} \circ g \circ j$ (teh inverse only in the image), which is an elelmnt of $G_{\mathbb{Q}}$. The kernel of j is $Gal(\overline{\mathbb{Q}}_p/j(\overline{\mathbb{Q}})$, Krasner Lemma tells us that the $\mathcal{P}$-completion of $j(\overline{\mathbb{Q}})$ is $\overline{\mathbb{Q}}_p$ (where $\mathcal{P}$ is a prime over p and in particular that all teh Galois are continuos.

The situation is different than in the case of the prime p as ∞ i.e. for the absolute archimedean value on $\mathbb{Q}$. The completion now is the field $\mathbb{R}$ and its algebraic closure is just $\mathbb{C}$. As before an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ gives rise to a map

$$G_{\mathbb{R}} = Gal(\mathbb{C}/\mathbb{R}) = \{id, \sigma\} \to G_{\mathbb{Q}}$$

it is an immersion of a group of two elements and as the embeddings of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ vary we obtain a coniugacy class of elements σ in $G_{\mathbb{Q}}$ of order 2 ...which we call *complex conjugations.*

Remark 1.2. Differences between the completions: $\overline{\mathbb{Q}}_p$ is not complete and it is an infinite extension of $\mathbb{Q}_p$. We can take its completion which maintains the fact it is algebraically closed. We indicate such a field as $\mathbb{C}_p$: of course by continuity and density $Gal_{continuos}(\mathbb{C}_p/\mathbb{Q}_p) = G_{\mathbb{Q}_p}$ (Ax-Tate and Krasner Lemma).

It is then clear that one may study $G_{\mathbb{Q}_p}$ inside $G_{\mathbb{Q}}$. But in which sense? Let's study at the same time also $\mathbb{C}_p$ and $\overline{\mathbb{Q}}_p$. All such a fields have an absolute value and the elements of absolute value less than or equal to 1 form a closed subring that we will indicate by $\mathbb{Z}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ and $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ (for $\mathbb{Q}_p$, $\mathbb{C}_p$ and $\overline{\mathbb{Q}}_p$) . They are local ring and their maximal ideal are $p\mathbb{Z}_p$, $\mathcal{M}_{\mathbb{C}_p}$ and $\mathcal{M}_{\overline{\mathbb{Q}}_p}$. Note that while for $\mathbb{Z}_p$ we have the maximal ideal generated by p for the other we have all the roots hence they are not discrete valued... hence not generated by only one element. The quotients $\mathcal{O}_{\mathbb{C}_p}/\mathcal{M}_{\mathbb{C}_p} = \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathcal{M}_{\overline{\mathbb{Q}}_p}$ is an algebraic closure of $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$ (separable one?). We may denote them by $\overline{\mathbb{F}}_p$. We then have a surjective map

$$G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}$$

Its kernel is the inertia group of $G_{\mathbb{Q}_p}$ and we will denote it as usual with $I_{\mathbb{Q}_p}$. It is a normal subgroup of $G_{\mathbb{Q}_p}$. The group $G_{\mathbb{F}_p}$ is procyclic and it has a generator which is called *the Frobenius element*. It is associated to $x \to x^p$ morphism (we are in ch=p!). This is a first decomposition of our Galois group.... we will use another one... so far we have $Frobenius$ element and the Inertia... see [SE] for a complete description...

Remark 1.3. We may see in this decomposition a first instance of a geometry setting. In fact one may think about $Spec\mathbb{Z}_p$, it has only two prime ideal: a maximal ideal $p\mathbb{Z}_p$ and the generic point associated to (0). For both of them we have a fiber: the residue field for the generic point is $\mathbb{Q}_p$, while for the closed point is $\mathbb{F}_p$. We may think to $G_{\mathbb{Q}_p}$ as an object at the generic point which has a specialization as $G_{\mathbb{F}_p}$....in this specialization it loses something...the inertia...

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Another way of studying a group is connected with the study of its representations. Representations arise naturally from arithmetic algebraic geometry. Actually suppose that we have $X/\mathbb{Q}$ a projective and smooth variety , take an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$. Then we may consider the topological space associated to the complex points of X: $X(\mathbb{C})$. It is a topological space, hence we can define its singular cohomology $H^i(X(\mathbb{C}), \mathbb{Z})$. If we extend the coefficients to $\overline{\mathbb{Q}}_l$ (l a prime of $\mathbb{Z}$), we can consider $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l)$, then a theorem of Artin (SGA4 exp XI thm 4.4) is going to tell us that we have an isomorphism between

$$H^i(X(\mathbb{C}), \mathbb{Q}_l) \simeq H^i_{et}(X \times_{\mathbb{Q}} \mathbb{Q}, \overline{\mathbb{Q}}_l)$$

and the last cohomology has an action of $G_{\mathbb{Q}}$ hence a natural representation!! (Definition of étale cohomology? some time to explain it in the afternoon. In fact the proof is rather articulate. One starts with the definition of étale coh. over $\mathbb{Q}_l$: limit over $\mathbb{Z}/l^n\mathbb{Z}$ and then tensor by $\frac{1}{l}$. Then one note that $H^i_{et}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l) \simeq H^i_{et}(X \times_{\mathbb{Q}} \mathbb{C}, \overline{\mathbb{Q}}_l)$ by flat base change (Milne Étale Cohomology , page 232, 4.3). Then finally we have iso singular/etale for schemes over $\mathbb{C}$: i.e. for any n we have $H^i_{et}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}/l^n\mathbb{Z}) \simeq H^i(X(\mathbb{C}), \mathbb{Z}/l^n\mathbb{Z})$ an dthen the limit (SGA4 exp XI thm 4.4))

Remark 1.4. If $X = E$ is an elliptic curve, then we may describe

$$H^1_{et}(E \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l) \simeq \mathrm{Hom}_{\mathbb{Z}_l}(\lim_{\leftarrow} E[l^r](\overline{\mathbb{Q}}), \overline{\mathbb{Q}}_l) \simeq \overline{\mathbb{Q}}_l^2$$

. This is l-adic Galois reprentation of $G_{\mathbb{Q}}$.

Of course we can take the restriction to $G_{\mathbb{Q}_p}$ and we have a l-adic representation (p can be equal to l). But this is not the only way of getting Galois representations.

a) Suppose to have $X/\mathbb{Q}_p$ a scheme defined over $\mathbb{Q}_p$, we can take its $H^i_{et}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_l)$, which is l-adic representation of $G_{\mathbb{Q}_p}$. Of course we can take $H^i_{et}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_p)$, we will have a p-adic representation of $G_{\mathbb{Q}_p}$. If $X/\mathbb{Q}_p$ is proper and smooth then we would have $dim_{\mathbb{Q}_p} H^i_{dR}(X) = dim_{\mathbb{C}} H^i_{dR}((X \times_{\mathbb{Q}_p} \mathbb{C})(\mathbb{C}))$ what about the link with étale cohomology? in this case all the cohomologies have the same dimension. But what about the existence of a functor which gives an isomorphism between all these vector spaces of the same dimension? Perhaps by enlarging the vector space where they are defined to a *a field of periods*. This leads to the *the mysterious functor*.

b) Consider $X/\mathbb{Z}_p$, projective and smooth. we have two fibers $X_s/\mathbb{F}_p$ and a generic $X_\eta/\mathbb{Q}_p$. Then we may consider $H^i_{et}(X_s \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Z}_l)$. It is l-adic representation of $G_{\mathbb{F}_p}$. Note that in this case we have $(l \neq p)$, $dim H^i_{et}(X_s \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Z}_l) = dim H^i_{sing}(X_\eta(\mathbb{C})) = dim H^i_{dR}(X_\eta)$, where we consider an immersion of $\mathbb{Q}_p \to \mathbb{C}$ and we then consider the topological space associated to the $\mathbb{C}$ points. There is an l-adic proof of the Weil conjecture: the Zeta function of X_s is calculated via the l-adic cohomology and the Frobenius action on it: such a cohomology can be seen as the specialization at the special point s of a variety defined over $\mathbb{Z}_p$. By the base change theorem all the fibers have the same dimension and for the generic one we have Artin's theorem. Hence the first identification. Secondly one has a second proof of such Weil conjecture which is given by the crystalline cohomology (which is not the p-adic etale!!!) and such a cohomology coincides (in this case) with the de Rham cohomology of the generic fiber... hence the second identity (even with the de Rham theorem..... of course...).

