A minimalist two-level foundation for constructive mathematics

Maria Emilia Maietti
Dipartimento di Matematica Pura ed Applicata
University of Padova
via Trieste n. 63 - 35121 Padova, Italy
maietti@math.unipd.it

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Abstract

We present a two-level theory to formalize constructive mathematics as advocated in a previous paper with G. Sambin.

One level is given by an intensional type theory, called Minimal type theory. This theory extends a previous version with collections.

The other level is given by an extensional set theory that is interpreted in the first one by means of a quotient model.

This two-level theory has two main features: it is minimal among the most relevant foundations for constructive mathematics; it is constructive thanks to the way the extensional level is linked to the intensional one which fulfills the "proofs-as-programs" paradigm and acts as a programming language.

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1 Introduction

In a previous paper with G. Sambin [MS05] we argued about the necessity of building a foundation for constructive mathematics to be taken as a common core among relevant existing foundations in axiomatic set theory, such as Aczel-Myhill's CZF theory [AR], or in category theory, such as the internal theory of a topos (for example in [Mai05]), or in type theory, such as Martin-Löf's type theory [NPS90] and Coquand's Calculus of Inductive Constructions [Coq90].

There we also argued what it means for a foundation to be constructive. The idea is that a foundation to develop mathematics is constructive if it satisfies the "proofs-as-programs" paradigm, namely if it enjoys a realizability model where to extract programs from proofs. If such a semantics is defined in terms of Kleene realizability [Tv88], then the foundation turns out to be consistent with the formal Church thesis (for short CT) and the axiom of choice (for short AC). In [MS05] we took such a consistency property as our formal notion of "proofs-as-programs" foundation. This notion appears very technical in comparison with the intuitive proofs-as-programs paradigm. Actually, one of our aims is to explore which commonly conceived proofs-as-programs theories satisfy our formal notion of consistency with CT and AC together.

To this purpose in [MS05], we first noticed that theories satisfying extensional properties, like extensionality of functions, can not satisfy our proofs-as-programs requirement. This is due to the well known result by Troelstra [Tro77] that in intuitionistic arithmetics on finite types extensionality of functions is inconsistent with CT+AC (see [TvD88] or [Bee85]). At present, only theories presented in terms of an intensional type theory, such as Martin-Löf's one in [NPS90], seem to fit into our paradigm.

This led us to conclude that in a proofs-as-programs theory one can only represent extensional concepts by modelling them via intensional ones in a suitable way, for example as done in [SV98, Alt99, AMS07, Hof97].

Therefore in [MS05] we ended up in saying that a *constructive* foundation for mathematics must be equipped with *two levels* of the following kind: with an *intensional* level that acts as a programming language and is the actual proofs-as-programs theory; with an *extensional* level that acts as the set theory where to formalize mathematical proofs. Then, the constructivity of the whole foundation relies on the fact that the extensional level must be implemented over the intensional level, but not only this. Indeed, in [MS05] following Sambin's forget-restore principle in [SV98] we required that extensional concepts must be abstractions of intensional ones as result of forgetting irrelevant computational information. Such information is then restored when we implement extensional concepts back at the intensional level. Here we push forward this by saying that the needed extensional concepts can be obtained by just abstracting over equalities of intensional ones. Hence, it is sufficient *to build a quotient model over the intensional theory* in order to be able *to implement the extensional level in the intensional one.* In this paper we present an example of such a two-level constructive foundation that is also minimal among the most relevant constructive ones.

Intensional level. The intensional level of our two-level foundation is essentially obtained by extending the typed calculus introduced in [MS05], and called Minimal Type Theory (for short mTT), with the notion of "collection" and corresponding constructors needed to implement the power collection of a set as a quotient. We still call mTT such an extension.

Then, the original version of Minimal Type Theory in [MS05] essentially represents the set-theoretic part of the mTT version presented here, called mTT$_{set}$. We say "essentially" because here we adopt a version of type theory *with explicit substitution rules and without the ξ-rule for lambda-terms as in [Mar75].*

The reason to adopt such a modified version is due to the fact that, as suggested to us by P. Martin-Löf and T. Streicher, its set-theoretic fragment mTT$_{set}$ directly enjoys Kleene's realizability interpretation of intuitionistic connectives [Tv88], and hence it turns out to satisfy our proofs-as-programs requirement. We expect that this is also the case for the whole mTT by further extending the realizability interpretation to support collections.

Instead, we still do not know whether the mTT version in [MS05] - that includes the ξ-rule - satisfies our proofs-as-programs requirement, namely whether it is consistent with the formal Church thesis and the axiom of choice. This problem can be reduced to asking whether intensional Martin-Löf's type theory in [NPS90], even with no universes, is consistent with the formal Church thesis

$$(CT) \qquad \forall f \in Nat \to Nat \quad \exists e \in Nat \qquad (\, \forall x \in Nat \;\; \exists y \in Nat \;\; T(e,x,y) \;\&\; U(y) =_{\mathrm{N}} f(x)\,)$$

where $T(e,x,y)$ is the Kleene predicate expressing that y is the computation executed by the program numbered e on the input x and $U(y)$ is output of the computation y. This technical problem seems to be still open. Luckily, we realized that we do not need to solve such a problem: indeed, we can take a version of mTT with explicit substitutions and without the ξ-rule, to satisfy our proofs-as-programs requirement in an easier way, because *the absence of the ξ-rule does not affect the properties of the quotient model we will build over it.* In particular, the quotient model will validate extensionality of functions as lambda-terms anyway. This was first noticed in categorical terms in [CR00, BCRS98].

Extensional level. The extensional level of our two-level foundation is taken to be an *extensional* dependent type theory with quotients, called emTT. This extends that presented in [Mai07] with collections and related constructors needed to represent the power collection of a set with ε-relation and comprehension used in everyday mathematical practice. The set-theoretic part of emTT includes the fragment without universes of extensional Martin-Löf's type theory in [Mar84].

Quotient model. We will interpret our extensional theory emTT in a quotient model built over mTT. This model is based on the well-known notion of total setoid à la Bishop [Bis67] and the interpretation shows that the design of emTT over mTT satisfies Sambin's forget-restore principle in [SV98]. Indeed, the interpretation represents the process of restoring all the irrelevant computational information missing at the extensional level. Moreover, it turns judgements of emTT, which are undecidable as those of extensional Martin-Löf's type theory in [Mar84], into judgements of mTT that are all decidable. This forget-restore process is very evident when looking at the design of emTT-propositions and their interpretation into mTT. In fact, whilst in emTT, as in mTT, all propositions are identified with collections of their proofs, in emTT, despite of mTT, they are inhabited by at most only one proof in order to express the fact that emTT-*propositional proofs are indeed irrelevant.* This allows us to introduce a canonical proof-term true to express that ϕ is valid in emTT if and only if we can derive $\mathsf{true} \in \phi$ in emTT. It is only when we interpret in mTT a derived judgement of the form $\mathsf{true} \in \phi$ that we need *to restore a specific proof-term* containing all the forgotten computational information about its derivation.

Benefits of adopting a two-level theory. We hope that making explicit an extensional level over an intensional type theory, as we do here with emTT over mTT, will be useful to formalize mathematics in intensional type theory. Indeed, in the current practice of formalizing mathematics in intensional type theory, one ends up to work with setoid constructions, and hence to work *within a quotient model.* Here we extract a *theory valid in one of such setoid models.* Therefore, one is then dispensed to work directly in the model with all the heavy type-theoretic details regarding setoids and families of setoids. He can develop and formalize his theorems in a simpler extensional theory like our emTT. The interpretation of the extensional level into the intensional one, given once and for all, guarantees that a formalization of theorems at the extensional level is then inherited at the intensional one.

Open issues. Our extensional level emTT does *not include* all the type-theoretic constructors that our quotient model can support. For example, our quotient model over mTT supports effective quotients on generic collections and not only effective quotients over sets as in emTT. Moreover, in our quotient model every object, which is a quotient of an intensional set over an arbitrary equivalence relation, is covered by a quotient copy of an intensional set, namely by a quotient of an intensional set over the identity relation. In the case we build our two-level foundation by taking Martin-Löf's type theory as our intensional level, the above observation has a very important consequence: the axiom of choice, which is not valid over generic quotients, turns out to be valid over copies of intensional sets in the quotient model. This implies that we can consistently add an axiom expressing that "every extensional set is covered by a set satisfying the axiom of choice" to the extensional theory abstracted over Martin-Löf's type theory. This axiom was first noticed by P.Aczel and expressed as the Presentation axiom in his CZF theory (see [AR]). We leave to future research how to formalize such an axiom in an extensional type theory abstracted over Martin-Löf's one, and more generally how to formalize the precise *internal language of our quotient model* over mTT or extensions, namely the theory that fully captures all the type-theoretic constructors that can be modelled via quotients in it.

Minimality. The presence of two levels in our foundation facilitates its comparison with other foundations, given that we can choose the most appropriate level at which to make the comparison. To establish the minimality of our foundation we will compare intensional theories, such as Martin-Löf's one or the Calculus of Inductive Constructions, with its intensional level, while extensional theories, such as the internal calculus of a generic topos (as devised, for example, in [Mai05]) or Aczel-Myhill's CZF theory, with its extensional level. Also logic enriched type theory in [GA06] can be compared with our mTT. Indeed it appears as a fragment of our mTT except that, being just a many-sorted logic on Martin-Löf's type theory, its propositions are not inhabited with proofs and they are not seen as collections (or sets) of their proofs as in our mTT. In mTT we use such a property to represent useful constructions on subsets.

Two-level theories, where one level is related to the other via a quotient completion, already appeared in the literature. One of this is Hyland's effective topos [Hyl82]. There the underlying theory is given by a tripos [HJP80], namely a realizability model of many-sorted intuitionistic logic indexed on classical set theory. Then the topos is obtained by freely adding quotients to a regular category associated to the tripos [Car95]. However, the effective topos can be seen as obtained by a quotient completion on a lex category, too [Car95]. This latter completion is closer to our quotient completion. The precise correspondence between our quotient completion and the ones existing in the literature of category theory is left to future work with the study of a general notion of quotient completion.

Summary. The main contributions of this paper are the following:

- We introduce a two-level foundation for constructive mathematics where both levels are given by type theories à la Martin-Löf: one called mTT is *intensional* and the other called emTT is *extensional.* They both essentially extend with collections previously introduced theories respectively in [MS05] and in [Mai07]. The foundation is *minimal* among the most relevant known constructive foundations in type theory, or in set theory, or in category theory, to be compared with ours at the appropriate level.

- The extensional level emTT is interpreted in a quotient model à la Bishop built over the intensional one mTT by means of *canonical isomorphisms*. This is because equality of emTT types gets interpreted into an isomorphism of intensional types. As a consequence, emTT can be viewed as a language *to reason within our quotient model* over mTT. As an application, we get that the emTT-formulation of the axiom of choice turns out to be interpreted in mTT as exactly Martin-Löf's extensional axiom of choice in [ML06, Car04], and hence it is not valid. Even the axiom of unique choice is expected not to be valid in emTT as advocated in [MS05].

- We adopt an intensional version of mTT *without the ξ-rule for lambda-terms*, after noticing from [CR00, BCRS98] that its presence is irrelevant to interpret the extensional level via quotients. As said in [Mar75], the absence of the ξ-rule opens the way to interpret mTT via Kleene's realizability interpretation [Tv88], and hence to show that it satisfies our proofs-as-programs requirement of consistency with CT+AC, as required to the intensional level according to our notion of *constructive two-level foundation*. This lets us to avoid the problem of proving consistency with CT+AC for intensional theories with the ξ-rule as in [MS05, NPS90], which is still open.

2 The intensional level mTT

Here we briefly describe the intensional level of our two-level constructive foundation. It consists of an intensional type theory in the style of Martin-Löf's one in [NPS90], which essentially extends that presented in [MS05] with the notion of "collection". Indeed, the version in [MS05] called Minimal Type theory (for short mTT) essentially corresponds to the set-theoretic part of that presented here, which we still call mTT. We says "essentially" since the set-theoretic part of our new mTT, called here mTT_{set}, has different rules about equality from that in [MS05].

We thought of modifying the original calculus in [MS05] for the following reasons. First, we wanted to extend the calculus in [MS05] with collections and the necessary constructors to support power collections of sets via quotients. This opens the way to formalize various mathematical theorems where power collections are used, like, for example, those about formal topology [Sam03, Sam09, Samar]. As a consequence we had to equip the modified calculus with propositions closed under usual intuitionistic connectives and quantification over generic collections. We then called *small propositions* those closed only under quantification over sets beside usual intuitionistic connectives. Lastly, we modified the equality rules in order to easily satisfy the proofs-as-programs paradigm, namely the consistency of mTT with the axiom of choice and the formal Church thesis.

More in detail, the typed calculus mTT is written in the style of Martin-Löf's type theory [NPS90] by means of the following four kinds of judgements:

$$A\ type\ [\Gamma] \qquad A = B\ type\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad a = b \in A\ [\Gamma]$$

that is the type judgement (expressing that something is a specific type), the type equality judgement (expressing when two types are equal), the term judgement (expressing that something is a term of a certain type) and the term equality judgement (expressing the definitional equality between terms of the same type), respectively, all under a context Γ. The contexts Γ of these judgements are formed as in [NPS90] and they are telescopic [dB91] since types are *dependent*, namely they are allowed to depend on variables ranging over other types. The precise rules of mTT are given in the appendix 6.

Types include collections, sets, propositions and small propositions and hence the word *type* is only used as a meta-variable, namely

$$type \in \{col, set, prop, prop_s\}$$

Therefore, in mTT types are actually formed by using the following judgements:

$$A\ set\ [\Gamma] \qquad A\ col\ [\Gamma] \qquad A\ prop\ [\Gamma] \qquad A\ prop_s\ [\Gamma]$$

As in [MS05], the general idea is to define a many-sorted logic, but now sorts include both sets and collections. The main difference between sets and collections is that *sets are those collections that are inductively generated*, namely those whose most external constructor is equipped with introduction and elimination rules, and all of their collection components are so. According to this view we will allow elimination rules of sets to act also towards collections.

Our sets will be closed under the empty set, the singleton set, strong indexed sums, dependent products, disjoint sums, lists. These constructors are formulated as in Martin-Löf's type theory with the modification that their elimination rules vary on all types. In order to view sets as collections, we add the rule **set-into-col**

$$\frac{A\ set}{A\ col}$$

The logic of the theory is described by means of propositions and small propositions. Small propositions are those propositions closed only under intuitionistic connectives and quantification over sets. To express that a small proposition is also a proposition we add the subtyping rule **prop$_s$-into-prop**

$$\frac{A\ prop_s}{A\ prop}$$

As explained in [MS05], since we restrict our consideration only to mathematical propositions, it makes sense to identify a proposition with the collection of its proofs. To this purpose we add the rule **prop-into-col**

$$\frac{A\ prop}{A\ col}$$

However, proofs of small propositions are inductively generated. Hence, small propositions, as propositions in [MS05], are though of as sets of their proofs by means of the rule **prop$_s$-into-set**

$$\frac{A\ prop_s}{A\ set}$$

The rules **prop$_s$-into-set** and **prop-into-col** allow us to form the strong indexed sum of a small propositional function $\phi(x)\ prop_s\ [x \in A]$, or simply of a propositional function,

$$\Sigma_{x\in A}\phi(x)$$

both on sets and on collections. Given that we will define a subset as the equivalence class of a small propositional function, then the **prop$_s$-into-set** rule is relevant to turn a small propositional function on a set into a set, and hence to represent functions between subsets as in [SV98] and to represent families indexed on a subset as advocated in [Samar]. The same can be said about subcollections. Moreover, the identification of a proposition with the collection (or set) of its proofs allows also to derive all the induction principles for propositions depending on a set, because set elimination rules can act towards all collections including propositions.

In order to interpret the power collection of a set as a quotient of propositional functions, to mTT we add the collection of small propositions $\mathsf{prop_s}$ and the collection of functions from a set towards $\mathsf{prop_s}$. Since such function collections towards $\mathsf{prop_s}$ are instances of dependent product collections, to easily show some meta-theoretic properties about mTT we will consider an extension of mTT, called mTT^{dp}, with generic dependent product collections (see the appendix 6 for its rules). Finally, by still keeping the minimality of our two-level foundation, we close mTT collections under strong indexed sums in order to give a simple categorical interpretation of the extensional level in the model we will build over mTT.

It is worth noting that the subtyping relation of propositions into collections and that of small propositions into sets, via the rules **prop-into-col** and **prop$_s$-into-set** respectively, are very different from the subtyping relation of sets into collections via the **set-into-col** rule, or that of small propositions into propositions via the **props-into-prop** rule. Indeed, the subtyping rules **prop-into-col** and **prop$_s$-into-set** do not affect the elimination rules of propositions and small propositions: they express a *merely inclusion*. Instead the subtyping rules **set-into-col** and **props-into-prop** take part in the definition of sets and in that of small propositions, because elimination rules of sets and of small propositions act respectively towards all collections and towards all propositions.

There are important motivations, already explained in [MS05], behind the fact that elimination rules of propositions act only towards all propositions and not towards all collections, as well as those of small propositions do not act towards all sets.

First, as said in [MS05], propositions have their own distinct origin, and only *a posteriori* they are recognized as collections of their proofs. Then, a more technical reason is that we want to prevent the validity of the axiom of choice. Indeed, as described in [MS05], the axiom of choice turns out to be valid if we allow elimination rules of propositions towards all collections or of small propositions towards all sets, because the existential quantification would be then equivalent to the corresponding strong indexed sum on the same constituents [Luo94]. The reason to reject the general validity of the axiom of choice in mTT is to get a minimalist foundation compatible with the existing ones, including the internal theory of a generic topos where the axiom of choice is not always valid. All this attention to avoiding the validity of the axiom of choice was paid in [MS05] because there we were trying to get our minimal foundation by modifying Martin-Löf's intensional type theory in [NPS90] (here called MLTT). MLTT is not minimal just because it validates the axiom of choice, given that it follows the isomorphism "propositions as sets" and hence it identifies the existential quantifier with the strong indexed sum. To discharge such an isomorphism, and hence the validity of the axiom of choice, it is sufficient to introduce a primitive notion of propositions with the mentioned restrictions on their elimination rules. As result of this process our mTT version with collections, as well as that in [MS05], can be naturally embedded in Martin-Löf's intensional type theory [NPS90], if we interpret sets as sets in a fixed universe, for example in the first universe U_0 in [NPS90], and collections as generic sets. Then, propositions are interpreted as sets, always by following the isomorphism "propositions as sets", and, after identifying small propositions with sets in U_0, the collection of small proposition will be of course interpreted as U_0 itself.

In [MS05] we chose to work with MLTT in order to get a minimalist proofs-as-programs foundation for its intensionality. Indeed, extensional theories, such the internal calculus of a topos or Aczel-Myhill's CZF, can not satisfy our "proofs-as-programs" requirement (see [MS05]). Actually, in [MS05] we designed a version of mTT of which we still do not know whether it satisfies our proofs-as-programs requirement, namely whether it is consistent with the axiom of choice and the formal Church thesis. This problem can be reduced to asking whether intensional Martin-Löf's type theory in [NPS90] is consistent with the formal Church thesis. Here, we do not solve such problems but we adopt a version of mTT that hopefully satisfies our requirement by means of Kleene's realizability interpretation [Tv88].

We got to this version after a suggestion by T. Streicher and P. Martin-Löf already reported in [Mar75]. There it is said that Kleene's realizability interpretation of set-theoretic constructors validates the first order version of Martin-Löf's type theory with explicit substitution rule for terms

$$\text{sub)}\quad \frac{\begin{array}{l} c(x_1,\ldots,x_n) \in C(x_1,\ldots,x_n)\ \ [\,x_1 \in A_1,\,\ldots,\,x_n \in A_n(x_1,\ldots,x_{n-1})\,] \\ a_1 = b_1 \in A_1\ \ldots\ a_n = b_n \in A_n(a_1,\ldots,a_{n-1}) \end{array}}{c(a_1,\ldots,a_n) = c(b_1,\ldots,b_n) \in C(a_1,\ldots,a_n)}$$

in place of the usual term equality rules in [NPS90]. Hence this modified first-order version of MLTT is consistent with the formal Church thesis and, in turn, satisfies our proofs-as-programs requirement, given that the axiom of choice is a theorem there. Instead, the version with the original term equalities in [NPS90] is not validated by Kleene's realizability interpretation just because of the presence of the ξ-rule

$$\xi\ \frac{c = c' \in C\ [x \in B]}{\lambda x^B.c = \lambda x^B.c' \in \Pi_{x\in B} C}$$

Now, given that, as we say above, we can interpret our version of mTT in Martin-Löf's type theory with at least one universe, if we perform the same change of equality rules for mTT then Kleene's realizability interpretation surely validates the set-theoretic version of mTT and hopefully also the whole version [1], by providing a proof of mTT-consistency with the axiom of choice and formal Church thesis (the axiom of choice holds because the realizability interpretation interprets the existential quantifier as the strong indexed sum).
Luckily, we can take this modified version of mTT without the ξ-rule as the *intensional level* of our desired two-level foundation with no effect on the interpretation of its extensional level. Indeed, the above change of term equality rules does not affect the properties of the quotient model we are going to build over mTT and where we will interpret emTT. This was first noticed in categorical terms in [CR00, BCRS98] (see remark 4.23).

Note that our mTT does not include the boolean universe U_b used in [Mai07] to derive disjointness of binary sums. Indeed, in the presence of a collection of small propositions this is derivable anyway (see the proof of theorem 4.20).

Our present version of mTT, as that in [MS05], can be still though of as a predicative version of the Calculus of Inductive Constructions based on [Coq90] with types and propositions: mTT collections and

[1] The realizability interpretation of the collection of small propositions, as well as of the first universe in Martin-Löf's type theory in [NPS90], is a delicate point. We expect to interpret it as the subset of codes of small propositions.

sets are simply interpreted as types, propositions as themselves, and the collection of small propositions as the type of all propositions.

Of course, it makes sense to compare mTT just with intensional type theories as those already mentioned.

3 The extensional level emTT

Here we briefly describe the extensional level of our desired two-level foundation. This is the level where we will actually formalize constructive mathematics.

As well as the intensional level, it consists of a type theory, called emTT, written now in the style of Martin-Löf's *extensional* type theory in [Mar84].

The main idea behind the design of emTT goes back to that of toolbox in [SV98], and it follows the *forget-restore principle* conceived by G. Sambin and introduced there. According to this principle, extensional concepts must be abstractions of intensional concepts as result of forgetting irrelevant computational information, that can be restored when implementing these extensional concepts back at the intensional level. Here, we think it is enough to require that extensional constructors must be obtained by abstracting only over equalities of corresponding constructors in mTT. Hence, it seems sufficient to add quotients to mTT to be able to represent our extensional level. This leads us to conclude that the extensional level should be considered as a fragment of the internal theory of a model built over mTT by just adding quotients.

In the literature it is well known how to add quotients to a type theory by building setoids (see [Hof97, BCP03]) on it. The extensional theory emTT we propose here can be interpreted in a suitable model of total setoids à la Bishop that we will describe in the following.

We need to warn that emTT is not precisely the *internal language* of the quotient model we adopt, namely it is not fully complete with it, or in other words it does not necessarily capture all the constructions valid in the model, especially at the level of collections. For example, in emTT collections are not closed under effective quotients while in the model they are, instead. Hence, different extensional levels may be considered over mTT, and even over the same quotient model. A criterion to decide what to put in emTT at the level of collections is that of preserving minimality, for example with respect to Aczel's CZF.

emTT extends the set-theoretic version introduced in [Mai07], called here emTT$_{set}$, with collections and related constructors needed to represent the power collection of a set with ε-relation and comprehension used in everyday mathematical practice.

emTT is essentially obtained as follows: we first take the *extensional* version of mTT, in the same way the so-called extensional Martin-Löf's type theory in [Mar84] is obtained from the intensional version in [NPS90], with the warning of replacing the collection of small propositions $\mathsf{prop_s}$ with its quotient under equiprovability $\mathcal{P}(1) \equiv \mathsf{prop_s}/\leftrightarrow$; then we collapse propositions into *mono collections* according to the notion in [Mai05]; and finally we add effective quotient sets as in [Mai05]. The precise rules of emTT are given in the appendix 7. The form of judgements to describe emTT are those of mTT.
One of the main differences between emTT and mTT amounts to being that between the extensional version of Martin-Löf's type theory in [Mar84] and the intensional one in [NPS90]. It consists in the fact that while type judgements in the intensional version are decidable, those in the extensional one are no longer so [Hof97]. Another difference is that in emTT propositions are *mono* as in [Mai05], that is they are inhabited by at most one proof by introducing in emTT the following rule:

$$\textbf{prop-mono)}\ \frac{A\ prop\ [\Gamma] \qquad p \in A\ [\Gamma] \quad q \in A\ [\Gamma]}{p = q \in A\ \ [\Gamma]}$$

Propositions are then *mono collections* and small propositions are *mono sets*. This property allows us to forget proof-terms of propositions, namely to make *proofs of propositions irrelevant*, by introducing a canonical proof-term called true for them:

$$\textbf{prop-true)}\ \frac{A\ prop \qquad p \in A}{\mathsf{true} \in A}$$

This canonical proof-term allows us to interpret true-judgements in [Mar84, Mar85] directly in emTT as follows:

$$A\ true\ [\Gamma; B_1\ true, \ldots, B_m\ true\,] \equiv \mathsf{true} \in A\ [\Gamma,\, y_1 \in B_1, \ldots, y_m \in B_m\,]$$

Then, we can prove that, according to this interpretation, all true judgements of the logic in [Mar84, Mar85] are valid in emTT.

A key feature of extensional type theory in [Mar84] is the presence of *Extensional Propositional Equality*, written $\mathsf{Eq}(A, a, b)$ to express that a is equal to b. This is stronger than Propositional Equality $\mathsf{Id}(A, a, b)$ in intensional type theory [NPS90], and also in mTT (see appendix 6), because the validity of $\mathsf{Eq}(A, a, b)$ is equivalent to the definitional equality of terms $a = b \in A$ (both under the same context). Furthermore, it is also mono. We add $\mathsf{Eq}(A, a, b)$ to emTT as a proposition, which is small when A is a set (see appendix 7).

Another key difference between emTT and mTT is that in emTT we can form effective quotient sets (see [Mai05]). Then, in the presence of quotient effectiveness it is crucial to require that propositions are mono collections, as well as that small propositions are mono sets. Indeed, if we identify small propositions with sets simply, or propositions with collections, quotient effectiveness may lead to classical logic (see [Mai99]), because it yields to a sort of choice operator, and hence it is no longer a constructive rule.

Moreover, observe that the set-theoretic part of our emTT, called emTT$_{set}$, is a variation of the internal type theory of a list-arithmetic locally cartesian closed pretopos, as devised in [Mai05]. Indeed, emTT$_{set}$ is not exactly that because in the internal theory of a generic pretopos *small propositions are identified with mono sets*, while in emTT *small propositions are only some primitive mono sets*, and it does not necessarily follow that that all mono sets are small propositions. In this way we avoid the validity of *the axiom of unique choice*, which would instead be valid under the identification of small propositions with mono sets (see [Mai05]).

Lastly, in emTT we add the necessary constructors to represent the power collection of a set as the "power set" in the internal theory of a topos devised in the style of Martin-Löf's type theory in [Mai05]. To this purpose we put in emTT the power collection of the singleton set $\mathcal{P}(1)$. This is represented as the quotient of the collection of small propositions under equiprovability. Therefore a subset is represented as an equivalence class of small propositions. Then, we add collections of functions towards $\mathcal{P}(1)$ in order to represent the power collection of a set A:

$$\mathcal{P}(A) \equiv A \to \mathcal{P}(1)$$

Therefore, a subset of A, being an element of $\mathcal{P}(A)$, is represented as a function from A to $\mathcal{P}(1)$.
Moreover, we can represent ε-relation and comprehension used in everyday mathematical practice as follows. To this purpose we need to assume the extensional equality $\mathsf{Eq}(\mathcal{P}(1), U, V\,)$ on $\mathcal{P}(1)$ to be a small proposition given that the equality on $\mathcal{P}(1)$ is intended to be defined as equiprovability of the small propositions characterizing the equivalence classes U, V, and this is still a small proposition. Then, given $W \in \mathcal{P}(A)$ and $a \in A$ we define the small proposition

$$a \varepsilon W \equiv \mathsf{Eq}(\mathcal{P}(1),\, W(a),\, [\mathsf{tt}]\,)$$

where $\mathsf{tt} \equiv \bot \to \bot$ is the truth constant (it may be represented by any tautology), and $[\mathsf{tt}]$ its equivalence class under equiprovability.
Furthermore, for any derivable $B(x) \in \mathsf{prop_s}\ [x \in A]$ we define

$$\{\, x \in A \mid B(x)\,\} \in \mathcal{P}(A) \equiv \lambda x^A.[B(x)] \in \mathcal{P}(A)$$

Then, thanks to the rules eq-$\mathcal{P}$) and eff-$\mathcal{P}$) of $\mathcal{P}(1)$ and the equality rules expressing extensionality of function collections in appendix 7, we can prove that the equality between subsets determined by propositional functions is the usual extensional one as in [SV98]:

$$\{\, x \in A \mid B(x)\,\} = \{\, x \in A \mid C(x)\,\} \in \mathcal{P}(A) \text{ holds in emTT iff } \forall_{x \in A}\, B(x) \leftrightarrow C(x) \text{ holds in emTT.}$$

Furthermore, also the comprehension axiom

$$a \varepsilon \{x \in A \mid B(x)\,\} \leftrightarrow B(a)$$

holds in emTT because $a\varepsilon\{x \in A \mid B(x)\,\}$ is equal to $\mathsf{Eq}(\,\mathcal{P}(1), [B(a)]\,, [\mathsf{tt}]\,)$, which is valid if and only if $B(a)$ is valid, too, again by the rules eq-P) and eff-P) of $\mathcal{P}(1)$ and those of Eq in appendix 7.

Thanks to the rule η-P) and those about function collections, we can prove that for any subset $W \in \mathcal{P}(A)$ we can derive

$$W = \{ x \in A \mid x \varepsilon W \} \in \mathcal{P}(A)$$

Alternative rules to form the power collection of a set can be deduced from the analysis in [Mai05] on how to represent the subobject classifier in the internal type theory of a topos.

Finally, it is worth noting that, thanks to proof irrelevance of propositions, we can implementing functions between subsets as in [SV98] without running into the problem pointed out in [Car03].

The desire of *representing power collections of sets* together with *proof irrelevance of propositions* is a key motivation to work in a two-level foundation. Indeed, such constructions can not be directly represented in mTT because mTT has no quotients. On the other hand it is sufficient to build a quotient model over mTT to interpret them, and to interpret the whole emTT.

In order to better present the proofs about this quotient model over mTT, it is more convenient to show them for an extended two-level theory. The extensional level of this extended two-level theory is taken to be an extension of emTT, called emTT^{dp}. emTT^{dp} is first obtained by extending emTT with dependent product collections, given that function collections towards $\mathcal{P}(1)$ can be seen as instances of them (see the rules in appendix 7). Consequently, the intensional level is necessarily taken to be mTT^{dp}, namely mTT extended with dependent product collections, in order to support the interpretation of corresponding collections in emTT^{dp}. Moreover, in emTT^{dp} we also include effective quotients on collections, (that we do not include in emTT!) in the attempt to capture the largest extensional theory à la Martin-Löf valid in our quotient model over mTT^{dp}. However emTT^{dp} does not seem to be the internal language of our quotient model over mTT^{dp} yet, as well as emTT is not at all that of our quotient model over mTT. We leave to future work how to determine the fully complete theory of our quotient model over mTT^{dp}, and eventually that over mTT.
It is worth noting that our quotient models over mTT and over mTT^{dp} support also the interpretation of the collection of small propositions $\mathsf{prop_s}$, if added to emTT and to emTT^{dp} respectively. Indeed, $\mathsf{prop_s}$ turns out to be interpreted as $\mathcal{P}(1)$, because equality of emTT propositions will be interpreted into equiprovability of mTT propositions. We chose of putting in emTT only the quotient collection of small propositions under equiprovability $\mathcal{P}(1)$, and not the collection of small propositions $\mathsf{prop_s}$, in order to make emTT easily interpretable in the internal theory of a topos. Indeed, if we consider the formulation $\mathcal{T}_{top}$ of the internal theory of a topos devised in [Mai05] in the style of Martin-Löf's extensional type theory, then emTT sets and collections are translated into $\mathcal{T}_{top}$ types, emTT small propositions and propositions into $\mathcal{T}_{top}$ mono types, that represent $\mathcal{T}_{top}$ propositions, and $\mathcal{P}(1)$ is translated as the $\mathcal{T}_{top}$ type representing the subobject classifier.

Note that, as in mTT also in emTT we do not include the boolean universe U_b used in [Mai07] to derive disjointness of binary sums. Indeed, in the presence of $\mathcal{P}(1)$ sum disjointness is derivable anyway [2].

Our emTT is also compatible with the notion of a predicative topos [MP02], being its set-theoretic part a fragment of the internal theory of a locally cartesian closed pretopos. In particular, if we perform over Martin-Löf's type theory with universes the quotient model we will build over mTT to interpret emTT, then we will get a predicative topos (see [MP00, MP02]). Given that our intensional level mTT can be interpreted in Martin-Löf's type theory with universes, this yields that emTT can be also interpreted in the predicative topos over it.

Finally, emTT is certainly compatible with Aczel's CZF [AR] by interpreting sets a CZF sets, collections as classes, propositions as subclasses of the singleton and small propositions as subsets of the singleton (in order to make the rules **prop-into-col** and **prop$_s$-into-set** valid). In particular, the power collection $\mathcal{P}(A)$ of a set A is interpreted as the corresponding power collection of subsets.

4 The quotient model

In what follows we are going to define a model of quotients over mTT. This is based on the well-known notion of setoid [Bis67, Hof97], namely a set with an equivalence relation over it, that we apply to any type we consider. We will then interpret emTT in such a model.

[2] The proof of disjointness in emTT is similar to that for mTT mentioned in the proof of theorem 4.20.

The model will be presented in a categorical shape. The reason is that we know the categorical semantics of emTT and, if the quotient model turns out to be an instance of it, then we can interpret emTT into the model, and hence into mTT. This model represents also a way to freely add quotients to mTT in a suitable sense. The precise categorical formulation of its universal property is left to future work.
We first start with defining the category of "extensional collections" namely collections equipped with an equivalence relation. In the case where the collection is a set, this notion is known as "total setoid" in the literature [Hof97, BCP03].