Remark 1.5. If the scheme $X/\mathbb{Z}_p$ has not good reduction i.e. the special fiber is not smooth then it is not clear anymore the identification (at least at level of dimension) of the etale and the de Rham cohomology. what about the etale l-adic in this singular case? Think about $xy - p$...the semistable case.

We will see other natural representations in the last paragraph. We are led now to discuss different kinds of representations of $G_{\mathbb{Q}_p}$: the l-adic with $l \neq p$ and the $p$adic ones. We have two cases. Remember that $G_{\mathbb{F}_p}$ is the profinite completion of $\mathbb{Z}$. We have a natural surjection $G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{F}_p}$.

i) First $\mathbf{l \neq p}$, in this case we may see that a Grothendieck theorem (Monodromy Theorem) tells us that the inertia is trivial on an open subset. Hence the Galois representation is given by the action of the Weil-Deligne group + a nilpotent operator ([FOO], 1.3). This uses the decomposition of $G_{\mathbb{Q}_p}$ using the inertia.

ii) $\mathbf{l = p}$ i.e. the p-adic Galois Representations: the problem is more involved. Here we cannot use anymore the decomposition of the Galois Group via the inertia.....and this is exactly where the theory of *perfectoid fields (spaces)* is coming from! One would like to decompose $G_{\mathbb{Q}_p}$ on a part which is the the Galois group of a field of ch. p (and not $\mathbb{F}_p$). This way of acting will be the first instance of *Tilting*.

I would like to finish with an observation. In the scheme theory we can see a field K as a point Spec$K$. Of course the first step further would be Spec$A$ where A is a K algebra or in general an algebra. In this sense we may seen K as associated to a point and then we would like to generalize to Spec$A$. But what do we want to generalize? What is the geometrical meaning of $G_{\mathbb{Q}_p}$? Or $G_{\mathbb{Q}}$? Because this is what we want to study. The meaning is the *fundamental group* viewed as covering transformations that in the schemes setting (the case of one point) is given by transformations of some particular fields extensions or, more generally, of étale coverings. The fundamental group is associated to the Galois group, $G_{\mathbb{Q}_p}$: the étale coverings of Spec$\mathbb{Q}_p$ are the finite (separable) extensions of $\mathbb{Q}_p$. We want to find an intermediate extension between $\mathbb{Q}_p$ and $\overline{\mathbb{Q}}_p$ such that its Galois Group gives a relevant decomposition of the absolute Galois Group.

i.e. Starting with a finite extension of $\mathbb{Q}_p$, K (but to avoid confusion, we may think directly to $\mathbb{Q}_p$), we would like to find K_∞

$$K \subset K_\infty \subset \overline{\mathbb{Q}}_p = \overline{K}$$

such that $Gal(\overline{\mathbb{Q}}_p/K_\infty)$ is the galois Group of a field of charatcteristic p. i.e. the etale covering of SpecK_∞ are the étale coverings of a Spec$K_\infty^\flat$ where $K_\infty^\flat$ is a field of ch.p!!! i.e. the finite separable extensions of K_∞ are the finite separable extensions of $K_\infty^\flat$. And $G_{K_\infty} = G_{K_\infty^\flat}$. Why? Because p-adic Galois representations of fields in ch.p are easy to handle.... Of course here we speak about *one point*.

2 The first case

Here we want to deal with the first case i.e. that one studied by Fontaine-Winterberger [FW] (that seems to be connecetd with Krasner, see the ICM [SC3]). We consider $\mathbb{Q}_p$ and its algebraic closure $\overline{\mathbb{Q}}_p$ and its Galois Group. The idea: to consider a sub-Galois extensions $\mathbb{Q}_p \subset \mathbb{Q}_p^\infty \subset \overline{\mathbb{Q}}_p$ such that its Galois group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^\infty}) = G_{\mathbb{Q}_p^\infty}$ is the Galois Group of a field of ch. p. Why? Because in this case a Galois representation of $G_\mathbb{Q}$ can be seen as a representation of $G_{\mathbb{Q}_p^\infty}$ (i.e. of a field of ch.p) plus an action of $Gal(\mathbb{Q}_p^\infty/\mathbb{Q})$, which we hope to be easier. But where to be found such a $\mathbb{Q}_p^\infty$? (The notation with ∞ is not par hasard....). First of all: why the case in ch.p should nbe easier? In fact we have a result of Fontaine [FO]. In 1.2 of that article, Fontaine uses E a field of ch.p and consider E^{sep} its separable closure. The Galois group of E is denoted by G_E and we take V a Z_p adic representation of G_E. (or, more generally, $\mathbb{Q}_p$-representation? 0.1 of [FO].) We consider $\mathcal{E}$ be a complete field for a discrete valuation of ch.0, whose valuation ring is denoted by $\mathcal{O}_\mathcal{E}$, absolutely not ramified (i.e. p is the generator of the maximal ideal) such that its residue field is $E = \mathcal{O}_\mathcal{E}/p\mathcal{O}_\mathcal{E}$. This is called a *Cohen Ring* (field) (EGA IV IHES 20, paragraph.18). If we consider $\mathcal{E}_{nr}$: a maximal algebraic extension which is not ramified over $\mathcal{E}$ then its residue field is an algebraic closure of the residue field E. We indicate by $\hat{\mathcal{E}}_{nr}$ its completion. Then we have $G_E = Gal(\mathcal{E}_{nr}/\mathcal{E}) = Gal(\hat{\mathcal{E}}_{nr}/\mathcal{E})$. (In general $\mathcal{E}$ is called the Cohen Ring of E if E is perfect it is called the fraction field of the Witt vectors.....). Let's define in the ring $\mathcal{O}_\mathcal{E}$ an endomorphism which we will call of Frobenius given by σ whose reduction mod.p is $\sigma x = x^p$. We denote its unique extension to $\mathcal{E}_{nr}$ as φ. The theory of local fields (see [SE]) tells us that $G_E = Gal(\mathcal{E}_{nr}/\mathcal{E}) = Gal(\hat{\mathcal{E}}_{nr}/\mathcal{E})$. Hence if we start with a V a $\mathbb{Z}_p$-representation of G_E (always the module is a free of finite type..!!) then we may build:

$$\mathbf{D}_\mathcal{E}(V) = (\mathcal{O}_{\hat{\mathcal{E}}_{nr}} \otimes_{\mathbb{Z}_p} V)^{G_E}$$

and $\mathbf{D}_\mathcal{E}(V)$ is a $\mathcal{O}_\mathcal{E}$-module endowed with a φ-action (semilnear with respect the Frobenius lifting in $\mathcal{E}$). This action is étale (i.e. the associated linear map is bijective). We denote by $\mathbf{\Phi M}^{et}_{\mathcal{O}_\mathcal{E}}$ the category of étale (finite) $\mathcal{O}_\mathcal{E}$-modules of finite type. Then

Theorem 2.1. *The two functors*

$$\mathbf{D}_\mathcal{E} : \mathbf{Rep}_{\mathbb{Z}_p}(G_E) \to \mathbf{\Phi M}^{et}_{\mathcal{O}_\mathcal{E}}$$

and $\mathbf{V}_\mathcal{E}$, which for every étale φ-module M is given by

$$\mathbf{V}_\mathcal{E}(M) = (\mathcal{O}_{\hat{\mathcal{E}}_{nr}} \otimes_{\mathcal{O}_{\hat{\mathcal{E}}}} M)_{\varphi=1}$$

give an equivalence of categories.