Def. 4.1 The category Q(mTT) is defined as follows:
ObQ(mTT): objects are pairs $(A, =_A)$ where A is a collection in mTT, called "support", and

$$x =_A y \; prop \; [x \in A, y \in A]$$

is an equivalence relation on the collection A. This means that in mTT there exist proof-terms witnessing reflexivity, symmetry and transitivity of the relation:

$$\begin{array}{l} \mathsf{rfl}(x) \in x =_A x \;\; [\, x \in A\,] \\ \mathsf{sym}(x,y,u) \in y =_A x \;\; [\, x \in A,\ y \in A,\ u \in x =_A y\,] \\ \mathsf{tra}(x,y,z,u,v) \in x =_A z \;\; [\, x \in A,\ y \in A,\ z \in A,\ u \in x =_A y,\ v \in y =_A z\,] \end{array}$$

We call $(A, =_A)$ *extensional collection.*
Mor**Q(**mTT**)**: morphisms from an object $(A, =_A)$ to $(B, =_B)$ are mTT terms $f(x) \in B$ $[x \in A]$ preserving the corresponding equality, i.e. in mTT there exists a proof-term

$$\mathsf{pr}_1(x,y,z) \in f(x) =_B f(y) \;\; [\, x \in A,\ y \in A,\ \ z \in x =_A y\,]$$

Moreover, two morphisms $f, g : (A, =_A) \to (B, =_B)$ are equal if and only if in mTT there exists a proof-term

$$\mathsf{pr}_2(x) \in f(x) =_B g(x) \;\; [\, x \in A\,]$$

The category Q(mTT) comes naturally equipped with an indexed category (or split fibration) satisfying the universal property of comprehension (see [Jac99] for its definition) thanks to the closure of mTT collections under strong indexed sums:

Def. 4.2 The indexed category:

$$\mathcal{P}_q : \mathrm{Q(mTT)}^{\mathsf{OP}} \to \mathsf{Cat}$$

is defined as follows. For each object $(A, =_A)$ in Q(mTT) then $\mathcal{P}_q(\,(A, =_A)\,)$ is the following category: $\mathsf{Ob}\mathcal{P}_q(\,(A, =_A)\,)$ are the propositions $P(x)$ $prop$ $[x \in A]$ depending on A and preserving the equality on A, namely for such propositions there exists a proof-term:

$$\mathsf{ps}(x,y,d) \in P(x) \to P(y) \; [x \in A,\ y \in A,\ d \in x =_A y]$$ [3]

Two objects $P(x)$ $prop$ $[x \in A]$ and $Q(x)$ $prop$ $[x \in A]$ are equal if a proof of $P(x) \leftrightarrow Q(x)$ $prop$ $[x \in A]$ can be derived in mTT.
Morphisms in $\mathsf{Mor}\mathcal{P}_q(\,(A, =_A)\,)$ are given by the following order

$$\mathcal{P}_q(\,(A, =_A)\,)(\,P(x)\,,\,Q(x)\,) \;\equiv\; P(x) \leq Q(x)$$

$$\text{iff there exists a proof-term } \mathsf{pt}(x) \in P(x) \to Q(x) \; [x \in A]$$

Moreover, for every morphism $f : (A, =_A) \to (B, =_B)$ in Q(mTT) given by $f(x) \in B$ $[x \in A]$ then $\mathcal{P}_q(f)$ is the substitution functor, i.e. $\mathcal{P}_q(f)(\,P(y)\,) \;\equiv\; P(f(x))$ for any proposition $P(y)$ $prop$ $[y \in B]$ (recall that $\mathcal{P}_q$ is contravariant).

Lemma 4.3 *$\mathcal{P}_q$ is an indexed category satisfying the universal property of comprehension.*

[3] Indeed, from this, by using the symmetry of $x =_A y$ it follows that $P(x)$ is equivalent to $P(y)$ if $x =_A y$ holds.

Proof. To describe the universal property of comprehension, we consider the Grothendieck completion $Gr(\mathcal{P}_q)$ of $\mathcal{P}_q$ (see [Jac99] for its definition) and the functor $T : \mathrm{Q(mTT)} \to Gr(\mathcal{P}_q)$ defined as follows:

$$T(\,(A,=_A)\,) \equiv (\,(A,=_A)\,,\,\mathsf{tt}\,) \qquad\qquad T(f) \equiv (f, id_{\mathsf{tt}})$$

where tt is the truth constant. T satisfies the following universal property of comprehension (cfr. [Jac99]): for every $Gr(\mathcal{P}_q)$-object $(\,(A,=_A)\,,\,P\,)$ there exists a Q(mTT)-object

$$Cm(\,(\,(A,=_A)\,,\,P\,)\,) \equiv (\Sigma_{x\in A}P(x)\,,\,=_{Cm}\,)$$

where $z_1 =_{Cm} z_2 \;\equiv\; \pi_1(z_1) =_A \pi_1(z_2)$ for $z_1, z_2 \in \Sigma_{x\in A}P(x)$, such that, for each $Gr(\mathcal{P}_q)$-morphism $(f,p) : T(\,(C,=_C)\,) \longrightarrow (\,(A,=_A)\,,\,P\,)$ there is a *unique morphism* $[f,p] : (C,=_C) \longrightarrow Cm(\,(\,(A,=_A)\,,\,P\,)\,)$ in Q(mTT) such that in $Gr(\mathcal{P}_q)$

$$(\eta_1,\eta_2)\,\cdot\,T(\,[f,p]\,) = (f,p)$$

where $\eta_1 \equiv \pi_1^P$ and $\eta_2 \equiv \pi_2^P$ are the first and second projections of the indexed sum $\Sigma_{x\in A}P(x)$.

Note that $Cm(-)$ on objects does not define an operation from $Gr(\mathcal{P}_q)$-objects to Q(mTT)-objects unless the Axiom of Choice is used.

Def. 4.4 ($\mathcal{P}_q$-proposition) Given a proposition $P \in \mathsf{Ob}\mathcal{P}_q(\,(A,=_A)\,)$ then the first component of the comprehension adjunction counit $(\eta_1,\eta_2) : (\,(\Sigma_{x\in A}P(x)\,,\,=_{Cm}\,)\,,\,\mathsf{tt}\,) \to (\,(A,=_A)\,,\,P\,)$ given by the first projection

$$\eta_1 \equiv \pi_1^P : (\Sigma_{x\in A}P(x)\,,\,=_{Cm}\,) \to (A,=_A)$$

is a monic morphism in Q(mTT) and it is called a *$\mathcal{P}_q$-proposition.*

Def. 4.5 ($\mathcal{P}_q$-small proposition) A $\mathcal{P}_q$-proposition $\eta_1 \equiv \pi_1^P : (\Sigma_{x\in A}P(x)\,,\,=_{Cm}\,) \to (A,=_A)$ is called a *$\mathcal{P}_q$-small proposition* if the proposition $P \in \mathcal{P}_q(\,(A,=_A)\,)$ is small.

In order to make clear how the set-theoretic part of Q(mTT) can interpret the set-theoretic fragment emTT_{set} of emTT, we single out the category of extensional sets from that of extensional collections.

Def. 4.6 The category $\mathrm{Q(mTT)}_{set}$ is defined as the full subcategory of Q(mTT) equipped with extensional sets $(A,=_A)$ where the support is a set and the equivalence relation is small, namely A *set* and $x =_A y\ prop_s\ [x \in A, y \in A]$ are derivable in mTT.

Def. 4.7 We define the functor $\mathcal{P}_{q_{set}} : \mathrm{Q(mTT)}_{\mathsf{set}}^{\mathsf{OP}} \to \mathsf{Cat}$ as the restriction of $\mathcal{P}_q$ on $\mathrm{Q(mTT)}_{set}$ to small propositions, namely for every extensional set $(A,=_A)$ we have that $\mathcal{P}_{q_{set}}(\,(A,=_A)\,)$ is the full subcategory of $\mathcal{P}_q(\,(A,=_A)\,)$ containing only small propositions.

Then, in an analogous way to lemma 4.3 we can prove:

Lemma 4.8 *$\mathcal{P}_{q_{set}}$ is an indexed category satisfying the universal property of comprehension.*

Analogously to definition 4.4, we can define a $\mathcal{P}_{q_{set}}$*-proposition*, which is indeed also small.

In order to prove that the category Q(mTT) has all the necessary structure to interpret emTT, in particular that it is closed under certain dependent products, it will be useful to know that Q(mTT)-morphisms correspond to extensional dependent collections defined in an analogous way to dependent sets in [Bis67, Pal05] (see also [Dyb96]) as follows:

Def. 4.9 (extensional dependent collection) Given an object $(A,=_A)$ of Q(mTT), abbreviated with $A_=$, we define an *extensional dependent collection* on $(A,=_A)$ written

$$B_=(x)\ [\,x \in A_=\,]$$

as a dependent collection $B(x)\ col\ [\,x \in A\,]$, called "dependent support", together with an equivalence relation

$$y =_{B(x)} y'\ prop\ [x \in A,\ y \in B(x),\ y' \in B(x)].$$

Moreover, for any $x_1, x_2 \in A$ such that $x_1 =_A x_2$ holds there must exist a *substitution morphism*

$$\sigma_{x_1}^{x_2}(d,y) \in B(x_2)\ [x_1 \in A,\ x_2 \in A,\ d \in x_1 =_A x_2,\ y \in B(x_1)]$$

preserving the equality on $B(x_1)$, namely there exists a proof of

$$\sigma_{x_1}^{x_2}(d,y) =_{B(x_2)} \sigma_{x_1}^{x_2}(d,y')\ prop\ [x_1 \in A,\ x_2 \in A,\ d \in x_1 =_A x_2,$$
$$y \in B(x_1),\ y' \in B(x_1),\ w \in y =_{B(x_1)} y'].$$

and *not depending* on $d \in x_1 =_A x_2$ in the sense that we can derive a proof of

$$\sigma_{x_1}^{x_2}(d_1,y) =_{B(x_2)} \sigma_{x_1}^{x_2}(d_2,y)\ prop\ [x_1 \in A,\ x_2 \in A,\ d_1 \in x_1 =_A x_2,\ d_2 \in x_1 =_A x_2,\ y \in B(x_1)]$$

Furthermore, $\sigma_x^x(\mathsf{rfl}(x), -)$ is the identity, namely there exists a proof of

$$\sigma_x^x(\mathsf{rfl}(x),y) =_{B(x)} y\ prop\ [x \in A,\ y \in B(x)]$$

and the $\sigma_{x_1}^{x_2}$'s are closed under composition, namely there exists a proof of

$$\sigma_{x_2}^{x_3}(\, d_2\, ,\, \sigma_{x_1}^{x_2}(d_1,y)\,) =_{B(x_3)} \sigma_{x_1}^{x_3}(\, \mathsf{tra}(x_1,x_2,x_3,d_1,d_2)\, ,\, y\,)\ prop$$
$$[x_1 \in A,\ x_2 \in A,\ x_3 \in A,\ y \in B(x_1),\ d_1 \in x_1 =_A x_2,\ d_2 \in x_2 =_A x_3].$$

Note that the no dependency on the proofs of the extensional collection equality $=_A$ allows to use the following abbreviations:

$$\sigma_{x_1}^{x_2}(y)\ \equiv \sigma_{x_1}^{x_2}(d,y)$$

for $x_1 \in A,\ x_2 \in A,\ d \in x_1 =_A x_2,\ y \in B(x_1)$.

Categorically speaking, we can see that an extensional dependent collection is given by a functor from a suitable groupoid category to Q(mTT). Indeed, for any extensional type $(A, =_A)$ we can define the category $\mathcal{G}(\,(A, =_A)\,)$ as follows: its objects are the elements of A (with their definitional equality as equality) and for $a_1, a_2 \in A$ then $a_1 \leq a_2$ holds if and only if we can derive $p \in a_1 =_A a_2$ in mTT. Then, any extensional dependent collection on $(A, =_A)$ is given by a functor

$$\sigma_B : \mathcal{G}(\,(A, =_A)\,) \to \mathrm{Q(mTT)} \qquad \text{such that} \qquad \sigma_B(x)\ \equiv\ B_=(x) \qquad \sigma_B(x_1 \leq x_2)\ \equiv\ \sigma_{x_1}^{x_2}$$

where $\sigma_{x_1}^{x_2}$ stands for the extensional morphism given by

$$\sigma_{x_1}^{x_2}(y) \in B(x_2)\ [x_1 \in A,\ x_2 \in A,\ d \in x_1 =_A x_2,\ y \in B(x_1)]$$

Note that $\mathcal{G}(A, =_A)$ is a groupoid category and hence its image along σ_B is a groupoid category, too. In particular we get that every $\sigma_{x_1}^{x_2}$ actually gives rise to an *isomorphism* between $(\, B(x_1)\, ,\, =_{B(x_1)}\,)$ and $(\, B(x_2)\, ,\, =_{B(x_2)}\,)$ for given $x_1, x_2 \in A$ and $d \in a_1 =_A a_2$.

Analogously, we can give the definition of extensional dependent set:

Def. 4.10 An extensional dependent collection $B_=(x)\ [\, x \in A_=\,]$ is an *extensional dependent set* if its support is a dependent set and its equivalence relation is small, namely we can derive

$$B(x)\ set\ [\, x \in A\,] \qquad\qquad y =_{B(x)} y'\ prop_s\ [x \in A,\ y \in B(x),\ y' \in B(x)]$$

Analogously, we can give the definition of *extensional dependent proposition* and of *extensional dependent small proposition*, where the latter must be also an extensional dependent set. In the following we will often speak of extensional dependent type to include any of them.

Def. 4.11 Let us call $Dep_{\mathrm{Q(mTT)}}(\,(A, =_A)\,)$ the category whose objects are extensional dependent collections $B_=(x)\ [x \in A_=]$ on the extensional collection $(A, =_A)$, and whose morphisms are *extensional terms*

$$b(x,y) \in B_=(x)\ [\, x \in A_=,\ y \in C_=(x)\,]$$

that are dependent terms $b(x,y) \in B(x)\ [\, x \in A,\ y \in C(x)\,]$ preserving the equality on A and that on $C(x)$, namely in mTT there exists a proof of

$$\sigma_{x_1}^{x_2}(\, b(x_1,y_1)\,) =_{B(x_2)}\ b(x_2,y_2)\ prop\ [\, x_1 \in A,\ x_2 \in A,\ w \in x_1 =_A x_2,$$
$$y_1 \in C(x_1),\ y_2 \in C(x_2),\ z \in \sigma_{x_1}^{x_2}(y_1) =_{C(x_2)} y_2\,]$$

and two extensional terms $b(x,y) \in B_=(x)\ [\, x \in A_=,\ y \in C_=(x)\,]$ and $b'(x,y) \in B_=(x)\ [\, x \in A_=,\ y \in C_=(x)\,]$ are equal if and only if in mTT there exists a proof of

$$b(x_1,y_1) =_{B(x_1)}\ b'(x_1,y_1)\ prop\ [\, x_1 \in A,\ y_1 \in C(x_1)\,]$$

Now, we are ready to prove that Q(mTT)-morphisms correspond to extensional dependent collections. Categorically, this can be expressed by saying that the slice category of Q(mTT) over an extensional collection $(A, =_A)$ (see [Jac99] for the definition of slice category) is equivalent to the category of extensional dependent collections on $(A, =_A)$:

Proposition 4.12 *The category* $\mathrm{Q(mTT)}/(A, =_A)$ *is equivalent* [4] *to* $Dep_{\mathrm{Q(mTT)}}(\,(A, =_A)\,)$.

Proof. We just describe how to associate an extensional dependent collection to a Q(mTT)-morphism and conversely.
Given a Q(mTT)-morphism $f : (C, =_C) \to (A, =_A)$ then the support of the extensional dependent collection associated with it is

$$\Sigma_{y \in C}\, f(y) =_A x \; col \; [x \in A]$$

with equality

$$z =_{\Sigma f} z' \;\equiv\; \pi_1(z) =_C \pi_1(z')$$

for $x \in A$ and $z, z' \in \Sigma_{y \in C}\, f(y) =_A x$. Then, we define

$$\sigma_{x_1}^{x_2}(z) \;\equiv\; \langle \pi_1(z), \mathsf{tra}(\pi_2(z), d) \rangle$$

for $d \in x_1 =_A x_2$. Clearly $\sigma_{x_1}^{x_1}(z) =_{\Sigma f} z$ and also the transitivity property holds.

Conversely, given an extensional dependent type $B_=(x)\ [x \in A_=]$, we consider the extensional type $(\,\Sigma_{x \in A} B(x)\,, =_\Sigma\,)$ where for $z, z' \in \Sigma_{x \in A} B(x)$

$$z =_\Sigma z' \;\equiv\; \exists d \in \pi_1(z) =_A \pi_1(z') \quad \sigma_{\pi_1(z)}^{\pi_1(z')}(\pi_2(z)) =_{B(\pi_1(z'))} \pi_2(z')$$

and we build the Q(mTT)-morphism

$$\pi^B : (\Sigma_{x \in A} B(x), =_\Sigma) \to (A, =_A).$$

called *comprehension of the extensional dependent collection* $B_=(x)\ [x \in A_=]$ and given by $\pi^B(z) \;\equiv\; \pi_1(z)$ for $z \in \Sigma_{x \in A} B(x)$.

Def. 4.13 We call *dset-morphism* a Q(mTT)-morphism

$$\pi^B : (\Sigma_{x \in A} B(x), =_\Sigma) \to (A, =_A).$$

that is the comprehension of an extensional dependent set $B_=(x)\ [x \in A_=]$, namely of an extensional dependent collection whose support is a dependent set and whose equivalence relation is small.

In an analogous way, we can define the category of extensional collections $\mathrm{Q(mTT}^{dp})$ on mTT^{dp}, and we can prove the same properties shown so far for Q(mTT).
Then, in order to describe the categorical structure of Q(mTT), as well as that of $\mathrm{Q(mTT}^{dp})$, it is also useful to know that these models are closed under finite products and equalizers (for their definition see, for example, [Mac71]), namely that they are lex categories:

Lemma 4.14 *The category* Q(mTT), *as well as* $\mathrm{Q(mTT}^{dp})$, *is lex (i.e. with terminal object, binary products, equalizers).*

Proof. In the following we indicate with $c : (D, =_D) \to (C, =_C)$ an arrow given by $c(x) \in C\ [x \in D]$ in Q(mTT) .
The *terminal* object is $(\,\mathsf{N}_1, \mathsf{Id}(\mathsf{N}_1, x, y)\,)$. Then, for any object $(A, =_A)$, the unique arrow to the terminal object is $\star \in \mathsf{N}_1\ [x \in A]$. The uniqueness can be proved thanks to the fact that for any $d \in \mathsf{N}_1$ we can prove that $\mathsf{Id}(\mathsf{N}_1, d, \star)$ holds by elimination rule on N_1.
The *binary product* of $(A, =_A)$ an $(B, =_B)$ is $(A \times B, =_\times)$ where

$$z =_\times z' \;\equiv\quad \pi_1(z) =_A \pi_1(z') \;\wedge\; \pi_2(z) =_B \pi_2(z')$$

[4] Equivalence is here formulated as the existence of a dense fullly faithful functor from $Dep_{\mathrm{Q(mTT)}}(\,(A, =_A)\,)$ to $\mathrm{Q(mTT)}/(A, =_A)$ unless the axiom of choice is used in the metatheory.

The projections are $\pi_1(z) \in A$ $[z \in A \times B]$ and $\pi_2(z) \in B$ $[z \in A \times B]$. The pairing of two arrows $a : (C, =_C) \to (A, =_A)$ and $b : (C, =_C) \to (B, =_B)$ is $\langle a(z), b(z) \rangle \in A \times B$ $[z \in C]$.
An *equalizer* of $b_1, b_2 : (A, =_A) \to (B, =_B)$ is $(\, \Sigma_{y \in A}\, b_1(y) =_B b_2(y)\, ,\, =_{eq}\,)$ where

$$z =_{eq} z' \equiv \quad \pi_1(z) =_A \pi_1(z')$$

for $z, z' \in \Sigma_{y \in A}\, b_1(y) =_B b_2(y)$. The embedding morphism is $\pi_1(z) \in A$ $[\, z \in \Sigma_{y \in A}\, b_1(y) =_B b_2(y)\,]$.
Moreover, for any arrow $c : (D, =_D) \to (A, =_A)$ equalizing b_1, b_2 and hence yielding to a proof

$$p(z) \in b_1(c(z)) =_B b_2(c(z)) \ [z \in D]$$

the unique arrow towards the equalizer is $\langle c(z), p(z) \rangle \in \Sigma_{y \in A}\, b_1(y) =_B b_2(y)$ $[z \in D]$.
Note that here we have used the existence of proof-terms witnessing the equality between morphisms in Q(mTT).

Now, after knowing how binary products are defined in Q(mTT), we give the definition of *categorical* equivalence relation induced by an equivalence relation of mTT (also small):

Def. 4.15 ($\mathcal{P}_q$-equivalence relation) Given an equivalence relation $R \in \mathcal{P}_q(\, (A, =_A) \times (A, =_A)\,)$, namely a proposition $R(x, y)$ prop $[x \in A, y \in A]$ that preserves $=_A$ on both dependencies and is also an equivalence relation, we call *$\mathcal{P}_q$-equivalence relation* the corresponding $\mathcal{P}_q$-proposition, namely

$$\eta_1 \equiv \pi_1^R : (\, \Sigma_{z \in A \times A}\, R(\, \pi_1 z\, ,\, \pi_2 z\,)\, ,\, =_{Cm}\,) \to (A \times A, =_\times)$$

Analogously, we can define a *$\mathcal{P}_q$-small equivalence relation* and a *$\mathcal{P}_{q_{set}}$-equivalence relation*, which is indeed also small.
In order to describe the categorical structure of Q(mTT), as well as that of Q(mTT^{dp}), it will be useful to know that pullbacks preserve dset-morphisms and $\mathcal{P}_q$-propositions, which in turn follows from the fact that a pullback along the comprehension of an extensional dependent collection is isomorphic to one of this form.

Lemma 4.16 *In* Q(mTT)*, as well as in* Q(mTT^{dp})*, the pullback of the comprehension of an extensional dependent collection $B(x)_=$ $[x \in A_=]$*

$$\pi_1^B : (\, \Sigma_{x \in A} B(x)\, ,\, =_\Sigma\,) \to (A, =_A)$$

along a generic morphism $\delta : (D, =_D) \to (A, =_A)$ is isomorphic to the comprehension of an extensional dependent collection $B'(x)_=$ $[x \in D_=]$ where $B'(x)$ $[x \in D] \equiv B(\delta(x))$ $[x \in D]$

$$\pi_1^{B'} : (\Sigma_{x \in D}\ B(\, \delta(x)\,)\, ,\, =_{B'(x)}\,) \to (D, =_D).$$

whose equality is $w =_{B'(x)} w' \equiv w =_{B(\delta(x))} w'$ for $x \in D, w, w' \in B'(x)$ and its substitution morphism is defined by using that of $B(x)$ as follows: for $x_1 \in D, x_2 \in D$, $w \in x_1 =_D x_2, y \in B'(x_1)$

$$\sigma_{x_2}^{x_2}(y) \equiv \sigma_{\delta(x_1)}^{\delta(x_2)}(y).$$

From this we deduce the following corollary:

Corollary 4.17 *In* Q(mTT)*, as well as in* Q(mTT^{dp})*, the pullback of a dset-morphism*

$$\pi_1^B : (\, \Sigma_{x \in A}\, B(x)\, , =_{Cm}) \to (A, =_A)$$

along a morphism $\delta : (D, =_D) \to (A, =_A)$ is isomorphic to a dset-morphism. Analogously, the pullback of a $\mathcal{P}_q$-proposition is isomorphic to a $\mathcal{P}_q$-proposition, and that of a $\mathcal{P}_q$-equivalence relation to a $\mathcal{P}_q$-equivalence relation. The same holds with respect to $\mathcal{P}_q$-small propositions and $\mathcal{P}_q$-small equivalence relations, as well as with respect to $\mathcal{P}_{q_{set}}$-propositions and $\mathcal{P}_{q_{set}}$-equivalence relations in Q(mTT)$_{set}$.

Proof. This follows immediately from lemma 4.16 after noticing that $B'(x)$ $[x \in D]$ is of the same kind of type as $B(x)$ $[x \in A]$, and it is an equivalence relation if $B(x)$ is so. In particular, to apply lemma 4.16 properly on $\mathcal{P}_q$-propositions, note that any $P \in \mathsf{Ob}\mathcal{P}_q(\, (A, =_A)\,)$ induces an extensional dependent proposition $P(x)_=$ $[x \in A_=]$ whose support is $P(x)$ *prop* $[x \in A]$, whose equality is the trivial one: $w =_{P(x)} w' \equiv \mathsf{tt}$, and whose substitution morphisms are defined as $\sigma_{x_1}^{x_2}(y) \equiv \mathsf{Ap}(\,\mathsf{ps}(x_1, x_2, d)\,, y)$ for $x_1, x_2 \in A$ and $d \in x_1 =_A x_2, y \in P(x_1)$.

Now, we are going to recall the categorical notion of *dependent product* and its stability under pullback (see [See84, Joh02a]). To this purpose we fix the notation about pullback: in a lex category $\mathcal{C}$ we indicate the projections of a pullback of $b : B \to A$ along $d : D \to A$ as follows

$$\begin{tikzcd} D\times_A B \arrow[r, "b^*(d)"] \arrow[d, "d^*(b)"'] & B \arrow[d, "b"] \\ D \arrow[r, "d"'] & A \end{tikzcd}$$

Furthermore, for any morphism $m : b' \to b$ in $\mathcal{C}/A$ we indicate with $d^*(m) : d^*(b') \to d^*(b)$ the unique morphism in $\mathcal{C}/D$ such that $d^*(b) \cdot d^*(m) = d^*(b')$ and $b^*(d) \cdot d^*(m) = m \cdot (b')^*(d)$ hold.

Then, we are ready to recall the definition of stable dependent product:

Def. 4.18 Given a lex category $\mathcal{C}$, we say that $\mathcal{C}$ is closed under *the dependent product of a morphism* $c : C \to B$ *over a morphism* $b : B \to A$, if there exists a $\mathcal{C}$-morphism $\Pi_b c : \Pi_B C \to A$ with a $\mathcal{C}/B$-morphism $Ap : b^*(\Pi_b c) \to c$ such that, for every $\mathcal{C}$-morphism $d : D \to A$ and any $\mathcal{C}/B$-morphism $m : b^*(d) \to c$, there exists a unique $\mathcal{C}/A$-morphism $\widehat{m} : d \to \Pi_b c$ in $\mathcal{C}/A$ such that $Ap \cdot b^*(\widehat{m}) = m$ in $\mathcal{C}/B$.

Moreover, a *dependent product of a morphism* $c : C \to B$ *over a morphism* $b : B \to A$ is *stable under pullback* if for every morphism $q : Q \to A$ then $q^*(\Pi_b c) : Q \times_A \Pi_B C \to Q$ together with $(b^*(q))^*(Ap)$ is a dependent product of $(b^*(q))^*(c) : (Q \times_A B) \times_B C \to Q \times_A B$ over $q^*(b) : Q \times_A B \to Q$.

Then, we recall the definition of exponential in a slice category and its stability under pullback:

Def. 4.19 Given a lex category $\mathcal{C}$, for any object A in $\mathcal{C}$ we say that $\mathcal{C}/A$ is closed under *the exponential of* $c : C \to A$ *to* $b : B \to A$, also called the *function space from* $b : B \to A$ *to* $c : C \to A$, if $\mathcal{C}$ is closed under the dependent product of $b^*(c) : C \times_A B \to B$ over $b : B \to A$.

Moreover, *the exponential of* $c : C \to A$ *to* $b : B \to A$ is *stable under pullback*, if the corresponding dependent product is stable under pullback.

Now, we are ready to describe the categorical structure of Q(mTT) and Q(mTT^{dp}) sufficient to interpret emTT and emTT^{dp} respectively (for the involved categorical definitions not presented here see loc. cit. in [Mai05]):

Theorem 4.20 *The following hold:*

- *The category* Q(mTT) *is lex with parameterized lists of extensional sets, with stable finite propositionally disjoint sums* [5] *of extensional sets and with stable effective quotients with respect to* $\mathcal{P}_q$*-equivalence relations.*

  Furthermore, Q(mTT) *is also closed under stable dependent products of any dset-morphism over another dset-morphism.*

  Moreover, in Q(mTT) *there is an object* $(\,\mathsf{prop_s}, \leftrightarrow)$*, where* $\leftrightarrow (p, q) \equiv (p \to q) \wedge (q \to p)$ *for* $p, q \in \mathsf{prop_s}$*, classifying* $\mathcal{P}_q$*-small propositions, namely for every object* $(A, =_A)$ *there is a bijection*

  $$\mathrm{Q(mTT)}(\,(A, =_A)\,, (\,\mathsf{prop_s}, \leftrightarrow)\,) \simeq Sub_{\mathcal{P}_q\text{-small}}(\,(A, =_A)\,)$$

  between morphisms from $(A, =_A)$ *to it and the collection of subobjects represented by* $\mathcal{P}_q$*-small propositions on* $(A, =_A)$.

  And, in Q(mTT) *there exist local stable exponentials of* $(\,\mathsf{prop_s}, \leftrightarrow)$ *to dset-morphisms, namely, for any extensional collection* $(A, =_A)$*, the slice category of* Q(mTT) *over* $(A, =_A)$ *is closed under exponentials of* $\pi_1 : (A, =_A) \times (\,\mathsf{prop_s}, \leftrightarrow) \to (A, =_A)$ *to any dset-morphism, and these exponentials are stable under pullback.*

  Lastly, the indexed category $\mathcal{P}_q$ *validates first-order intuitionistic logic with equality, namely it is an intuitionistic hyperdoctrine in the sense of [See83] (see also [Law70, Law69, Pit00]).*

[5] The sums are only propositionally disjoint in the sense that the pullback vertex of the sum injections is the comprehension of the falsum, which is an initial object only in the fibres of $\mathcal{P}_q$.

- *The category* Q(mTT^{dp}) *enjoys the same properties as* Q(mTT)*, but in addition is also locally cartesian closed (i.e. with dependent products).*

Proof. Thanks to lemma 4.14 we already know that Q(mTT) and Q(mTT^{dp}) are lex, and we proceed to prove all the other properties.
The *list object* on an extensional set $(C, =_C)$ is $(List(C), =_{List(C)})$ with

$$z =_{List(C)} z' \equiv \exists_{l \in List(R)} \ \mathsf{Id}(\, List(C), \overline{\pi_1}(l), z) \ \wedge \ \mathsf{Id}(\, List(C), \overline{\pi_2}(l), z')$$

for $z, z' \in List(C)$ where $R \equiv \Sigma_{x \in C} \Sigma_{y \in C} \, x =_C y$ and $\overline{\pi_i} \equiv List(\pi_i)$ is the lifting on lists of the i-th projection for $i = 1, 2$. The empty list arrow is $\epsilon \in List(C) \ [w \in D]$ and the list constructor is $\mathsf{cons}(z, y) \in List(C) \ [z \in List(C), y \in C]$. Given the Q(mTT)-morphisms $a : (D, =_D) \to (M, =_M)$ and $l : (M, =_M) \times (C, =_C) \to (M, =_M)$, the recursor map is $El_{list}(\, u, a(w)\,, (x, y, z).l(\langle z, y \rangle)\,) \in M \ [w \in D, u \in List(C)]$. The uniqueness of the recursor map follows by elimination rule on lists.
The *initial* object is $(\, \mathsf{N}_0, \mathsf{Id}(\mathsf{N}_0, x, y)\,)$ and, for any object $(A, =_A)$, the unique arrow from the initial object to it is $\mathsf{emp_o}(x) \in A \ [\, x \in \mathsf{N}_0\,]$. The uniqueness of such an arrow follows by the elimination rule on N_0.
The *binary coproduct* of extensional sets $(B, =_B)$ and $(C, =_C)$ is $(B + C, =_{B+C})$ [6]
where

$$z =_{B+C} z' \equiv \begin{cases} b =_B b' & \text{if } \ z = \mathsf{inl}(b) \text{ and } z' = \mathsf{inl}(b') \ \text{ for } \ b, b' \in B \\ c =_C c' & \text{if } \ z = \mathsf{inr}(c) \text{ and } z' = \mathsf{inr}(c') \ \text{ for } c, c' \in C \\ \bot & \text{otherwise} \end{cases}$$

for $z, z' \in B + C$. The injections are $\mathsf{inl}(z) \in B + C \ [z \in B]$ and $\mathsf{inr}(z) \in B + C \ [z \in C]$ and the coproduct morphism of $b : (B, =_B) \to (D, =_D)$ and $c : (C, =_C) \to (D, =_D)$ is $El(z, \, (y_1).b(y_1), \, (y_2).c(y_2)\,) \in D \ [z \in B{+}C]$. Uniqueness of the coproduct morphism follows by elimination rule on the binary sum $B + C$. Sums are disjoint thanks to the disjointness of sums in mTT (this can be proved easily thanks to the fact that we can eliminate on disjoint sums toward $\mathsf{prop_s}$ by using $El_+(z, (x).\bot, (y).\mathsf{tt}\,) \in \mathsf{prop_s} \ [z \in A + B]$). Stability of coproducts under pullbacks follows by elimination on binary sums and from the fact that in mTT we can prove injectivity of sum injections [7] and sum disjointness.
The *classifying bijection* between morphisms from $(A, =_A)$ to $(\mathsf{prop_s}, \leftrightarrow)$ and subobjects given by $\mathcal{P}_q$-small propositions on $(A, =_A)$ follows by construction of $\mathcal{P}_q$-small propositions.
The fact that $\mathcal{P}_q$ validates first-order intuitionistic logic with equality follows by construction of the indexed functor.
Proving that Q(mTT) is closed under dependent products of dset-morphisms over dset-morphisms, as well as proving that local stable exponentials of $(\, \mathsf{prop_s}, \leftrightarrow\,)$ to dset-morphisms exist in Q(mTT), can be seen as a particular case of proving *local closure* of Q(mTT^{dp}). Indeed, the latter is equivalent to proving that Q(mTT^{dp}) is closed under *dependent products* of generic morphisms (usually expressed in the form that a right adjoint to the pullback functor induced by any morphism exists, see [Jac99, Joh02a]). We proceed, then, to prove that Q(mTT^{dp}) is closed under dependent products of any morphism over any other. To this purpose, thanks to the fact, shown in proposition 4.12, that any Q(mTT^{dp})-morphism $f : (B, =_B) \to (A, =_A)$ is isomorphic in Q(mTT^{dp})$/(A, =_A)$ to one of the form

$$\pi_1^B : (\Sigma_{z \in A} \, B(z), =_\Sigma) \to (A, =_A)$$

with $B_=(z) \ [z \in A_=]$ extensional dependent collection, it is enough to show the existence of dependent products for morphisms of the form π_1^B. In the next we will use the abbreviation $B_\Sigma \equiv \Sigma_{z \in A} \, B(z)$.
Then, a *dependent product* of

$$\pi_1^C : (\Sigma_{z \in B_\Sigma} C(z), =_{C_\Sigma}\,) \to (B_\Sigma, =_{B_\Sigma})$$

over $\pi_1^B : (B_\Sigma, =_\Sigma) \to (A, =_A)$, is $\pi_1^{\Pi_B^C} : (\, (\Pi_B C)_\Sigma \,, =_\Pi) \to (A, =_A)$ that is the comprehension (in the sense of proposition 4.12) of the extensional dependent collection $\Pi_B \, C(x)_= \ [x \in A_=]$ defined as follows: for $x \in A$

$$\Pi_B \, C(x) \equiv \Sigma_{h \in \Pi_{y \in B(x)} \, C(\langle x, y \rangle)} \quad \forall_{y_1 \in B(x)} \, \forall_{y_2 \in B(x)} \, \forall_{d \in y_1 =_{B(x)} y_2} \quad \sigma^{\langle x, y_2 \rangle}_{\langle x, y_1 \rangle} (\, \mathsf{Ap}(h, y_1)\,) =_{C(\langle x, y_2 \rangle)} \mathsf{Ap}(h, y_2)$$