But what about our problem? If we start with a p-adic representation of $G_{\mathbb{Q}_p}$, then one may hope to have an intermediate field $\mathbb{Q}_p^\infty$ as before such that the Galois

Group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^\infty) = G_{\mathbb{Q}_p^\infty}$ is the Galois Group of a field of ch.p.Which we indicate by E in that case (for notation reasons we will indicate $\Gamma = Gal(\mathbb{Q}_p^\infty/\mathbb{Q})$). if we take a Cohen ring for E i.e. $\mathcal{O}_{\mathcal{E}}$ of fraction field $\mathcal{E}$ endowed with a lifting of the Frobenius φ which commutes with the action of Γ then to a p-adic representation of $G_{\mathbb{Q}_p}$ we can associate a φ-$\mathcal{O}_{\mathcal{E}}$ module endowed with a Γ action...a (φ, Γ)-module i.e. an object of $\mathbf{\Phi M}^{et}_{\mathcal{O}_{\mathcal{E}}}$.

But where to find such an intermediate field? Via the *field of norms*. This has been the answer given by Fontaine (see also [FOO]).

If we take a field then $SpecK$ is just a point, but it should be clear that the set of all separable extension is associated to the étale extensions. I.e. to the coverings...and the Galois is just the fundamental group. A $SpecK$ is just a point...no topology...what should be the analogue if we don't have anymore just a point? *the perfectoid spaces associated to a perfectoid rings/field.*

3 The Construction

I want to stress again what we want: we want to translate a field in ch.0 to a field in ch.p in such a way they will have the same absolute Galois group in such a way we can use this *ping-pong* in order to get result from the (wild)side we like more i.e. where we know more. In the sense, where we know more. The inspiring example as we said was about a choice of a field in the algebraic closure of $\mathbb{Q}_p$. We denote by $\mathbb{Q}_{p^\infty}$ or $\mathbb{Q}_p(\zeta^\infty)$ both the infinity Galois extensions of $\mathbb{Q}_p$ obtained by adding the $p^{\frac{1}{n}}$ roots of p (resp. of 1) to $\mathbb{Q}_p$. Both are absolutely unramified infinity extensions of $\mathbb{Q}_p$ with the same residue field.....with really big ramification....deeply....not with a discrete valuation.... and such that if we take the ch.p quotient $\mathcal{O}_{\mathbb{Q}_{p^\infty}}/p\mathcal{O}_{\mathbb{Q}_{p^\infty}}$ or $\mathcal{O}_{\mathbb{Q}_p(\zeta^\infty)}/p\mathcal{O}_{\mathbb{Q}_p(\zeta^\infty)} = \mathbb{F}_p$ the Frobenius is surjective (being perfect....it is actually bijective..)

Proposition 3.1. *The field $\mathbb{Q}_{p^\infty}$ has the same (separable) absolute Galois group as $\mathbb{F}_p((t))$ (the fraction field of the power series in t, t is an indeterminate over $\mathbb{F}_p$).*

(we could have given a similar statement for $\mathbb{Q}_p(\zeta^\infty)$) We will try to prove this iso which is the basis of the theory. We pass from ch.0 to a non perfect field in ch. p. Note that we can say even more: that we have an equivalence of categories for the finite separable extensions. But: this is connected with the theory of Field of Norms for *non perfect field*, while we will try to study the perfect case which is what will be discussed in the perfectoid case.

Note that if we denote by $E = \mathbb{F}_p((u))$, then it is easy to describe a Cohen ring for it:

$$\mathcal{O}_{\mathcal{E}} = \{ \sum_{i=-\infty}^{+\infty} a_i t^i \mid a_i \in \mathbb{Z}_p \, , \lim_{i \to -\infty} \mid a_i \mid = 0\}$$

and $\mathcal{E} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Idea: to translate a field in ch.0 to a field in ch.p having the same Galois group. This has been studied by Fontaine and Winteberger [FW]. They were able to associate to a APF field (valuation field non archimedean with perfect residue field even not a not a priori complete with some hypotheses about ramification) a field in ch.p not a priori perfect, but with the same Galois Group. Such a correspondance was also at level of intermediate separable extensions. In the same article they were also able to associate a *perfect field of norms* in case the field is complete and this is exactly along this line that Scholze has introduced his perfectoid fields (spaces). For such a complete/perfect correspondance Fontaine-Winterberger do not have a complete Galois (also intermediate) correspondance using their APF fields, but this is exactly what Scholze did for Perfectoid fields!

Before giving the very definition of perfectoid spaces and the link to the old cases, we would like to say why proposition 3.1 is true. This is something that have been presented by Scholze in several examples [SC1], [SC2], here I would like to give other examples and the aim of my lectures will be to give all the tools to understand the example as an exercise. Note that $\mathbb{Q}_{p^\infty}$ and $\mathbb{F}_p((t))$ are both complete fields for a non arch. valuation). Nothing changes if we pass to the completion K of $\mathbb{Q}_{p^\infty}$ in ch.0 and to $K^\flat$ in ch.p the completion of the perfection $\mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$ (completion for the t-adic topology, it is also an absolute value). Hence it is enough to prove the relation between K and $K^\flat$. We will connect the two by considering p which is deeply ramified to t which is also deeply ramified. So we may suppose to replace p by the variable t....Note that if we take K° and $K^{\flat\circ}$ the subring of integral elements then we have

$$K^\circ/p = \mathbb{F}_p[p^{\frac{1}{p^\infty}}] \simeq \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t = K^{\flat\circ}/t$$

Using this one can define a non additive map $K^\flat \longrightarrow K$ sending $x \to x^\sharp$ which should send p to t and on $K^{\flat\circ}$is given by sending x to $\lim_{n\to\infty} y_n^{p^n}$ where $y_n \in K^\circ$ is a lifting of the images of the$x^{\frac{1}{p^n}}$'s in $K^\circ/p = K^{\flat\circ}/t$ In this terms we may seen

$$K^\flat = \lim_{\leftarrow} K^\circ$$

where the inverse limit is given by the p=power and the map is

$$x \to (x^\sharp, (x^{\frac{1}{p}})^\sharp, \ldots\ldots)$$

Note that $K^\flat$ is perfect: if we take $x \in K^\flat$, then $x^{\frac{1}{p^n}}$ are all uniquely defined by x, so the map above is in one sense justified. So if we start with a finite extension L of $K^\flat$ then we would like to associate an extension $L^\sharp$ of K. Say that L is the splitting field of $X^d + a_{d-1}X^{d-1} + \ldots a_0$, because $K^\flat$ is perfect, this is also the splitting field of is $X^d + a_{d-1}^{\frac{1}{p^n}}X^{d-1} + \ldots a_0^{\frac{1}{p^n}}$ for all $n \geq 0$ (same degree..) Then $L^\sharp$ can be defined as the splitting field of $X^d + (a_{d-1}^{\frac{1}{p^n}})^\sharp X^{d-1} + \ldots (a_0^{\frac{1}{p^n}})^\sharp$. for n large enough....: these fields stabilize for $n \to \infty$.... This is the "example" as explained by Scholze.

Remark 3.2. Here I would like to start with an example for $p \neq 2$. Consider $X^2 - t$, over $K^\flat$, because $K^\flat$ is perfect then the splitting field of $X^2 - t$ is the splitting field of $X^2 - t^{\frac{1}{p^n}}$. Infact

$$t^{\frac{1}{2p^n}} = (t^{\frac{1}{p^n}})^{\frac{p^n+1}{2}} p^{-\frac{1}{2}}$$

On the other hand one may define at this point the associated extension $L^\sharp$ as the splitting field of all the various liftings of $X^2 - t^{\frac{1}{p^n}}$, $n \geq 0$, which can be written easily as $f_n(X) = X^2 - p^{\frac{1}{p^n}}$, $n \geq 0$. A priori we would have a sequence of extensions L_n of K but one notes that the different ideal $\delta_{\frac{L_n}{K_n}}$ is generated by $f_n'(p^{\frac{1}{2p^n}})$: whose p-adic valuation is $\frac{1}{2p^n}$ and it tends to zero as $n \to \infty$. Because the different tells us about ramification then we see that the extension $\frac{L_n}{K_n}$ is getting less ramified as n goes to ∞. $K_n = \mathbb{Q}_p(p^{\frac{1}{p^n}})$ are the intermediate extensions of $\mathbb{Q}_p$ (automatically complete...). Then we are saying that at the limit we *don't have more ramification* but all is ruled by the square root of p.

The example used by Scholze is $X^2 - 7tX + t^5$ as polynomial. To give a direct proof of the proposition above.

Let's do some *math body building* about the situation we have at hands. Remember we need to translate something from ch.0 to ch.p ....but maintaining some properties...at least same Galois, or same finite separable extensions.

4 From ch.0 to ch.p

Here we follow [BC] (pararagraph 4). Consider $\mathbf{B}$ a $\mathbb{F}_p$ algebra.

$\Rrightarrow$ $(*)$ You may think (and this will be relevant for us) to any field K of characteristic 0 that is complete (we will indicate exactly when this condition is required!!) with respect to a fixed valuation of rank 1 (discrete or not...) that has a residue field k of characteristic $p \neq 0$. In particular we will say K a p-adic local field if it is a field as before but with perfect residue field and discrete valuation. Then for any K as before we can take $\mathcal{O}_K$ its ring of integers and then $\mathbf{B}$ can be chosen $\mathcal{O}_K/(p)$...... Of course this will be our "champion" but one can take also a genuine $\mathbf{B} = \mathbb{F}_p[t]$..... So, let's take

$$R(\mathbf{B}) = \lim_{\leftarrow} \mathbf{B} = \{(x_0, x_1, \dots) \in \prod_{n \geq 0} \mathbf{B} \mid x_{n+1}^p = x_n\}$$

It has naturally a ring structure and it is perfect: the power p is surjective and injective because if $(x_j)^p = 0$ then $x_{j-1} = x_j^p = 0$ for all j. We have a natural map $R(\mathbf{B}) \to \mathbf{B}$ which is given by $(x_j) \to x_0$. If $\mathbf{B}$ was perfect of ch.p, then $R(\mathbf{B}) = \mathbf{B}$.

We will be mainly interested in the case when we start with a field as in $(*)$ before. In that case we will use the notation $R(\mathcal{O}_K/(p))$. If we start with a K_1 as before which is not complete and such that K is its completion, then we have by density that $R(\mathcal{O}_K/(p)) = R(\mathcal{O}_{K_1}/(p))$. **Why** (p)**?** Of course we could have taken $(p) \subset \mathcal{A} \subset \mathcal{O}_K$ (we are saying that K is p-adically complete!!!!!now!!!!) such that $\mathcal{A}^N \subset (p)$. then we can construct $R(\mathcal{O}_K/(\mathcal{A}))$. Note that in all these definitions an element x is given by $x = (x_{(0)}, x_{(1)}, \dots, x_{(n)} \dots)$ and $x^{\frac{1}{p^n}}$ is given by $(x_{(n)}, x_{(n+1)}, \dots, x_{(m+m)} \dots)$. Moreover one can also take

$$\lim_{\leftarrow} \mathcal{O}_K$$

the inverse limit over the p-th power. This is indipendent of the ideal $\mathcal{A}$ as before.

Then we have a natural map

$$\lim_{\leftarrow} \mathcal{O}_K \to R(\mathcal{O}_K/(\mathcal{A}))$$

just the reduction. But this map is a multiplicative one!! **But it is a bijection!!**

Proof. . We are going to propose an inverse. In fact we can define: take $x = (x_n) \in R(\mathcal{O}_K/(\mathcal{A}))$, take any lifting $\widehat{x_{(n)}} \in \mathcal{O}_K$. then one can define for any n

$$l_n(x) = \lim_{m \to \infty} \widehat{x_{(n+m)}}^{p^m}$$

In fact for any $m' \geq m \geq 0$ we have $\widehat{x_{(n+m')}}^{p^{m'-m}} \simeq \widehat{x_{(n+m)}}$, mod.$\mathcal{A}$. And because $(p) \subset \mathcal{A}$ we have $\widehat{x_{(n+m')}}^{p^{m'}} \simeq \widehat{x_{(n+m)}}^{p^m}$, mod$\mathcal{A}^{m+1}$. Hence the limit is well defined and it is indipendent upon the lifting (ex). Hence the inverse map $x \to (l_n(x))$ is well defined and giving the inverse. $\square$

Remark 4.1. If we were using $(p) = \mathcal{A}$ then $l_n(x) = l_0(x^{\frac{1}{p^n}})$. And we have defined $l_0(x) = (x)^\sharp$ in the notation of section 1.

Remark 4.2. In particular $R(\mathcal{O}_K/(\mathcal{A})) \simeq R(\mathcal{O}_K/(p))$!! In any case the inverse from $R(\mathcal{O}_K/(\mathcal{A})) \simeq R(\mathcal{O}_K/(p))$ to

$$\lim_{\leftarrow} \mathcal{O}_K$$

is given by $(l_n(x)) = (x_{(n)})$. Because we have an identification we can put an additive structure on

$$\lim_{\leftarrow} \mathcal{O}_K$$

. The multiplicative structure is just $(xy)_{(n)} = x_{(n)}y_{(n)}$. But for the addition we have

$$(x+y)_{(n)} = \lim_{m \to \infty} (x_{(n+m)} + y_{(n+m)})^{p^m}$$

Note that it is important to have a complete valuation!!!! In order to have the inverse..if not we won't have...such a construction even if we have an identification..between $R(K)$ and $R(\hat{K})$ using the limit on the quotient. To go in reverse we need complete!