---

[6] More formally, thanks to disjointness of sums in mTT we can define

$$z =_{B+C} z' \equiv \begin{array}{l} (\ \exists_{x \in B} \, \exists_{x' \in B} \ \mathsf{Id}(B + C, z, \mathsf{inl}(x)) \wedge \mathsf{Id}(B + C, z', \mathsf{inl}(x')) \ \wedge \ x =_B x' \ ) \\ \vee \ (\ \exists_{y \in C} \, \exists_{y' \in C} \ \mathsf{Id}(B + C, z, \mathsf{inr}(y)) \wedge \mathsf{Id}(B + C, z', \mathsf{inr}(y')) \ \wedge \ y =_C y' \ ) \end{array}$$

[7] For example, injectivity of inl can be proved as follows. Consider the term $p(u, z) \in A \ [u \in A, z \in A + B]$ where $p(u, z) \equiv El_+(z, (x).x, (y).u)$. Then, if $\mathsf{Id}(\, A + B, \, \mathsf{inl}(a), \, \mathsf{inl}(a')\,)$ holds for $a, a' \in A$, by preservation of propositional equality we get that $\mathsf{Id}(\, A, \, p(a, \, \mathsf{inl}(a)\,)\,, \, p(a, \mathsf{inl}(a'))$ holds, too. Hence, from this we conclude that $\mathsf{Id}(\, A, \, a\,, \, a')$ holds.

and

$$z =_{\Pi_B^C(x)} z' \equiv \forall_{y\in B(x)} \ \mathsf{Ap}(\pi_1(z), y) =_{C(\langle x,y\rangle)} \mathsf{Ap}(\pi_1(z'), y) \qquad \text{for } z, z' \in \Pi_B C(x)$$

and for $x_1 \in A, x_2 \in A, w \in x_1 =_A x_2, u \in \Pi_B^C(x)$

$$\sigma_{x_1}^{x_2}(u) \equiv \langle \lambda y^{B(x_2)}.\, \sigma^{\langle x_2,y\rangle}_{\langle x_1, \sigma_{x_2}^{x_1}(y)\rangle}(\mathsf{Ap}(h, \sigma_{x_2}^{x_1}(y))), t\rangle$$

for $u \equiv \langle h,p\rangle$ and suitable t built by using p and proof-terms witnessing properties of substitution isomorphisms.
Then, supposing that the pullback domain $(B_\Sigma, =_{B_\Sigma}) \times_{(A,=_A)} ((\Pi_B C)_\Sigma, =_\Pi)$ is represented by

$$(\Sigma_{w\in B_\Sigma} \Sigma_{u\in(\Pi_B C)_\Sigma} \pi_1(w) =_A \pi_1(u), =_{\times(A,=_A)})$$

where $z =_{\times(A,=_A)} z' \equiv v =_{B_\Sigma} v' \wedge u =_\Pi u'$ assuming that $z \equiv \langle v, \langle u,p\rangle\rangle$ and $z' \equiv \langle v', \langle u',p'\rangle\rangle$, then the *application map*

$$App : (B_\Sigma, =_{B_\Sigma}) \times_{(A,=_A)} ((\Pi_B C)_\Sigma, =_\Pi) \to (C_\Sigma, =_{C_\Sigma})$$

is given by

$$App(z) \equiv \langle \langle a_2, \sigma_{a_1}^{a_2}(b)\rangle, \mathsf{Ap}(h, \sigma_{a_1}^{a_2}(b))\rangle \in C_\Sigma \qquad \text{for } z \in \Sigma_{w\in B_\Sigma} \Sigma_{u\in(\Pi_B C)_\Sigma} \pi_1(w) =_A \pi_1(u)$$

supposing $\pi_1(z) \equiv \langle a_1, b\rangle$ and $\pi_1(\pi_2(z)) \equiv \langle a_2, \langle h,p\rangle\rangle$. Note that $\sigma_{a_1}^{a_2}$ is well defined because $q \equiv \pi_2(\pi_2(z)) \in a_1 =_A a_2$.

Now, given a morphism $m : (D, =_D) \times_{(A,=_A)} (B_\Sigma, =_{B_\Sigma}) \to (C_\Sigma, =_{C_\Sigma})$ in $\mathrm{Q}(\mathrm{mTT}^{dp})/(B_\Sigma, =_{B_\Sigma})$ where we assume the support of the pullback domain $(D, =_D) \times_{(A,=_A)} (B_\Sigma, =_{B_\Sigma})$ to be $\Sigma_{z\in D} \Sigma_{w\in B_\Sigma} \delta(z) =_A \pi_1(w)$ for $\delta : (D, =_D) \to (A, =_A)$ as above, then, the abstraction map $\widehat{m}(d) \in (\Pi_B C)_\Sigma \ [d \in D]$ is given by

$$\widehat{m}(d) \equiv \langle \delta(d), \langle \lambda y^{B(\delta(d))}.\, \sigma^{\langle \delta(d), y\rangle}_{\pi_1(m(z_d))}(\pi_2(m(z_d))), p\rangle\rangle$$

where $z_d \equiv \langle d, \langle \langle \delta(d), y\rangle, \mathsf{ref}(\delta(d))\rangle\rangle$ and suitable p built by using proof-terms witnessing properties of substitution isomorphisms and equality preservation of m.
In particular, the exponential of $\pi_1 : (A, =_A) \times (\mathsf{props_s}, \leftrightarrow) \to (A, =_A)$ to the dset-morphism $\pi_1^B : (B_\Sigma, =_\Sigma) \to (A, =_A)$ in $\mathrm{Q}(\mathrm{mTT})/(A, =_A)$ is the comprehension of the extensional dependent collection $\mathcal{P}(B(x))_= \ [x \in A_=]$ where

$$\mathcal{P}(B(x)) \equiv \Sigma_{h\in B(x)\to \mathsf{props_s}} \ \forall_{y_1\in B(x)} \forall_{y_2\in B(x)} \ y_1 =_{B(x)} y_2 \ \to \ (\mathsf{Ap}(h,y_1) \leftrightarrow \mathsf{Ap}(h,y_2))$$

and its corresponding equality is

$$z =_{\mathcal{P}(B(x))} z' \equiv \forall_{y\in B(x)} \ \mathsf{Ap}(\pi_1(z), y) \leftrightarrow \mathsf{Ap}(\pi_2(z), y) \qquad \text{for } z, z' \in \mathcal{P}(B(x)).$$

Stability under pullback of dependent products of dset-morphisms over dset-morphisms, or of local exponentials of $(\mathsf{props_s}, \leftrightarrow)$, follows easily thanks to corollary 4.17.
The *quotient of a* $\mathcal{P}_q$*-equivalence relation in* $\mathrm{Q}(\mathrm{mTT}^{dp})$

$$r : (\Sigma_{z\in A\times A} R(\pi_1 z, \pi_2 z), =_{Cm}) \to (A \times A, =_\times)$$

is (A, R). The quotient map from $(A, =_A)$ to (A, R) is $z \in A \ [z \in A]$. Given a map $a : (A, =_A) \to (D, =_D)$ coequalizing the projections along r, namely $\pi_1 \cdot r$ and $\pi_2 \cdot r$, the unique map from (A, R) to $(D, =_D)$ factoring a is $a(z) \in D \ [z \in A]$ itself. The quotient map satisfies effectiveness by construction. The quotient stability under pullback follows easily thanks to corollary 4.17.

Furthermore, in an analogous way to theorem 4.20, we can prove that the category $\mathrm{Q}(\mathrm{mTT})_{\mathsf{set}}$ enjoys all the categorical properties necessary to interpret emTT-sets (for their definitions see, for example, [Mai05]). In particular, to prove local closure we will make use of a proposition analogous to 4.12 that can be proved in an analogous way as well:

Proposition 4.21 *The category of extensional sets* $Dep_{\mathrm{Q}(\mathrm{mTT})_{set}}((A, =_A))$*, defined analogously to definition 4.11, on an extensional set* $(A, =_A)$ *is equivalent to* $\mathrm{Q}(\mathrm{mTT})_{set}/(A, =_A)$.

Theorem 4.22 *The category* $\mathrm{Q(mTT)_{set}}$ *is lextensive (i.e. with terminal object, binary products, equalizers and stable finite disjoint coproducts), list-arithmetic (i.e. with parameterized list objects) and locally cartesian closed with stable effective quotients with respect to* $\mathcal{P}_{q_{set}}$*-equivalence relations, and the embedding* $I : \mathrm{Q(mTT)_{set}} \longrightarrow \mathrm{Q(mTT)}$ *preserves all such a structure. Moreover, the indexed category* $\mathcal{P}_{q_{set}}$ *validates first-order intuitionistic logic with equality, namely it is an intuitionistic hyperdoctrine in the sense of [See83], and the natural embedding* $i : \mathcal{P}_{q_{set}} \Longrightarrow \mathcal{P}_q \cdot I$ *preserves such a structure.*

Remark 4.23 Note that to prove theorems 4.20, 4.22 we do not need to use any preservation of definitional equality of discharged premises in the elimination constructors, which is indeed absent in mTT. For example, we did not need to use rule E-eq list) in the appendix 7 about l in $El_{list}(s, a, l)$. Indeed, uniqueness of lists does not need such an equality preservation. The same can be said for binary products and coproducts with respect to elimination rules of strong indexed sums and disjoint sums of mTT.

Moreover, the proof of local closure, or existence of suitable dependent products, went through without any use of the ξ-rule mentioned in section 2. This point was already noticed in categorical terms in [CR00, BCRS98] with the theorem stating that the exact completion of a lex category with weak dependent products has dependent products. In our case this can be read as follows. Let us define the category $\mathcal{C}(\mathrm{mTT})_{set}$ in this way: its objects are mTT-sets, its morphisms from A to B are terms $b(x) \in B\ [x \in A]$, and two morphisms $b_1(x) \in B\ [x \in A]$ and $b_2(x) \in B\ [x \in A]$ are equal if there exists a proof of $\mathsf{Id}(B, b_1(x), b_2(x))\ prop\ [x \in A]$ in mTT. The identity and composition are defined as in the syntactic categories in [Mai05].

Then, to prove the local closure in $\mathrm{Q(mTT)}_{set}$ it is enough that in $\mathcal{C}(\mathrm{mTT})_{set}$ the natural transformation

$$App \cdot (\pi_1^B)^*(-) : \mathcal{C}(\mathrm{mTT})_{set}/(A, =_A)(\,\delta\,,\,\pi_1^{\Pi_B C}\,) \to \mathcal{C}(\mathrm{mTT})_{set}/(B_\Sigma, =_{B_\Sigma})(\,(\pi_1^B)^*(\delta)\,,\pi_1^C\,)$$

is surjective without necessarily enjoying a retraction. In turn, to prove such a surjectiveness it is enough that λ-terms exist without necessarily satisfying the ξ-rule, which is instead necessary to define a proper retraction.

4.1 The interpretation of emTT

After theorem 4.20, in order to interpret emTT in Q(mTT) , and emTT^{dp} in $\mathrm{Q(mTT}^{dp})$, at a first glance it seems that we could simply use the interpretation in [Mai05] given by fibred functors (we remind from [Mai05] that this overcomes the problem, first solved in [Hof94], of interpreting substitution *correctly* when following the *informal* interpretation first given by Seely in [See84] and recalled in [Joh02b]). But this interpretation requires a choice of Q(mTT)-structure in order to interpret the various constructors. Unfortunately, we are not able to fix such a choice, if we take type equality as object equality. In particular we are not able to fix a choice of Q(mTT)-equalizers because they depend on the representatives of Q(mTT)-arrows. Indeed, the equalizer $Eq(f, g)$ of two Q(mTT)-arrows $f, g : (A, =_A) \to (B, =_B)$ may be defined as $\Sigma_{x\in A}\ f(x) =_A g(x)$, which depends on the chosen f, g and it is not intensionally equal to that built on another choice f', g' of the same two Q(mTT)-morphisms, namely for which $f =_{\mathrm{Q(mTT)}} f'$ and $g =_{\mathrm{Q(mTT)}} g'$ hold. Now, given that we are not able to fix a choice of Q(mTT)-arrows, we are not able to fix a choice of Q(mTT)-structure. Luckily, this problem can be avoided if we work in Q(mTT) up to isomorphisms. Indeed, if we take the category $\mathrm{Q(mTT)}/\simeq$ obtained from Q(mTT) by quotienting it over isomorphisms, then this category enjoys a unique choice of the structure needed to give the interpretation in [Mai05]. In this case, also the informal way of interpreting a dependent typed calculus in [See84] can be used, because it turns out to be correct. Our solution is to interpret emTT in a category $\mathrm{Q(mTT)}/\simeq$ obtained by quotienting Q(mTT) only over suitable isomorphisms, called *canonical isomorphisms*, to make the proof of the validity theorem go through. In particular, we interpret emTT-signatures of types and terms in $\mathrm{Q(mTT)}/\simeq$ by just using an interpretation of them into Q(mTT), or better into extensional dependent types and terms of mTT. This interpretation allows us to determine $\mathrm{Q(mTT)}/\simeq$-objects and morphisms where to interpret emTT-signatures by selecting their representatives up to canonical isomorphisms.

To this purpose, we need to generalize definition 4.9 of extensional dependent collection and that of extensional dependent set, as well as that of extensional dependent proposition and small proposition. To include all cases we speak of extensional dependent type.

Def. 4.24 An *extensional dependent type*

$$B_{=}(x_1, \dots x_n)\ [\, x_1 \in A_{1=}, \dots x_n \in A_{n=}\,]$$

is given by a dependent type

$$B(x_1, \dots x_n)\ type\ [\, x_1 \in A_1, \dots x_n \in A_n\,]$$

together with the following isomorphisms:

$$\sigma_{x_1,..x_n}^{x'_1,..x'_n}(p_1, \dots p_n, z) \in B(x'_1, \dots x'_n)\ [\, x_1 \in A_1, \dots x_n \in A_n\,, x'_1 \in A_1, \dots x'_n \in A_n\,,$$
$$p_1 \in x_1 =_{A_1} x'_1\,,\, p_2 \in \sigma_{x_1}^{x'_1}(x_2) =_{A_2} x'_2\, \dots\, p_n \in \sigma_{x_1,\dots x_{n-1}}^{x'_1,\dots,x'_{n-1}}(x_n) =_{A_n} x'_n, \quad z \in B(x_1, \dots x_n)]$$

not depending on the proof-terms p_i for $i = 1, \dots, n$ and preserving the equality of the various A_i for $i = 1, \dots, n$ in the sense of definition 4.9. Such isomorphisms are also closed under identity and composition as in definition 4.9. Analogously to definition 4.9 we will use the abbreviation

$$\sigma_{x_1,..x_n}^{x'_1,..x'_n}(z) \equiv \sigma_{x_1,..x_n}^{x'_1,..x'_n}(p_1, \dots p_n, z)$$

Then we define the notion of extensional isomorphism between two extensional dependent types:

Def. 4.25 Given two extensional terms $\tau_B^C(y) \in C_{=}\ [\Gamma_{=}, y \in B_{=}]$ and $\tau_C^B(z) \in B_{=}\ [\Gamma_{=}, y \in C_{=}]$, we say that τ_B^C and τ_C^B provide an *extensional isomorphism* between the extensional types $B_{=}\ [\Gamma_{=}]$ and $C_{=}\ [\Gamma_{=}]$, if we can derive proofs of

$$\tau_B^C(\tau_C^B(x)) =_{C(x)} x\ prop\ [\Gamma_{=}, x \in C_{=}] \quad \text{and} \quad \tau_C^B(\tau_B^C(x)) =_{B(x)} x\ prop\ [\Gamma_{=}, x \in B_{=}]$$

In the following, we simply indicate an extensional isomorphism with one of its parts, given that the inverse is uniquely determined up to extensional equality.

In the following, when we say that a proposition P implies another proposition Q in mTT, we mean that in mTT there a proof of $P \to Q$ under the context defining the propositions.

In the interpretation we will need to use suitable canonical isomorphisms between extensional dependent types defined as follows:

Def. 4.26 (canonical isomorphism) Let $\tau_B^C(y) \in C_{=}\ [\Gamma_{=}, y \in B_{=}]$ be part of an extensional isomorphism as defined in def. 4.25. We say when τ_B^C with its inverse τ_C^B is a *canonical isomorphism* by induction on the derivation of $B_{=}\ [\Gamma_{=}]$ and $C_{=}\ [\Gamma_{=}]$ as follows. For easiness, here we suppose to work in the larger calculus mTT^{dp}:

- If $B = C \equiv \mathsf{prop_s}$, or $B = C \equiv \mathsf{N}_0$, or $B = C \equiv \mathsf{N}_1$, then $\tau_B^C(y) =_B y$ holds under the appropriate context in mTT.

- If $B \equiv \Sigma_{x\in A} D(x)$ and $C \equiv \Sigma_{x \in A'} D'(x)$ where in mTT both $=_B$ and $=_C$ imply $=_\Sigma$ instantiated respectively for B and C as in proposition 4.12, with $D_{=}\ [\Gamma_{=}, x \in A_{=}]$ and $D'_{=}\ [\Gamma_{=}, x \in A'_{=}]$ and $A_{=}\ [\Gamma_{=}]$ and $A'_{=}\ [\Gamma_{=}]$ extensional dependent types having canonical substitution isomorphisms, then $\tau_B^C(y) =_C El_\Sigma(\, y, (w_1, w_2).\langle \tau_A^{A'}(w_1), \tau_D^{D'}(w_2)\rangle\ )$ holds under the appropriate context with $\tau_A^{A'}$ and $\tau_D^{D'}$ canonical isomorphisms where in particular $\tau_D^{D'}(y) \in D'(\tau_A^{A'}(x))\ [\Gamma_{=}, x \in A_{=}, y \in D_{=}(x)]$ (note that we still use the notation $\tau_D^{D'}$ even if the types are on different contexts). Moreover, also the substitution isomorphisms of $B_{=}$ and $C_{=}$ and τ_C^B are of this form.