Do we have other properties for such a "correspondance"?

Consider that we start with a field K as before and we consider an algebraic (separable) closure $\overline{K}$ and then its completion $\widehat{\overline{K}}$ which is algebraically closed (??). We can play the same game for $\widehat{\overline{K}}$. Note that the ideal $\mathcal{A}$ cannot be the maximal ideal $\mathcal{M}$ of $\widehat{\overline{K}}$ relative to its valuation... it doesn't exist N such that $\mathcal{M}^N \subset (p)$! We have to choose a different one (p) is ok....for example....$(p^{\frac{1}{p^n}})$ too... In any case we can play the usual game for $\widehat{\overline{K}}$ and to obtain the identification we had before between

$$\lim_{\leftarrow} \mathcal{O}_{\widehat{\overline{K}}}$$

and $R(\mathcal{O}_{\widehat{\overline{K}}}/(\mathcal{A}))$ for some choice of $\mathcal{A}$ ...among them (p) ..... We my call such a perfect ring of ch.p simply R (even if it is built using an algebraic closure).

we have

Proposition 4.3. *With the definition of R we had before: R is separated and complete for an absolute value and it is algebraically closed. It is not noetherian.*

Proof. we refer as usual to [BC] 4.3.3. We denote by $| \bullet |_p$ the normalized absolute value in $\widehat{\overline{K}}$ which is given by $|p|_p = 1/p$. Then we may define the abs.value $| \bullet |: R \to p^{\mathbb{Q}} \cup 0$ as $|x| = |(x_{(n)})| = |x_{(0)}|_p$. The key point is that $|x_{(n)}|_p = |x_{(0)}|_p^{p^n}$. The otehr point is the fact that we can see R as closed inside the product $\prod \mathcal{O}_{\widehat{\overline{K}}}/(p)$ endowed with the discrete topology. For the alg. closed see [BC] 4.3.5. $\square$

Note that the valuation is not discrete.

So, if we start with K we may associated $R(\mathcal{O}_K/(\mathcal{A}))$, briefly denoted by $R(\mathcal{O}_K) = R_K$, even if it complete or not.... in any case $R_K = R(\mathcal{O}_K) \simeq R(\mathcal{O}_{\widehat{K}}) = R_{\widehat{K}}$, its completion and it is indipendent upon the ideal $\mathcal{A}$ one is going to use in $\widehat{K}$.

So we have *a way to associate from ch.0 complete something perfect and complete in ch.p*. Naturally the same in ch.p: we associate something complete and perfect..... Naturally we may consider $K \subset L \subset \widehat{K}$, L as before (complete..) and we can choose an ideal $\mathcal{A}$ of $\mathcal{O}_L$ as before. We can define R_L and if we denote $\widehat{L}$ its completion we will have $R_L = R_{\widehat{L}}$ and a presentation

$$R_L = \{(x_{(n)}) \in \prod \mathcal{O}_{\widehat{L}} \mid (x_{(n+1)})^p = x^{(n)}\}$$

In particular we will have if $L \subset L'$ that $R_L \subset R_{L'}$ as perfects subrings and all are subrings of R and R is complete for a valuation. But this valuation has a meaning also on the various R_L and in particular on $Frac(R_L)$ its fraction field and this is complete for the $|\bullet|$-adic topology of R! It is separeted, complete and for a valuation. What about the **Galois groups**?

Here we cannot expect to have an exact correspondance and this is where we have to put hypotheses on the field where we started from.

5 Galois Correspondance

It is here, where the work of Fontaine-Winterberger [FW] and later Ramero-Gabber [GR], Faltings and finally Scholze has been directed. In [FW] we have for the first time the way of detecting a good set of p-adic fields where we could have a kind of Galois correspondance. This has been done using the method of the field of norms, which associated to a field even if *not complete* (and in all characteristic) another valuation field complete but not perfect in ch.p.

This correspondance was a very sophisticated one (at the prize of loosing the perfection of the ch.p field and not requiring to be complete), but required some hypothesis on the field K but not really easy to be generalized. It has been Scholze who has found the good definition and the good in order to have a "fine tuning" correspondance but also a correspondance that can be generalize. But changing the original definition of good fields. Here we are only over "one point".

Remark 5.1. If we start with $K = \mathbb{Q}_p$ then $R_L = \mathbb{F}_p$. ...we may consider then $\overline{\mathbb{Q}}_p = K$ and in this case we can define $R_{\overline{\mathbb{Q}}_p}$.

As we said originally we had [FW] work: they were not interested on the completed fields and on the generalization but only in associating a correspondance between for Galois Groups and for fields extensions. They worked with arithmetically profinite extensions L of a local field K (i.e. K discrete valuated and with perfect ?? residue field). Their champion K of such a local field was $\mathbb{Q}_p$ and/or some of its finite extensions. But not only this. Not complete a priori...... neither Galois.... even not of ch.0... Then the extensions they have in mind are of the type L such that their Galois closure has Galois group whose higher ramification groups are all open + some condition on the ramification. This is the case if the Galois group is a p-adic Lie group and the inertia is open (see [FW] and [BC] 13.3) . Clearly a chain of totally (wildly) ramified (or totally wildly ramified up to a finite) extensions of $\mathbb{Q}_p$ do the job.... a kind of deeply ramified... note that they ask the residue field to be perfect. In any case the exetnsion should be infinite.

Remark 5.2. $\mathbb{Q}_{p^\infty}$ and $\mathbb{Q}(\zeta^{\frac{1}{p^\infty}})$ are of this kind... the example we should have in mind is: let K a local p-adic field with $(K_n)_{n\geq 0}$ an increasing sequence of finite Galois extensions and we may define $K_\infty = \cup K_n$, We can define $\Gamma = Gal(K_\infty/K)$ assume that from n_0 on K_n/K_{n_0} is totally ramified and with $Gal(K_n/K_{n_0}) \simeq \mathbb{Z}/p^{n-n_0}\mathbb{Z}$. This has as a corollary that $Gal(K_\infty/K_{n_0})$ is a $\mathbb{Z}_p$-extension totally ramified and it satisfies our request. For these reasons we may speak of "deeply ramified" extension. A maximal totally ramified extension of $\mathbb{Q}_p$ or any of its finite extensions is one of them.

It is on the absolute Galois group of the field K_∞ as in 5.2 where we can have a kind of Galois correspondance for the Galois extensions.

Proposition 5.3. *Let $L_1 \subset L_2$ finite extensions of K_∞ in $\overline{K}$. We can associate as before, the perfect fields $Frac(R_{L_1})$ and $Frac(R_{L_2})$ inside $Frac(R) = Frac(R_{\overline{K}})$.*

Then the extension (separable: they are perfect) $Frac(R_{L_2})/Frac(R_{L_1})$ *is of finite degree* $[L_2 : L_1]$. *If* L_2/L_1 *is Galois, then we have an identification*

$$Gal(L_2/L_1) \simeq Gal(Frac(R_{L_2})/Frac(R_{L_1})$$

The basic thing is that we work with fields with an infinite p-parts in their ramification...deeply ramified. It does not say that we have an identification of any Galois extension of K_∞ with a Galois extension of $Frac(R_{K_\infty})$. But only at level of Galois groups...if we want the identification of the Galois extensions we need the imperfect field of norms (i.e. we associate something which is not perfect) or we should use *perfectoid fields*. In the proposition 6.3 we could have had a perfect correspondance if we ask K_∞ complete and we condider only extension of the similar kind (Perfectoid), as we will see by Scholze.

6 The non perfect fields

We want to define an imperfect field in ch.p in order to link the Galois extension. We do not carry all the calculations (we never did), but we only indicate the method one is going to use and to give a reason to for the old name *the field of norms*.

We consider L as an APF extensions of a local field K (perfect residue field). For our pourposes, K_∞ as before. We denote by $\mathcal{E}_{L/K}$ the set of finite extensions of K_0 (the maximal unramified extension of K inside L) and then we define $X_K(L)$ the inverse limit of all $E^* \in \mathcal{E}_{L/K}$ where the maps are given by the norms between $E'^* \subset E^*$. We add zero and we obtain $X_K(L)$.

Theorem 6.1. *$X_K(L)$ as before is a field, it admits an absolute value $\alpha = (\alpha_E)_{E \in \mathcal{E}_{L/K}} \in X_K(L)$, $\nu(\alpha) = \nu_E(\alpha_E)$. With all these data it is a local field of ch.p, hence endowed with a complete discrete valuation. The residue field k_L is isomorphic to the residue field of $X_K(L)$ (both perfect).*

.

Remark 6.2. Every complete discrete valuation field in ch.p E whose residue field is k_E is isomorphic to $k_E((t))$ where t is an indeterminated and seen as uniformizer for the t-adic valuation. In particular $X_{\mathbb{Q}_p}(\mathbb{Q}_{p^\infty}) \simeq \mathbb{F}_p((t))$. And t will correspond to a chain of uniformizers for all E^* compatible with the norms!

.

We finally have also a Galois correspondance. In fact we may consider L as an APF extension of K in $\overline{K}$. For each separable algebraic extension M of L, we can write $X_{L/K}(M)$ as the limit of $X_K(L')$ where L' is a finite separable extension of L in M (it is again APF). If M/L is finite then: $X_{L/K}(M) = X_K(M)$. We don't discuss the fact that such a functor sends separable extension of L in separable extensions of $X_K(L)$. In any case we have the fine tuning result:

Proposition 6.3. . *As before, let M_1, M_2 be separable extensions of L in $\overline{K}$. The set of the separable L- immersions of M_1 inside M_2 coincides with the set of separable $X_K(L)$-immersions of $X_{L/K}(M_1)$ in $X_{L/K}(M_2)$. Moreover If X' is an algebraic separable extension of $X_K(L)$ then there exists a separable algebraic extension M of L such that there exists a $X_K(L)$ isomorphism of $X_{L/K}(M)$ with X'. The conclusion is the fact that if $\overline{K}$ is a separable closure of K which contains L, then $\overline{X} = X_{L/K}(\overline{K})$ is a separable algebraic closure of $X_K(L)$ and $Gal(\overline{K}/L) = Gal(X_{L/K}(\overline{K})/X_K(L))$. Moreover we have an identification among the separable finite extensions of L and those of $X_K(L)$ (not only for the Galois ones)*

Hence for the non perfect field of norms the tilting from ch.0 to ch.p (for APF extensions or "deeply ramified") respects not only the Galois, but also gives an equivalence for separable finite extensions hence an identification of the Etale site!. What we are going to have is a caracterization of a certain type of fields (perfectoids or deeply ramified) where we can state a similar result. But with the following advantages :

-not only for perfect residue field

-a direct functor along the line of the *perfect field of norms*

- such a functor transforms a perfectoid in another perfectoid (remember that the field of norms works for ch.p i.e. sending ch.p in ch.p)

Remark 6.4. A final result which is going to be important for us is the fact that if we start with L, APF over a local field (even not complete, for example $\mathbb{Q}_{p^\infty} = L$ a maximal ramified extension of $\mathbb{Q}_p = K$ in its algebraic closure), then we may take its completion $\widehat{L}$, then link between the "field of norms" $X_K(L)$ can be seen inside the "perfect field of norms" i.e. that one associated with the limit of the p^{th}-powers of the completion $R_{\widehat{L}}$. I.e. and in particular $R_{\widehat{L}}$ is the completion of the perfectization of $X_K(L)$. So according to our example $X_{\mathbb{Q}_p}(\mathbb{Q}_{p^\infty}) \simeq \mathbb{F}_p((t))$. And what we have indicated as $K^\flat$ is $R_{\widehat{\mathbb{Q}}_{p^\infty}}$.

7 How to justify the iso of Galois extensions

Now we should be ready to justify Scholze's assertion. We want to link separable (they are all) extensions of $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$ with separable (they are all) extensions of $\widehat{\mathbb{Q}}_{p^\infty}$. The first is perfect and t has to be seen as a limit of elements of $\widehat{\mathbb{Q}}_{p^\infty}/(p) = \mathbb{Q}_{p^\infty}/(p)$. In particular *formally* we have $x \in K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, but also $x^{\frac{1}{p^n}} \in K^\flat$, $n \in \mathbb{Z}$ (they are all uniquely determined by x) Remember that we have an identification of $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$ with $\lim \mathcal{O}_{\widehat{\mathbb{Q}}_{p^\infty}}$ (limit on the p-th powers), if $x \in K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, it can be seen as $x = (x_{(0)}, x_{(1)}, x_{(2)}, \dots)$ in the limit ($x^p_{(j)} = x_{(j-1)}$, with $x^{\frac{1}{p}} = (y_{(0)}, y_{(1)}, y_{(2)}, \dots)$ where $y_{(i)} = x_{(i+1)}$ and so on for all the p^n-roots of x!! The map we have built up aims to link $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$ with $R_{\widehat{\mathbb{Q}}_{p^\infty}}$. First we have defined $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}}) \to \widehat{K} = \widehat{\mathbb{Q}}_{p^\infty}$ (here we need the be completed). Remember that being $K^\flat$ perfect $\lim K^\flat = K^\flat$ (limit on p-powers). We take $x \in K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, we take its reduction mod. t, then we have

$$\widehat{\mathbb{Q}}^\circ_{p^\infty}/p = \mathbb{F}_p[p^{\frac{1}{p^\infty}}] \simeq \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t = K^{\flat\circ}/t$$

we take any lift of it $\widehat{\mathbb{Q}}^\circ_{p^\infty}$ and we call it y_0, we consider $x^{\frac{1}{p}} \in K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$ (it is perfect..), then again take its reduction mod.p and take any lift in $\widehat{\mathbb{Q}}^\circ_{p^\infty}$, call it y_1. Then we have defined

$$x^\sharp = \lim_{n\to\infty} y_n^{p^n}.$$

as a map $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}}) \to \widehat{\mathbb{Q}}_{p^\infty}$. If we want the identification, we should take

$$K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}}) \to \lim_{\leftarrow} \widehat{\mathbb{Q}}^\circ_{p^\infty}$$

where the inverse limit is given by the p=power and the map is

$$x \to (x^\sharp, (x^{\frac{1}{p}})^\sharp, \dots\dots).$$

Let's go back to Scholze's example: $X^2 - 7tX + t^5$ as polynomial on $K^\flat = \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, and its splitting field. The first observation if the fact that the splitting field does not change of we take roots of the coefficients! I.e. if we consider $X^2 - 7t^{\frac{1}{p^n}}X + t^{\frac{5}{p^n}}$, $n \geq 0$. Then we should consider the linked splitting fields L_n over $\widehat{\mathbb{Q}}_{p^\infty}$ inside the its *complete*(!!) algebraic closure $\widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p$ of

$$X^2 - 7(t^{\frac{1}{p^n}})^\sharp X + (t^{\frac{5}{p^n}})^\sharp$$

What we need to show is the fact that the solutions we have to add converge to a solutions in $\widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p$ according to the $(-)^\sharp$-contruction. Because all the liftings are the same we can have a rapresentation of the previous polynomial as

$$X^2 - 7p^{\frac{1}{p^n}}X + p^{\frac{5}{p^n}}$$

over $\widehat{\mathbb{Q}}_{p^\infty}$, for each $n \geq 0$. What we want to show is the fact that the two solutions α_n and β_n converge

$$\lim_{n\to\infty} (\alpha_n)^{p^n} = \alpha, \quad \lim_{n\to\infty} (\beta_n)^{p^n} = \beta$$

to two elements in $\widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p$ which give the expected extension of $\widehat{\mathbb{Q}}_{p^\infty}$.

how: newton polygon and discriminant.