- If

  $$B \equiv \Sigma_{h \in \Pi_{x\in A} D(x)} \quad \forall_{x_1 \in A}\ \forall_{x_2 \in A}\ \forall_{d \in x_1 =_A x_2} \quad \sigma_{\overline{x}, x_1}^{\overline{x}, x_2}(\mathsf{Ap}(h, x_1)) =_{D(x_2)} \mathsf{Ap}(h, x_2)$$

  with $\overline{x}$ the variables in Γ and analogously

  $$C \equiv \Sigma_{h \in \Pi_{x\in A'} D'(x)} \quad \forall_{x_1 \in A'}\ \forall_{x_2 \in A'}\ \forall_{d \in x_1 =_{A'} x_2} \quad \sigma_{\overline{x}, x_1}^{\overline{x}, x_2}(\mathsf{Ap}(\,h,\, x_1\,)) =_{D'(x_2)} \mathsf{Ap}(\,h,\, x_2\,)$$

where $=_B$ implies in mTT the following equivalence relation for $z, z' \in B$

$$\forall_{x \in A} \ \mathsf{Ap}(\,\pi_1(z)\,,\,x\,) =_{D(x)} \mathsf{Ap}(\,\pi_1(z')\,,\,x\,)$$

and $=_C$ is of analogous form, with $D_=\ [\Gamma_=, x \in A_=]$ and $D'_=\ [\Gamma_=, x \in A'_=]$ and $A_=\ [\Gamma_=]$ and $A'_=\ [\Gamma_=]$ extensional dependent types having canonical substitution isomorphisms, then we can derive a proof of

$$\tau_B^C(y) =_C \langle\, \lambda w^{A'} .\, \sigma^{w}_{\tau_A^{A'}(\tau_{A'}^{A}(w))}(\,\tau_D^{D'}(\,\mathsf{Ap}(\,\pi_1(y)\,,\,\tau_{A'}^{A}(w))\,)\,)\,,\,p\rangle$$

for some proof-term p under the appropriate context with $\tau_A^{A'}$ and $\tau_D^{D'}$ canonical isomorphisms where in particular $\tau_D^{D'}(y) \in D'(\tau_A^{A'}(x))\ [\Gamma_=, x \in A_=, y \in D_=(x)]$. Moreover, also the substitution isomorphisms of $B_=$ and $C_=$ and τ_C^B are of this form.

- If $B \equiv \mathit{List}(D)$ and $C \equiv \mathit{List}(D')$ where in mTT both $=_B$ implies $=_{List(D)}$ and $=_C$ implies $=_{List(D)}$ defined as in the proof of theorem 4.20 with $D_=\ [\Gamma_=]$ and $D'_=\ [\Gamma_=]$ extensional dependent types having canonical substitution isomorphisms, then $\tau_B^C(y) =_C \mathit{El}_{List}(y, \epsilon, (y_1, y_2, z).\,\mathsf{cons}(\,z\,,\,\tau_D^{D'}(y_2)\,)\,)$ holds under the appropriate context with $\tau_D^{D'}$ canonical isomorphism. Moreover, also the substitution isomorphisms of $B_=$ and $C_=$ and τ_C^B are of this form.

- If $B \equiv A + D$ and $C \equiv A' + D'$ where in mTT both $=_B$ implies $=_{A+D}$ and $=_C$ implies $=_{A'+D'}$ defined as in the proof of theorem 4.20 with $A_=\ [\Gamma_=]$ and $A'_=\ [\Gamma_=]$ and $D_=\ [\Gamma_=]$ and $D'_=\ [\Gamma_=]$ extensional dependent types having canonical substitution isomorphisms, then $\tau_B^C(y) =_C \mathit{El}_+(y, (y_1).\,\mathsf{inl}(\,\tau_A^{A'}(y_1)\,)\,,\,(y_2).\,\mathsf{inr}(\,\tau_D^{D'}(y_2)\,)\,)$ holds under the appropriate context with $\tau_A^{A'}$ and $\tau_D^{D'}$ canonical isomorphisms. Moreover, also the substitution isomorphisms of $B_=$ and $C_=$ and τ_C^B are of this form.

- If $B\ \mathit{prop}\ [\Gamma]$ and $C\ \mathit{prop}\ [\Gamma]$ are derivable and $=_B$ and $=_C$ are the trivial relation equating all proofs, then any τ_B^C with τ_C^B is canonical.

Remark 4.27 Note that the above definition of canonical isomorphisms can be adapted to work for mTT by simply adding a case analogous to the third one where

$$D(x) \equiv \mathsf{prop_s} \qquad z =_{D(x)} z' \equiv z \leftrightarrow z' \ \text{ for } z, z' \in \mathsf{prop_s}$$

with $\sigma^{\overline{x}, x_2}_{\overline{x}, x_1}(w) \equiv w$ for $w \in D(x_1)$ and in particular

$$B \equiv \Sigma_{h \in A \to \mathsf{prop_s}} \quad \forall_{x_1 \in A} \forall_{x_2 \in A} \ x_1 =_A x_2 \to (\,\mathsf{Ap}(h, x_1) \leftrightarrow \mathsf{Ap}(h, x_2)\,)$$

and furthermore C is defined analogously.

Note that canonical isomorphisms are closed under composition, and more importantly between two extensional dependent types there is at most only one canonical isomorphism extensionally:

Proposition 4.28 *Canonical isomorphisms enjoy the following properties:*

- *If* $\tau_A^B(y) \in B_=\ [\Gamma_=, y \in A_=]$ *and* $\tau_B^C(y) \in C_=\ [\Gamma_=, y \in B_=]$ *are canonical isomorphisms, then* $\tau_B^C(\tau_A^B(y)) \in C_=\ [\Gamma_=, y \in A_=]$ *is also a canonical isomorphism.*

- *If* $\tau_A^B(y) \in B_=\ [\Gamma_=, y \in A_=]$ *and* ${\tau'}_A^B(y) \in B_=\ [\Gamma_=, y \in A_=]$ *are both canonical isomorphisms, then we can prove that they are equal as extensional terms, namely that we can derive a proof of*

$$\tau_A^B(y) =_B {\tau'}_A^B(y)\ \mathit{prop}\ [\Gamma_=, y \in A_=]$$

- *If* $\tau_A^B(y) \in B_=\ [\Gamma_=, y \in A_=]$ *is a canonical isomorphism, then* $\tau_A^B(y) \in B'_=\ [\Gamma_=, y \in A'_=]$ *is also a canonical isomorphism provided that it is an extensional isomorphism with* $='_A$ *implying* $=_A$ *and* $='_B$ *implying* $=_B$.

Proof. The proof is by induction on the definition of canonical isomorphism by showing all the points of the main statement in the same time together with the facts that for any canonical isomorphism $\tau_A^B(y) \in B_= \; [\Gamma_=, y \in A_=]$ then all all the substitutions morphisms of $A_=$ and $B_=$ are canonical and that for any extensional dependent type $B_= \; [\Gamma_=]$ with canonical substitution morphisms then the identity $\tau_B^B(w) \equiv w \in B_= \; [\Gamma_=]$ is also a canonical isomorphism, and finally canonical isomorphisms have canonical inverses.

Then we define the quotient category of Q(mTT) over canonical isomorphisms in which we will interpret emTT. The equivalence relation generated by canonical isomorphisms coincides with extending the previous collection of canonical isomorphisms by including all identity morphisms between objects of Q(mTT) [8].

Def. 4.29 We call Q(mTT)/$\simeq$ the category obtained by quotienting Q(mTT) over canonical isomorphisms: namely an object of Q(mTT)/$\simeq$ is given by the equivalence class of a Q(mTT)-object $(A, =_A)$

$$[\,(A, =_A)\,]$$

where two objects are defined to be equivalent if they are isomorphic via a canonical isomorphism, and a morphism from $[\,(A, =_A)\,]$ to $[\,(B, =_B)\,]$ is the equivalence class given by a Q(mTT)-morphism $f : (C, =_C) \to (D, =_D)$

$$[f] : [(A, =_A)] \to [(B, =_B)]$$

such that $(C, =_C)$ is canonically isomorphic to $(A, =_A)$, i.e. via a canonical isomorphism, and $(D, =_D)$ canonically isomorphic to $(B, =_B)$, where two such morphisms f, g are defined to be equivalent, if, supposing $f : (C, =_C) \to (D, =_D)$ and $g : (M, =_M) \to (N, =_N)$ then

$$g \cdot \tau_C^M =_{\mathrm{Q(mTT)}} \tau_D^N \cdot f$$

for canonical isomorphisms τ_C^M and τ_D^N, given that $(C, =_C)$ turns out to be canonically isomorphic to $(M, =_M)$ and $(D, =_D)$ to $(N, =_N)$. The equality between two morphisms coincides with the equality of their equivalence classes. The unit and composition are those inherited from Q(mTT).

The category is well defined thanks to the first two properties of canonical isomorphisms in prop. 4.28, which continue to hold with the addition of all identity morphisms.

We define the category Q(mTT^{dp})/$\simeq$ analogously and we will use it to interpret emTT^{dp}.

Why canonical isomorphisms. We will use canonical isomorphisms to interpret equality between emTT(emTT^{dp})-types. The reason is the following. The underlying assumption is that, in order to be able to interpret quotient types, we interpret emTT-dependent types as mTT-extensional dependent types and emTT-terms as mTT-extensional terms. Then, it follows that the definitional equality between emTT-terms must be interpreted in the existence of a proof that the two terms are equal according to the equality associated with their type interpretation. Now, suppose that the mTT-extensional dependent collection $B^I_=(x) \; [x \in A^I_=]$ interprets the emTT collection $B(x) \; col \; [x \in A]$, and that the judgement $a_1 = a_2 \in A$ holds in emTT and is valid in Q(mTT), namely that $a_1^I =_{A^I} a_2^I$ holds in mTT. Now, in emTT we get also that $B(a_1) = B(a_2)$ holds, too, but in mTT we just know that

$$B^I(a_1)_= \text{ is isomorphic to } B^I(a_2)_=$$

via a substitution isomorphism. Therefore, we are forced to interpret type equality as isomorphism of types, and to this purpose we introduce in mTT the judgement $A_= =_{ext} B_= \; [\Gamma_=]$ for saying that $A_= \; [\Gamma_=]$ is isomorphic to $B_= \; [\Gamma_=]$ via an mTT-extensional isomorphism. But then, in order to make the rule conv) in appendix 6 valid as well, we may need to correct the interpretation b^I of an emTT-term b via an isomorphism. Indeed, supposing that $a^I \in A^I_{1=} \; [\Gamma^I_=]$ in mTT interprets the derived emTT-judgement $a \in A_1 \; [\Gamma]$ and that the mTT-judgement $A^I_{1=} =_{ext} A^I_{2=} \; [\Gamma^I_=]$ interprets the derived emTT-judgement $A_1 = A_2 \; col \; [\Gamma]$, in order to make the interpretation of the emTT-judgement $a \in A_2 \; [\Gamma]$ derived by conv) valid, given that in mTT we only know that $A^I_{1=} =_{ext} A^I_{2=} \; [\Gamma^I_=]$ holds, we need to introduce in mTT the judgement

$$a^I \in_{ext} A^I_{2=} \; [\Gamma^I_=]$$

[8] We still get a well defined quotient category, if we make canonical also all the morphisms between objects Q(mTT) which include the identity morphism between their supports as a representative (and hence the equivalence relations of the two objects are equivalent).

to express that a^I is *extensionally* of type A_2^I if it belongs to a type isomorphic to it, namely if in mTT we can derive

$$\tau_{A_1^I}^{A_2^I}(a^I) \in A^I_{2=}\ [\Gamma^I_=]$$

via a canonical isomorphism $\tau_{A_1^I}^{A_2^I}$. But now, given that the interpretation of emTT-terms in mTT depends on isomorphisms, then the interpretation of emTT-types depends on isomorphisms, too, because types may depend on terms. Luckily, we are able to give the interpretation by making use of *canonical isomorphisms only*, and hence we require the isomorphisms used so far to be canonical. The fact that canonical isomorphisms between two interpreted types are at most one extensionally will allows us to prove the validity theorem.

Before interpreting emTT-signatures, we say when two extensional dependent types under a common context and on different contexts are isomorphic:

Def. 4.30 In mTT we say that the extensional dependent type $B_=\ [\Gamma_=]$ is *extensionally equal* to the extensional dependent type $C_=\ [\Gamma_=]$ with the judgement

$$B_= =_{ext} C_=\ [\Gamma_=]$$

if they are isomorphic via a canonical extensional isomorphism as in definition 4.26.
We will generally call the isomorphism components $\tau_B^C(y) \in C_=\ [\Gamma_=, y \in B_=]$ and $\tau_C^B(z) \in B_=\ [\Gamma_=, z \in C_=]$.

Def. 4.31 Given the mTT-extensional dependent types $B_=\ [\Gamma_=]$ and $D_=\ [\Delta_=]$ where $\Gamma_= \equiv x_1 \in A_{1=}, \ldots, x_n \in A_{n=}$ and $\Delta_= \equiv y_1 \in C_{1=}, \ldots, y_n \in C_{n=}$ we introduce the judgement

$$B_=\ [\Gamma_=] =_{ext} D_=\ [\Delta_=]$$

to express that in mTT we can derive $A_1 =_{ext} C_1$ and $A_{i=} =_{ext} D_i(\tau_{A_1}^{C_1}(x_1), \ldots, \tau_{A_{i-1}}^{C_{i-1}}(x_{i-1}))_=\ [x_1 \in A_{1=}, \ldots, x_{i-1} \in A_{i-1=}]$ for canonical isomorphisms $\tau_{A_i}^{C_i}$ for $i = 2, \ldots, n$ if $n \geq 2$, and also

$$B_= =_{ext} \widetilde{D}_=\ [\Gamma_=]$$

via a canonical isomorphism τ_B^D where $\widetilde{D} \equiv D\,[y_1/\tau_{A_1}^{C_1}(x_1), \ldots, y_n/\tau_{A_n}^{C_n}(x_n)]$ whose equality is $w =_{\widetilde{D}} w' \equiv z =_D z'\,[y_1/\tau_{A_1}^{C_1}(x_1), \ldots, y_n/\tau_{A_n}^{C_n}(x_n), z/w, z'/w]$.

Then, we are ready to define when a term belongs to a type in an extensional way:

Def. 4.32 Given an mTT-extensional term $b \in D_=\ [\Delta_=]$ and an mTT-extensional dependent type $B_=\ [\Gamma_=]$ where $\Gamma_= \equiv x_1 \in A_{1=}, \ldots, x_n \in A_{n=}$ and $\Delta_= \equiv y_1 \in C_{1=}, \ldots, y_n \in C_{n=}$, and supposing that $B_=\ [\Gamma_=] =_{ext} D_=\ [\Delta_=]$ with canonical isomorphisms $\tau_{A_i}^{C_i}$ for $i = 1, \ldots, n$, if $n \geq 1$, and τ_B^D, then in mTT we say that b *is extensionally of type* $B_=\ [\Gamma_=]$ with the judgement

$$b \in_{ext} B_=\ [\Gamma_=]$$

if in mTT we can derive

$$\widetilde{b} \in \widetilde{B}_=\ [\Gamma_=]$$

where $\widetilde{b} \equiv \tau_D^B(\, b(\,\tau_{A_1}^{C_1}(x_1), \ldots, \tau_{A_n}^{C_n}(x_n)\,)\,)$.

Then, we define extensional equality between terms:

Def. 4.33 Given two mTT-extensional terms

$$b \in_{ext} B_=\ [\Gamma_=] \qquad c \in_{ext} B_=\ [\Gamma_=]$$

in mTT we say that they are *extensionally equal terms* with the judgement

$$b =_{ext} c \in_{ext} B_=\ [\Gamma_=]$$

if and only if in mTT we can derive a proof

$$p \in \widetilde{b} =_{\widetilde{B}} \widetilde{c} \quad [\,\Gamma\,]$$

where $\widetilde{b}, \widetilde{c}$ and $\widetilde{B}$ are defined respectively as in definitions 4.32 and 4.31.

Interpretation of emTT **types and terms.** Now we are ready to define the interpretation of emTT type and term signatures as mTT-extensional dependent types, with canonical substitution morphisms, and terms, respectively. Then, a type judgement will be interpreted as an extensional dependent type

$$(\, B\ type\ [\,\Gamma\,]\,)^I \ \equiv\ B^I_=\ [\,\Gamma^I_=\,]$$

and a term judgement as an extensional term which belongs *only extensionally* to the interpretation of the assigned type as follows:

$$(\, b \in B\ [\Gamma])^I \ \equiv\ b^I \in_{ext} B^I_=\ [\,\Gamma^I_=\,]$$

In order to give an idea on how the interpretation is defined, suppose to interpret $\mathsf{Eq}(B, b_1, b_2)\ prop\ [\Gamma]$ assuming that $(\, B\ type\ [\,\Gamma\,]\,)^I \ \equiv\ B^I_=\ [\Gamma^I_=]$. Then, the term signatures b_1 and b_2 under context Γ are assumed to be interpreted as mTT-extensional terms $b_1^I \in C_=\ [\Gamma^I_=]$ and $b_2^I \in M_=\ [\Gamma^I_=]$. Now, to give the interpretation we do not require as usual that $C_=$ is equal in mTT to $M_=$ and to $B^I_=$, but only that it is isomorphic to them via canonical isomorphisms. Hence we put $(\,\mathsf{Eq}(B, b_1, b_2)\ prop\ [\Gamma]\,)^I \equiv \widetilde{b_1^I} =_{B^I} \widetilde{b_2^I}\ prop\ [\Gamma^I]$ where we have corrected the interpretation of b_1^I and b_2^I to match the type B^I as in definition 4.32 by means of canonical isomorphisms.