8 Other motivations

We would like to mention some motivations of the study the representations of the Galois group of $\mathbb{Q}$ or $\mathbb{Q}_p$ and, in some cases, how the tilting to ch.p has been useful. Other applications of the perfectoid fields and, more generally, to perfectoid spaces (read rings) will be considered along the school.

a) As we tried to say there is a long history about the use of Galois representations. Chronologically we had: l-adic Tate module of the l^n-torsion points for an elliptic curve over some global field (read $\mathbb{Q}$). In general later we had Grothendieck and his school that where able to associate via étale methods l-adic cohomology groups to algebraic varieties hence l-adic Galois representations. If we start with a smooth and projective variety over a number field (global...) K , then for any prime l one is able to attach (and for every $i = 0, 1, \ldots, 2\mathrm{dim}X$),

$$\rho_l : G_K = Gal(\overline{K}/K) \to Aut_{\mathbb{Q}_l}(H^I_{et}(X_{\overline{K}}, \mathbb{Q}_l)) = GL_{d_i}(\mathbb{Q}_l)$$

where $d_i = dimH^I_{et}(X_{\overline{K}}, \mathbb{Q}_l)$. This is really indipendent upon l. If we choose $l \neq l'$, hence we obtain ρ_l and $\rho_{l'}$. If we take than a place ν not in the finite set of places where X has bad reduction and not equal to l and l' , one can state an equality of the ch.polynomials of $\rho_l(Frob_\nu)$ and of $\rho_{l'}(Frob_\nu)$. And both are in $\mathbb{Q}[T]$ (not to mentions that the roots are...Weil Numbers...hence the Riemann Hypothesys). A natural questions can be posed: what about the compatibility in the case the place ν divides l (i.e. p-adic Galois represnetations of $G_{\mathbb{Q}_p}$)?? Or in the case of bad reduction? More explicitely, if $l = \eta = p$ and $K = \mathbb{Q}$ what about the restrcition of ρ to $G_{\mathbb{Q}_p}$? I.e. a p-adic Galois representations? This has lead to the example we gave at the beginning and it has been driving us all along our presentation.

b) Fontaine equivalence beteween $G_{\mathbb{Q}_p}$ p-adic represenions and (φ, Γ)-modules associated to a decomposition $\mathbb{Q}_p \subset \mathbb{Q}_{p^\infty} \subset \overline{\mathbb{Q}}^{alg}$ and the fact that

-$\Gamma = Gal(\mathbb{Q}_{p^\infty}/\mathbb{Q}_p)$

- $Gal(\overline{\mathbb{Q}}^{alg}/\mathbb{Q}_{p^\infty})$ is a Galois group of a field of ch.p according to the field of norms. namely $G_{\mathbf{E}}$ where $\mathbf{E} = \mathbb{F}_p((t))$ (t is an indeterminate).

-Then a $\mathbb{Q}_p$ representation of $G_{\mathbf{E}}$ is nothing but a module over a Cohen ring of $\mathbf{E}$ i.e. an absolute non ramified valuation ring (complete) such that its residue field is $\mathbf{E}$ endowed with a Frobenius action. Hence we have a perfect candidate *the Amice Ring* i.e. $\mathcal{E}$ (formal in positive and convergent in negative power series).). Moreover Cherbonnier and Colmez [CC] have proved that the equivalnec eis over the Robba Ring $\mathcal{R}$. (this is only true for the cyclotomic perfectoid field...). In fact we have to remember that the first equivalence was between, for all K local fields (i.e. finite extension of $\mathbb{Q}_p$, for us directly $\mathbb{Q}_p$) then the category of p-adic representations of $G_{\mathbb{Q}_p}$ is equivalent to the category of (φ, Γ)-modules over $\mathcal{E}$, where $\mathcal{E}$ is the field associated to the Cohen ring

$$\mathcal{O}_{\mathcal{E}} = \{ \sum_{i=-\infty}^{+\infty} a_i t^i \mid a_i \in \mathbb{Z}_p \, , \lim_{i \to -\infty} \mid a_i \mid = 0\}$$

and the frobenius action φ commutes with the action of $\Gamma = Gal(\mathbb{Q}_{p^\infty}/\mathbb{Q}_p) \simeq \mathbb{Z}_p$ (a maximal totally ramified extension, and teh iso is given by the cyclotomic character χ). In particular $\varphi(t) = (1+t)^p - 1$ and for $g \in Gal(\mathbb{Q}_{p^\infty}/\mathbb{Q}_p)$ $g(t) = (1+t)^{\chi(g)} - 1$. The action is semilinear. Cherbonnier-Colmez [CC] were able (in the case of the totally ramified $\mathbb{Q}_{p^\infty}$ or, in general, on the cyclotomic tower over any local field) to extend the action to the *Robba Ring/field* :

$$R = \{ \sum_{i=-\infty}^{+\infty} a_i t^i \mid a_i \in \mathbb{Q}_p \, , \exists \epsilon < 1, | \lim_{i \to -\infty} |a_i| \epsilon^i = 0\}$$

(i.e. they converge in some annulus of radius $\epsilon \leq | - |_p < 1$, while, in general, the elements of the Cohen $\mathcal{E}$ have not radius of convergence!). This has led Berger to put some "differental "...at least on the de Rham Galois representation. But this another story.