This explains why we give an interpretation $(-)^I : \text{emTT} \to \text{mTT}$ of emTT-type and term signatures as extensional dependent types with canonical substitution morphisms and extensional terms in mTT, that is not only partial as usual interpretations of dependent type theories (see first paragraph in appendix 8), but also uses canonical isomorphisms.

Analogously, we define an interpretation $(-)^I : \text{emTT}^{dp} \to \text{mTT}^{dp}$ of emTT^{dp}-type and term signatures as mTT^{dp} extensional dependent types with canonical substitution morphisms and extensional terms.

The interpretations are properly defined in appendix 8. Here, we just show the interpretation of the power collection of the singleton $\mathcal{P}(1)$ and of dependent product sets, quotient sets and function collections towards $\mathcal{P}(1)$ with their terms. Therefore, to interpret these emTT-types as extensional dependent types of mTT, we need to specify the support of their interpretation with related equality and substitution morphisms. Note that in the case we are interpreting a type or a term that requires to have already interpreted more than one term, we need to match the types of such terms and we assume to correct them via *canonical isomorphisms.* In the following, we simply write $b^{\widetilde{I}}$ instead of $\widetilde{b^I}$.
The power collection of the singleton is interpreted as the extensional collection classifying $\mathcal{P}_q$-small propositions in theorem 4.20, namely as the mTT-collection of small propositions equipped with equiprovability as equality:

Power collection of the singleton :
$\mathcal{P}(1)^I\ col\ [\Gamma^I] \ \equiv\ \mathsf{prop_s}\ [\Gamma^I]$
and $z =_{\mathsf{prop_s}^I} z' \ \equiv\ (z \to z') \wedge (z' \to z)$ for $z, z' \in \mathsf{prop_s}$
$\sigma_{\overline{x}}^{\overline{x}'}(w) \ \equiv\ w$ for $\overline{x}, \overline{x'} \in \Gamma^I, w \in \mathsf{prop_s}$.
$([A])^I \ \equiv\ A^I$ for A small proposition.

The dependent product set is interpreted similarly to the extensional collection behind the dependent product construction in theorem 4.20, namely as the strong indexed sum of functions preserving the corresponding equalities. Two elements of this indexed sum are considered equal if their first components, which are lambda-functions, send a given element to equal elements. This interpretation validates both β and η equalities for functions and also the ξ-rule.

Dependent Product set :
$(\,\Pi_{y\in B} C(y)\,)^I\ set\ [\Gamma^I] \equiv\ \Sigma_{h\in \Pi_{y\in B^I}\ C^I(y)} \quad \forall_{y_1\in B^I}\ \forall_{y_2 \in B^I}\ \forall_{d\in y_1 =_{B^I} y_2}\ \sigma_{\overline{x}, y_1}^{\overline{x}, y_2}(\,\mathsf{Ap}(h, y_1)\,) =_{C^I(y_2)} \mathsf{Ap}(h, y_2)$
and $z =_{\Pi} z' \ \equiv\ \forall_{y\in B^I}\ \ \mathsf{Ap}(\,\pi_1(z)\,, y) =_{C^I(y)} \mathsf{Ap}(\,\pi_1(z')\,, y)$ for $z, z' \in (\,\Pi_{y\in B}\, C(y)\,)^I$
$(\,\lambda y^B.c\,)^I \ \equiv\ \langle\, \lambda y^{B^I}.c^{\widetilde{I}}\,,\, p\,\rangle$ where $p \in \forall_{y_1\in B^I}\ \ \forall_{y_2\in B^I}\ \forall_{d\in y_1 =_{B^I} y_2}\ \sigma_{\overline{x}, y_1}^{\overline{x}, y_2}(\, c^{\widetilde{I}}(y_1)\,) =_{C^I(y_2)} c^{\widetilde{I}}(y_2)$
$(\,\mathsf{Ap}(f, b)\,)^I \ \equiv\ \mathsf{Ap}(\,\pi_1(f^{\widetilde{I}})\,, b^{\widetilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \ \equiv\ \langle\, \lambda y'^{B^I(\overline{x'})}.\,\sigma_{\overline{x}, \sigma_{\overline{x'}}^{\overline{x}}(y')}^{\overline{x'}, y'}(\,\mathsf{Ap}(\,\pi_1(w)\,,\, \sigma_{\overline{x'}}^{\overline{x}}(y')\,)\,), p\rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\,\Pi_{y\in B}\, C(y)\,)^I(\overline{x})$.
where p is the proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

The quotient on a set whose interpretation has support A^I is interpreted as the extensional set with same support A^I, but whose equality is the interpretation of the quotient equivalence relation, in a similar way to the construction of quotients in theorem 4.20. Then the interpretation of the quotient map is simply given by the identity and effectiveness becomes trivially validated:

Quotient set :

$(\,A/R\ set\ [\Gamma])^I \ \equiv\ A^I\ set\ [\Gamma^I]$

and $z =_{A/R^I} z' \ \equiv\ R^I(z,z')$ for $z,z' \in A^I$

$(\,[a]\,) \ \equiv\ a^I$ and $El_Q(p,l)^I \ \equiv\ l^{\widetilde{I}}(p^{\widetilde{B}})$

$\sigma_{\overline{x}}^{\overline{x'}}(w)$ is defined as the substitution isomorphism of $A^I_{=}\,[\,\Gamma^I_{=}]$.

A function collection towards $\mathcal{P}(1)$ is interpreted as the extensional collection behind the local exponential construction of the $\mathcal{P}_q$-small proposition classifier in theorem 4.20:

Function collection toward $\mathcal{P}(1)$:

$(\,B \to \mathcal{P}(1)\ col\ [\Gamma]\,)^I \ \equiv\ \Sigma_{h\in B^I \to \mathsf{props}} \quad \forall_{y_1\in B^I}\ \ \forall_{y_2\in B^I}\ \ y_1 =_{B^I} y_2 \ \to\ (\,\mathsf{Ap}(h,y_1)\ \leftrightarrow\ \mathsf{Ap}(h,y_2)\,)$

and $z =_{\mathcal{P}} z' \ \equiv\ \forall_{y\in B^I}\ \ \mathsf{Ap}(\,\pi_1(z)\,,y)\ \leftrightarrow\ \ \mathsf{Ap}(\,\pi_1(z')\,,y)$ for $z,z' \in (\,B \to \mathcal{P}(1)\,)^I$

$(\,\lambda y^B.c\,)^I \ \equiv\ \langle\,\lambda y^{\widetilde{B}}.c^{\widetilde{I}}\,,\,p\,\rangle$ where $p \in \forall_{y_1\in B^I}\ \ \forall_{y_2\in B^I}\ \ y_1 =_{B^I} y_2 \ \to\ (\,c^{\widetilde{I}}(y_1)\ \leftrightarrow\ c^{\widetilde{I}}(y_2)\,)$

$(\,\mathsf{Ap}(f,b)\,)^I \ \equiv\ \mathsf{Ap}(\,\pi_1(f^{\widetilde{I}})\,,b^{\widetilde{I}})$

$\sigma_{\overline{x}}^{\overline{x'}}(w) \ \equiv\ \langle\,\lambda y'^{B^I(\overline{x'})}.\,\sigma^{\overline{x'},y'}_{\overline{x},\sigma^{\overline{x}}_{\overline{x'}}(y')}(\,\mathsf{Ap}(\,\pi_1(w)\,,\,\sigma^{\overline{x}}_{\overline{x'}}(y')\,)\,),\,p\,\rangle$ for $\overline{x},\overline{x'} \in \Gamma^I$ and $w \in (\,B \to \mathcal{P}(1)\,)^I(\overline{x})$ where p is a proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

Remark 4.34 Note that we need to close mTT collections under strong indexed sums in order to interpret function collections toward $\mathcal{P}(1)$, and hence in turn to interpret power collections of sets as described in section 3.

After giving the interpretation of emTT-signatures (and emTT^{dp}-signatures) into mTT-extensional dependent types and terms, we can interpret emTT (and emTT^{dp}) judgements in the category Q(mTT)/$\simeq$ (Q(mTT^{dp})/$\simeq$) by following the idea behind the naive interpretation of dependent types in [See84] and in the completeness theorem in [Mai05]. To this purpose we need first to transform extensional dependent types into Q(mTT)-arrows as in proposition 4.12.

Def. 4.35 Given an mTT-context $\Gamma_{=}$ we define its indexed closure $Sig(\Gamma_{=})$ as a Q(mTT)-object together with suitable projections $\pi^n_j(z)$ for $z \in Sig(\Gamma_{=})$ and $j = 1,\ldots,n$ by induction on the length n of $\Gamma_{=}$ as follows:

- If $\Gamma_{=}\ \equiv\ x \in A_{=}$ then
$$Sig(\,x \in A_{=}\,) \equiv (A, =_A) \qquad \text{and} \qquad \pi^1_1(z)\ \equiv\ z$$
- If $\Gamma_{=}\ \equiv\ \Delta, x \in A$ of $n+1$ length then
$$Sig(\,\Delta_{=}, x \in A_{=}\,)\ \equiv\ (\,\Sigma_{z\in Sig(\Delta_{=})}\ A[x_1/\pi^n_1(z),\ldots x_n/\pi^n_n(z)]\,,\,=_{Sig}\,)$$
where $w =_{Sig} w'\ \equiv\ \exists_{d\in \pi_1(w) =_{Sig} \pi_1(w')}\ \ \sigma^{\pi_1(w')}_{\pi_1(w)}(\pi_2(w)) =_{A_{w'}} \pi_2(w')$
for $w,w' \in \Sigma_{z\in Sig(\Delta_{=})}\ A\,[x_1/\pi^n_1(z),\ldots x_n/\pi^n_n(z)]$ and $\pi^{n+1}_j(w)\ \equiv\ \pi^n_j(\pi_1(w))$ for $j = 1,\ldots,n$ and $\pi^{n+1}_{n+1}(w)\ \equiv\ \pi_2(w)$ and $A_{w'}\ \equiv\ A\,[x_1/\pi^n_1(\pi_1(w')),\ldots x_n/\pi^n_n(\pi_1(w'))]$.

Def. 4.36 (Interpretation in Q(mTT)/$\simeq$**)** The interpretation of emTT judgements in the category Q(mTT)/$\simeq$
$$Int : \text{emTT} \to \text{Q(mTT)}/\simeq$$
is defined by using the interpretation of emTT-signatures into mTT-extensional dependent types and terms in appendix 8 as follows.
An emTT-dependent type is interpreted as the projection in Q(mTT)/$\simeq$ of its interpretation as mTT-extensional dependent type according to the idea of turning a dependent collection into an arrow in proposition 4.12:
$$Int(\,B\ type\ [\,\Gamma\,]\,)\ \ \equiv [\pi_1]\ :\ [\,Sig(\,\Gamma^I_{=}, B^I_{=}\,)\,] \to [\,Sig(\,\Gamma^I_{=}\,)\,]$$

Then, an emTT-type equality judgement is interpreted as the morphism equality of type interpretations in Q(mTT)/$\simeq$

$$Int(\, A = B\ type\ [\Gamma]\,) \ \equiv\ Int(\, A\ type\ [\,\Gamma\,]\,) =_{\mathrm{Q(mTT)}/\simeq} Int(\, B\ type\ [\,\Gamma\,]\,)$$

which amounts to proving that their interpretations as mTT-extensional dependent types are extensionally equal, namely that in mTT we can derive

$$A^I_= \ =_{ext}\ B^I_= \ [\,\Gamma^I_=\,]$$

An emTT-term is interpreted as a section of the corresponding type and it is built out of its interpretation as mTT-extensional term:

$$Int(\, b \in B\ [\Gamma]\,) \ \equiv\ [\,\langle z, \overline{b^I}\rangle\,] : [\,Sig(\,\Gamma^I_=\,)\,] \to [\,Sig(\,\Gamma^I_=, B^I_=\,)\,]$$

where $\overline{b^I}$ is obtained by substituting its free variables $x_1,\ldots,x_n$ with $\pi^n_j(z)$ for $j=1,\ldots,n$ as in definition 4.35. The given interpretation amounts to deriving in mTT

$$b^I \in_{ext} B^I\ [\Gamma^I]$$

Note that the interpretation of a term is a section of its type interpretation because $[\pi_1]\cdot[\langle z, \overline{b^I}\rangle] =_{\mathrm{Q(mTT)}/\simeq} \mathsf{id}$ holds.
Finally, an emTT-term equality judgement is interpreted as the equality of Q(mTT)/$\simeq$-morphisms interpreting the terms:

$$Int(\, a = b \in B\ [\Gamma]\,) \ \equiv\ Int(\, a \in B\ type\ [\,\Gamma\,]\,) =_{\mathrm{Q(mTT)}/\simeq} Int(\, b \in B\ type\ [\,\Gamma\,]\,)$$

which amounts to deriving in mTT

$$a^I \ =_{ext}\ b^I \in_{ext} B^I\ [\Gamma^I]$$

Analogously, we define the interpretation of emTT^{dp}-judgements in Q(mTT^{dp})/$\simeq$.

In order to prove the validity theorem we need to know how to interpret weakening and substitution. For easiness we just show how to interpret substitution.
Note that in the following, given a context $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ with $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ then for every $a \in A_n\ [\Sigma]$ and for any type $B\ type\ [\Gamma]$ we simply write the substitution of x_n with a in B in the form $B[x_n/a]\ type\ [\Sigma, \Delta_a]$ instead of the more correct form $B[x_n/a_n][x_i/x'_i]_{i=n+1,\ldots,k}\ type\ [\Sigma, \Delta_a]$ where $\Delta_a \equiv x'_{n+1} \in A'_{n+1}, ..., x'_k \in A'_k$ and $A'_j \equiv A_j\ [x_n/a_n][x_i/x'_i]_{i=n+1,\ldots,j-1}$ for $j = n+2, ..., k$, if $n+2 \leq k$, otherwise $A'_{n+1} \equiv A_{n+1}\ [x_n/a_n]$. If Δ is the empty context, then Δ_a is empty, too. Similar abbreviations are used also for terms.

Lemma 4.37 *For any* emTT *judgement B type* $[\Gamma]$ *interpreted in* Q(mTT)/$\simeq$ *as*

$$[\pi_1]\ :\ [\,Sig(\,\Gamma^I_=\,,\ w \in B^I_=\,)] \to [\,Sig(\,\Gamma^I_=\,)\,]$$

and $b \in B\ [\Gamma]$ *interpreted in* Q(mTT)/$\simeq$ *as*

$$[\langle z, \overline{b^I}\rangle]\ : [\,Sig(\,\Gamma^I_=\,)\,] \to [\,Sig(\,\Gamma^I_=\,,\ w \in B^I_=\,)]$$

substitution *is interpreted as follows: supposed* $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ *with* $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ *if not empty, for every* emTT *judgement* $a \in A_n\ [\Sigma]$ *interpreted as*

$$[\,\langle z, \overline{a^I}\rangle\,]\ :\ [\,Sig(\,\Sigma^I{}_=\,)\,] \to [\,Sig(\,\Sigma^I{}_=, x_n \in A^I_{n=}\,)\,]$$

then

$$I(\,B[x_n/a]\ type\ [\Sigma, \Delta_a]\,) =_{\mathrm{Q(mTT)}/\simeq} [\pi_1]\ :\ [\,Sig(\,\Sigma^I{}_=, \Delta^I_{a=}\,,\ w \in B^I[x_n/\widetilde{a^I}]_=\,)\,] \to [\,Sig(\,\Sigma^I{}_=, \Delta^I_{a=}\,)\,]$$

and

$$Int(\,b[x_n/a] \in B[x_n/a]\ type\ [\Sigma, \Delta_a]\,) =_{\mathrm{Q(mTT)}/\simeq} [\,\overline{\langle z, b^I[x_n/\widetilde{a^I}]\rangle}\,] : [Sig(\,\Sigma^I{}_=, \Delta^I_{a=}\,)\,] \to [\,Sig(\,\Sigma^I{}_=, \Delta^I_{a=}, w \in B^I[x_n/\widetilde{a^I}]_=\,)\,]$$

where the support of $B^I[x_n/\widetilde{a^I}]_=$ *is* $B^I[x_n/\widetilde{a^I}][x_i/x'_i]_{i=n+1,\ldots,k}$ *and* $z =_{B^I[x_n/\widetilde{a^I}]} z' \equiv (z =_{B^I} z')[x_n/\widetilde{a^I}][x_i/x'_i]_{i=n+1,\ldots,k}$ *defines its equality, if Δ is not empty.*

Proof. By induction on the interpretation of the signature.

An analogous lemma holds for the interpretation of emTT^{dp}-signatures in Q(mTT^{dp})/$\simeq$.

Theorem 4.38 (validity of emTT **into** Q(mTT)/$\simeq$**)** *The calculus* emTT *is valid with respect to the interpretation in definition 4.36 of* emTT*-signatures in* Q(mTT)/$\simeq$*:*
If A type $[\Gamma]$ is derivable in emTT*, then Int(A type $[\Gamma]$) is well defined. If $a \in A\ [\Gamma]$ is derivable in* emTT*, then Int($a \in A\ [\Gamma]$) is well defined. Supposing that A type $[\Gamma]$ and B type $[\Gamma]$ are derivable in* emTT*, if $A = B$ type $[\Gamma]$ is derivable in* emTT*, then Int($A = B$ type $[\Gamma]$) is valid.*
Supposing that $a \in A\ [\Gamma]$ and $b \in A\ [\Gamma]$ are derivable in emTT*, if $a = b \in A\ [\Gamma]$ is derivable in* emTT*, then Int($a = b \in A\ [\Gamma]$) is also valid.*

Proof. We can prove the statements by induction on the derivation of the judgements by making use of theorem 4.20.
Note that the rule conv) is validated because of the presence of canonical isomorphisms witnessing that two types are extensionally equal. Moreover, conversion rules are valid thanks to the properties of canonical isomorphisms in proposition 4.28 and thanks to the fact that they send canonical elements to canonical ones.