Remark 8.1. We may use different totally ramified extensions: not only the cyclotomic one. The Lubin-Tate theory gives to us different ways of introducing them. In fact what Cherbonnier-Comez did was to use the Lubin-Tate formal groups of height 1! But we could have started from $\mathbb{Q}_{p^n}$ and to have introduced other totally ramified extensions where to apply Fontaine theory. But not to extend the theory to the level of Cherbonnier-Colmez's one i.e. to the Robba's ring. This has been Kisin-Ren's work [KR] and more recently [BE].

b) Another use of the Galois representationhas been Colmez's proof of the Langlands correspondance for $n = 2$ (Completed after with Paskunas and Dopinescu). In fact the starting point is the fact that one can associate to a p-adic galois representation a (φ, Γ) module over the Robba Ring R. ....This has bounded the Langlands correposndance to p-adic representation of $G_{\mathbb{Q}_p}$ ...not for every local field!!! Not to mention the fact that the proof is linked only to $n = 2$ i.e. representation to $GL_2(\mathbb{Q}_p)$.

c) Other global Galois representations are given via modular and automorphic forms. Reference: [GO] . The classical space of complex modular forms of weight k on $\Gamma_1(Np^\nu)$ has a basis consisting of modular forms whose q-expansions at infinity have all the coefficients which are integers. By a modular form defined over $\mathbb{Z}_p$ we understand a $\mathbb{Z}_p$-linear combination of all of them. Later Katz gave a more intrinsic definition: there exists an algebraic curve defined over the localization of $\mathbb{Z}$ at (p): $\mathbb{Q} \cap \mathbb{Z}_p$, and a canonical invertible sheaf ω such that the global sections of ω^k correspond to the classical (cuspidal) modular forms of weight k on $\Gamma_1(Np^\nu)$ defined over $\mathbb{Z}_p$ (almost..). In such a space we have Hecke operators, T_l, l not dividing Np. One has also the operators U_ℓ, $\ell \mid N$. We consider the *eigenforms* i.e. the forms which are simultaneous eigenfunctions for all the Hecke operators. We have also a diamond operator. But the conclusion is the following: given a modular forms of weight k on $\Gamma_1(Np^\nu)$ defined over $\mathbb{Z}_p$ which is an eigenform for all T_ℓ, ℓ not dividing Np and for the diamond operators then one can construct a Galois representation

(Eichler, Deligne, Shimura, Serre) with value in $\mathbb{Z}_p$ of $G_{\mathbb{Q},S}$ (maximal extension of $\mathbb{Q}$ unramified outside S, in this case S are the primes which divide N, p and ∞). We see this as "deformation of representations" because we can change the prime where we take $G_{\mathbb{Q}_\ell}$. Again we would like to study all the reductions. What about the case $G_{\mathbb{Q}_p}$? Via the (φ, Γ)-modules...

We can have another application of the theory in families (geometrically this time). This is connected with the *Eigencurve*. Here we are talking about classical p-adic modular form. Again we don't want to go through the definition. But want also to mention that they can be seen as *limit of classical* modular forms of weight k on $\Gamma_1(Np^\nu)$ defined over $\mathbb{Z}_p$. again in this set we can define Hecke operators, U_p and diamonds. We can speak about eigenforms, but of course, we have new objects other than the old one.......we can associate to each of them a Galois representation as before. The fact is that all these new eigenforms form a geometric objects and we would like to study the defomations (in the curves or in the eigenvariety) of such a representations on this geometric object. Again the way to study is via the reduction to (φ, Γ)-modules over the Robba ring R. In particular, even if each point is a global p-adic Galois representation, its restriction to $G_{\mathbb{Q}_p}$ and its associated (φ, Γ)-modules, tell us if the point is associated to a classical modular forms of weight k on $\Gamma_1(Np^\nu)$ defined over $\mathbb{Z}_p$...([KI], [EM]). For all of this [GO] and the whole volume where it is found. For example, the Fontaine-Mazur conjecture (now a theorem in many cases by work of Kisin [KI] and Emerton [EM]) predicts that if the restriction is potentially semi-stable at p then it is a classical modular eigenform. This is an amazing conjecture because it predicts something global from a very local condition on the Galois representation (This strongly suggests that a good understanding of p-adic representations of p-adic Galois groups is necessary to understand the global Langlands correspondence). For example, what local condition at p singles out the global Galois representations coming from the eigencurve?

Definition 8.2. *We say that a p-adic $G_{\mathbb{Q}_p}$ representation is trianguline of rank 2 if the associated (φ, Γ)-modules $D_{rig}(V)$ is the successive extension of rank one (φ, Γ)-modules over R.*

In [KI2] Kisin proves that for any point on the eigencurve the associated Galois representation is trianguline at p. In [EM] Emerton shows that under mild technical restrictions this condition exactly determines those representations occuring on the eigencurve up to a twist. Hence, very very roughly speaking, the eigencurve is the moduli 2-dimensional representations of $G_{\mathbb{Q}}$ with prescribed ramification away from p, together with a choice of triangulation at p.

d) Herr's Galois cohomology. If we have a group G and a reprsentation V as a free A-modules (A is a commutative ring with unit), then we can see V as a module over the group algebra $\mathbb{Z}[G]$. As a module over such a group algebra we can define a functor (to abelian groups, say) given by $V \to V^G$, where V^G are the invariant objects under the G action of V. This a functor and one would like to study its derived functors. If the group G is a Galois group: we will indicate it as *Galois*

Cohomology. A reference [SE]. If $G = G_{\mathbb{Q}_p}$ and V is a $\mathbb{Z}_p$ representation we can have a different way of calculating such a groups via (φ, Γ)-modules. This has been given by Herr in [H] and [H1].

His method starts with a field E of ch.p, consider $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$ and define , with $\sigma : E^{sep} \to E^{sep}$ the Frobenius.

$$D(V) = (E^{sep} \otimes_{\mathbb{F}_p} V)^{G_E}$$

We have $dim_E D(V) = dim_{\mathbb{F}_p}(V)$ and the Frobenius map $\varphi = \sigma \otimes id_V$ acts on $D(V)$ in a semilinear way and $\varphi(D(V))$ generates $D(V)$. A finite dimensional vector space M over E is called étale Φ-module over E if there exists a σ- semilinear φ such that $\varphi(M)$ generates M. We denote such a category as $\mathbf{\Phi} M_E^{et}$.

Proposition 8.3. *(see [FO])The functor $V \to D(V)$ is an equivalence of category between $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ and $\mathbf{\Phi} M_E^{et}$.*

Hence $V^{G_E} = D(V)^{\varphi = id_{D(V)}} = H^0(G_E, V)$. For $M \in \mathbf{\Phi} M_E^{et}$ we consider the following complex $C(M)$ of abelian groups

$$0 \to M \to M \to 0$$

where the map between the M is $\varphi - id_M$. Then if we start with $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$, we can calculate $H^i(G_E, V)$ and we may also tale $D(V)$ and hence $C(D(V)$. We have [H]6.1.2

$$H^i(C(D(V)) = H^i(G_E, V)$$

Such a result can be generalized to the case of $\mathbb{Z}_p$-representations of $G_{\mathbb{Q}_p}$. We know that $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p})$ is equivalent to the category of (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$ as we saw in proposition 2.1. In the case we use the cyclotomic tower $\mathbb{Q}_{p^\infty}$ the the $\Gamma = Gal(\mathbb{Q}_{p^\infty}/\mathbb{Q}) \simeq \mathbb{Z}_p$ and we denote γ a generator. Then we can build the following complex $C_2(M)$ for a M a (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$

$$0 \to M \to M \oplus M \to M \to 0$$

where the first map $M \to M \oplus M$ is given by $\alpha(x) = ((\varphi - id_M)(x), (\gamma - id_M)(x))$ while the second map $M \oplus M \to M$, is given by $\beta(y, z) = ((\gamma - id_M)(y) - \varphi - id_M)(z)$. Again we can start with $V \in \mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p})$, we can associate a (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$: $\mathbf{D}_{\mathcal{E}}(V) = (\mathcal{O}_{\hat{\mathcal{E}}_{nr}} \otimes_{\mathbb{Z}_p} V)^{G_E}$ and then we can construct $C_2(\mathbf{D}_{\mathcal{E}}(V))$. Again Herr's result says that

$$H^i(C_2(\mathbf{D}_{\mathcal{E}}(V)) = H^i(G_{\mathbb{Q}_p}, V)$$

[H]6.4.4.

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