Analogously, we can prove:

Theorem 4.39 (validity of emTT^{dp} **into** Q(mTT^{dp})/$\simeq$**)** *The calculus* emTT^{dp} *is valid with respect to the interpretation in definition 4.36 of* emTT^{dp}*-signatures in* Q(mTT^{dp})/$\simeq$*:*
If A type $[\Gamma]$ is derivable in emTT^{dp}*, then Int(A type $[\Gamma]$) is well defined. If $a \in A\ [\Gamma]$ is derivable in* emTT^{dp}*, then Int($a \in A\ [\Gamma]$) is well defined. Supposing that A type $[\Gamma]$ and B type $[\Gamma]$ are derivable in* emTT^{dp}*, if $A = B$ type $[\Gamma]$ is derivable in* emTT^{dp}*, then Int($A = B$ type $[\Gamma]$) is valid.*
Supposing that $a \in A\ [\Gamma]$ and $b \in A\ [\Gamma]$ are derivable in emTT^{dp}*, if $a = b \in A\ [\Gamma]$ is derivable in* emTT^{dp}*, then Int($a = b \in A\ [\Gamma]$) is also valid.*

Remark 4.40 Note that the power collection of a set is interpreted in Q(mTT) as the quotient of suitable small propositional functions under equiprovability as in [SV98], but with the difference that here it is a particular construction on the top of mTT, while in [SV98] it is declared to be a type in the underlying intensional theory, which then loses the decidability of type judgements. Hence, in our interpretation of emTT over mTT a subset is interpreted as the equivalence class determined by a small propositional function, while in [SV98] (and also in [CS08]) it is identified with it.

Remark 4.41 Internal logic of Q(mTT^{dp}). As already announced, emTT is not at all the internal language of Q(mTT), because, for example, it does not include generic quotient collections whilst they are supported in the model. Even emTT^{dp} is not the internal language of Q(mTT^{dp}). One reason is the following. If we restrict to the set-theoretic fragment of emTT, called qmTT in [Mai07] and here emTT$_{set}$, then the interpretation of implication, of universal quantification and of dependent product set do not seem to be preserved by the functor $\xi : \mathrm{Q(mTT)}_{set} \to \mathcal{C}(\mathrm{emTT}_{set})$ sending an extensional type into its quotient (see also [vdB06]), where $\mathcal{C}(\mathrm{emTT}_{set})$ is the syntactic category of emTT$_{set}$ defined as in [Mai05].

However, if we take the *set-theoretic coherent fragment* cemTT of emTT$_{set}$, then we expect cemTT to be an *internal language of the quotient model* built out of *the corresponding set-theoretic coherent fragment* cmTT *of* mTT$_{set}$. The coherent fragment cemTT is obtained from the set-theoretic fragment emTT$_{set}$ by cutting out implication, universal quantification and dependent product sets. Also the fragment cmTT is obtained from mTT$_{set}$ in an analogous way. A proof of this is left to future work.

Remark 4.42 Connection with the exact completion of a weakly lex category. The construction of total setoids on mTT corresponds categorically to an instance of a generalization of the exact completion construction [CV98, CC82] of a weakly lex category.

The connection with the construction of the exact completion in [CV98, CC82] is clearer if we build our quotient model over Martin-Löf's type theory MLTT in [NPS90]. Then, such a quotient model built over MLTT, always with total setoids as in definition 4.1 and called Q(MLTT), happens to be equivalent to the exact completion construction in [CV98, CC82] performed on the weakly lex syntactic category, called C(MLTT), associated with MLTT as in remark 4.23. In particular, Q(MLTT) turns out to be a *list-arithmetic locally cartesian closed pretopos.*

Now, it is important to note that the quotient model Q(MLTT) does not seem to validate well-behaved quotients, if we identify *propositions as sets* as done in MLTT. Indeed, under this identification Q(MLTT), *but also* Q(mTT), supports first-order extensional Martin-Löf's type theory in [Mar84] and hence it *validates the axiom of choice*, as a consequence of the fact that universal and existential quantifiers are identified with dependent products Π and strong indexed sums Σ respectively. Then, effectiveness of quotients, being generally incompatible with the axiom of choice (see [Mai99]), does not seem to be validated.

To gain well-behaved quotients in Q(MLTT), one possibility is to reason by identifying *propositions with mono collections*, as well as *small propositions with mono sets* like in the logic of a pretopos (see [Mai05]). Instead, in Q(mTT) we get them by identifying propositions only with some mono collections, as well as small propositions only with some mono sets. Categorically this means that Q(mTT) is closed under effective quotients of categorical equivalence relations obtained via the properties of the comprehension adjunction as $\mathcal{P}_q$-equivalence relations. In fact, even if Q(mTT) supports quotients of all the monic equivalence relations, these do not seem to enjoy effectiveness. Always categorically speaking, this means that Q(mTT) does not seem to be a pretopos, even if it has quotients for all monic equivalence relations. Indeed, we are not able to prove that all the monic equivalence relations are in bijection with $\mathcal{P}_q$-equivalence relations, for which effective quotients exist (which explains why we introduced the concept of $\mathcal{P}_q$-equivalence relation!).

Finally note that effective quotients in Q(mTT) and Q(mTT)$_{set}$, and also in Q(mTT^{dp}), are enough to make these models regular (see [Joh02a] for the categorical definition): indeed one can define the image of $f : (A, =_A) \to (B, =_B)$ as the quotient of $(A, =_A)$ over its kernel, namely as $(\, A,\ f(x) =_B f(y)\,)$, after noticing that monic arrows are indeed injective.

Remark 4.43 In order to interpret quotients in mTT we also considered to mimic the exact completion on a regular category in [CV98, Hyl82]. But we ended up just in a list-arithmetic pretopos, for example not necessarily closed under dependent products. Indeed, given that in this completion arrows are identified with functional relations, if the axiom of choice is not generally valid as it happens in mTT, in order to define exponentials and dependent products we would need to use an impredicative quantification on relations that is not allowed in mTT for its predicativity.

4.2 The axiom of choice is not valid in emTT

The axiom of choice is not derivable in emTT. This is not surprising, if we consider that it may be incompatible with effective quotients (see [Mai99]). To show this fact, just observe that the *propositional axiom of choice* written in emTT as follows

$$(AC) \qquad \forall x \in A\ \exists y \in B\ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\ \ \forall x \in A\ R(x,\, \mathsf{Ap}(f,x)\,)$$

is exactly interpreted in Q(MLTT), and also in Q(mTT), as:

$$\forall x \in A^I\ \exists y \in B^I\ R^I(x,y) \quad \longrightarrow \quad \exists f \in (A \to B)^I\ \ \forall x \in A^I\ \ R^I(x, \mathsf{Ap}(f,x))$$

where we recall that

$$(A \to B)^I \equiv \Sigma_{h \in A^I \to B^I} \quad \forall_{x_1 \in A^I}\ \forall_{x_2 \in A^I}\ \ x_1 =_{A^I} x_2\ \to \mathsf{Ap}(h,x_1) =_{B^I} \mathsf{Ap}(h,x_2)$$

This interpretation in Q(mTT) amounts to be exactly equivalent to the *extensional axiom of choice* in [ML06, Car04] given that R^I satisfies the conditions:

- If $a =_{A^I} a'$ then $\forall_{z \in B^I}\ R^I(a,z) \to R^I(a',z)$ holds.
- If $b =_{B^I} b'$ then $\forall_{z \in A^I}\ R^I(z,b) \to R^I(z,b')$ holds.

Therefore, the arguments in [ML06, Car04] exactly show that the propositional axiom of choice *fails* to be valid in the quotient models Q(MLTT), and even more in Q(mTT) or in an Heyting pretopos, and hence also in emTT. Indeed, we can prove that the validity of the axiom of choice in emTT yields that all propositions are decidable as shown in [ML06, Car04, MV99], whose proof goes back to Goodman-Myhill's one in [GM78]. To prove this, we use a choice property valid for effective quotients thanks to the fact that propositions are mono:

Lemma 4.44 *In* emTT *for any quotient set A/R set* $[\Gamma]$ *we can derive a proof of*

$$\forall_{z\in A/R}\ \exists_{y\in A}\ [y] =_{A/R} z$$

Proof. Given $z \in A/R$, by elimination of quotient sets we get $\mathsf{El}_{\mathcal{Q}}(z, (x).\mathsf{true}) \in \exists_{y\in A}\ [y] =_{A/R} z$ because for $x \in A$ then $\exists_{y\in A}\ [y] =_{A/R} [x]$ holds by reflexivity of Propositional Equality by taking y as x and it is well defined since propositions are mono.

Proposition 4.45 *In* emTT *the validity of the axiom of choice*

$$(AC) \qquad \forall x \in A\ \exists y \in B\ R(x, y) \quad \longrightarrow \quad \exists f \in A \to B\ \forall x \in A\ R(x, \mathsf{Ap}(f, x))$$

on all emTT *sets A and B and small relation $R(x, y)$ $prop_s$ $[x \in A, y \in B]$ implies that all small propositions are decidable.*

Proof. We follow the proof in [Car04]. Let us define the following equivalence relation on the boolean set $Bool \equiv \mathsf{N}_1 + \mathsf{N}_1$ whose elements are called $\mathsf{true} \equiv \mathsf{inl}(\star)$ and $\mathsf{false} \equiv \mathsf{inr}(\star)$: given any proposition P we put

$$R(a, b) \equiv a =_{Bool} b \vee P$$

Then, thanks to lemma 4.44 in emTT we can derive a proof of

$$\forall_{z\in Bool/R}\ \exists_{y\in Bool}\ z =_{Bool/R} [y]$$

and by the validity of the axiom of choice we get a proof of

$$\exists f \in Bool/R \to Bool\ \ \forall z \in Bool/R\ \ z =_{Bool/R} [\mathsf{Ap}(f, z)]$$

that amounts to being an injective arrow by definition. Then, given that the equality in *Bool* is decidable (which follows by sum disjointness), we get that the equality in $Bool/R$ is decidable, too. Hence, $[\mathsf{true}] =_{Bool/R} [\mathsf{false}] \vee \neg[\mathsf{true}] =_{Bool/R} [\mathsf{false}]$ is derivable. Then, by effectiveness also $R(\mathsf{true}, \mathsf{false}) \vee \neg R(\mathsf{true}, \mathsf{false})$ is derivable, too. Now, given that $R(\mathsf{true}, \mathsf{false}) \leftrightarrow P$, then $P \vee \neg P$ is derivable, too, namely P is decidable. The logic of small propositions is then classical.

Note that a similar argument holds also at the level of propositions. Moreover, about pretopoi we can deduce the following:

Corollary 4.46 *A locally cartesian closed pretopos enjoying the validity of the propositional axiom of choice is boolean.*

Proof. The internal logic of a locally cartesian closed pretopos devised in [Mai05] is an extension of the set-theoretic fragment of emTT and validates lemma 4.44.

In Q(MLTT) the propositional axiom of choice survives at least for those quotients whose equivalence relation is the propositional equality of MLTT. Only the validity of the axiom of unique choice continues to hold in its generality in Q(MLTT) (see also [ML06]).

4.3 What links the two levels?

A question we need to address in forming a two-level foundation is to decide what mathematical link should tie the two levels. In [MS05] we said that the extensional level must be obtained from the intensional one by following Sambin's forget-restore principle expressed in [SV98].

Our example of two-level foundation fully satisfies such a principle. In particular, the interpretation of emTT into the quotient model over mTT makes visible the validity of such a principle. Indeed, in emTT we work with undecidable judgements, while those of mTT are decidable. The interpretation of emTT into mTT restores the forgotten information of emTT undecidable judgements by transforming them into decidable ones, once the lost information has been recovered. For example, the emTT-judgement $B = C\ set\ [\Gamma]$ is interpreted as the existence of a canonical isomorphism

$$\begin{array}{ccc} \text{emTT} & & \text{mTT} \\ I(\, B = C\ set\ [\Gamma]\,) \equiv & & \text{there exists a canonical isomorphism with components} \\ & & \tau^{C^I}_{B^I}(y) \in C^I_{=}\ [\Gamma^I_{=}, y \in B^I_{=}] \qquad \text{and} \qquad \tau^{B^I}_{C^I}(z) \in B^I_{=}\ [\Gamma^I_{=}, z \in C^I_{=}] \end{array}$$

This says that in order to interpret emTT-type equalities into mTT we need to restore the missing canonical isomorphisms and hence to prove suitable decidable mTT judgements.
Another example, already studied in [Mar84, Mar85, Val95], is the interpretation of the validity of a proposition $A\ prop\ [\Gamma]$, expressed by the emTT judgement $\mathsf{true} \in A$, as the existence of a proof-term:

$$\begin{array}{cc} \text{emTT} & \text{mTT} \\ I(\, \mathsf{true} \in A\,) \equiv & \text{there exists } p \in A^I \end{array}$$

In other words to interpret the validity of a proposition we need to restore a proof-term of its mTT-interpretation.

Such considerations reveal that the link between our two levels is *not* then *a merely interpretation* of the extensional level into the intensional one. Instead, the extensional level is designed over the intensional one *only by forgetting information about equality between types and terms* that can be *restored*, and hence it is implemented *only via a quotient construction performed on the intensional level.*

Therefore, this kind of two-level foundation rules out examples of two-level foundations where the extensional level is governed by a classical logic that is interpreted in the intensional one, like mTT, via a double-negation interpretation.

5 Future work

Given that emTT is not at all the internal language of our quotient model Q(mTT), we would like to find out whether such an internal language exists in terms of an extensional dependent type theory. In particular, it would be useful to find the internal language of the quotient model Q(MLTT) built over Martin-Löf's type theory in an analogous way to Q(mTT). An application of this would be to extend emTT with occurrences of the axiom of choice that are constructively admissible. Indeed, whilst the propositional axiom of choice is generally constructively incompatible with emTT, as recalled in section 4.2, there are extensional sets on which we can apply the axiom of choice without losing constructivity, as advocated by Bishop in [Bis67]. For example, in the quotient model Q(MLTT) over Martin-Löf's type theory the axiom of choice is valid on those extensional sets whose equivalence relation is the identity relation, including, for example, the extensional set of natural numbers. These extensional sets are actually copies of intensional sets at the extensional level.

We think that knowing the internal language of Q(MLTT) in terms of an extensional dependent type theory à la Martin-Löf would help to characterize such intensional sets in emTT without stating their existence in a purely axiomatic way in the style of Aczel's Presentation axiom in [AR].

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6 Appendix: The typed calculus mTT

We present here the inference rules to build types in mTT. The inference rules involve judgements written in the style of Martin-Löf's type theory [Mar84, NPS90] that may be of the form:

$$A\ type\ [\Gamma] \qquad A = B\ type\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad a = b \in A\ [\Gamma]$$

where types include collections, sets, propositions and small propositions, namely

$$type \in \{col, set, prop, prop_s\}$$

For easiness, the piece of context common to all judgements involved in a rule is omitted and typed variables appearing in a context are meant to be added to the implicit context as the last one.
Note that to write the elimination constructors of our types we adopt the higher-order syntax in [NPS90] [9].
We also have a form of judgement to build contexts:

$$\Gamma\ cont$$

whose rules are the following

$$\emptyset\ cont \qquad\qquad \text{F-c}\ \frac{A\ type\ [\Gamma]}{\Gamma, x \in A\ cont}\ (x \in A \notin \Gamma)$$

Then, the first rule to build elements of type is the assumption of variables:

$$\text{var)}\ \frac{\Gamma, x \in A, \Delta \quad cont}{x \in A\ [\Gamma, x \in A, \Delta]}$$

Among types there are the following embeddings: sets are collections and propositions are collections

$$\textbf{set-into-col)}\ \frac{A\ set}{A\ col} \qquad\qquad \textbf{prop-into-col)}\ \frac{A\ prop}{A\ col}$$

Moreover, collections are closed under strong indexed sums:

Strong Indexed Sum

$$\text{F-}\Sigma)\ \frac{C(x)\ \ col\ [x \in B]}{\Sigma_{x\in B} C(x)\ col} \qquad \text{I-}\Sigma)\ \frac{b \in B \quad c \in C(b) \qquad C(x)\ col\ [x \in B]}{\langle b, c\rangle \in \Sigma_{x\in B} C(x)}$$

$$\text{E-}\Sigma)\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ d \in \Sigma_{x\in B} C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) \in M(d)}$$

$$\text{C-}\Sigma)\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(\ \langle b, c\rangle, m\ ) = m(b, c) \in M(\langle b, c\rangle)}$$

[9] For example, note that the elimination constructor of disjunction $El_{\vee}(w, a_B, a_C)$ binds the open terms $a_B(x) \in A\ [x \in B]$ and $a_C(y) \in A\ [y \in C]$. Indeed, given that they are needed in the disjunction conversion rules, it follows that these open terms must be encoded into the elimination constructor. To encode them we use the higher-order syntax as in [NPS90] (see also [Gui]). According to this syntax the open term $a_B(x) \in A\ [x \in B]$ yields to $(x \in B)\, a_B(x)$ of higher type $(x \in B)\, A$. Then, by η-conversion among higher types, it follows that $(x \in B)\, a_B(x)$ is equal to a_B. Hence, we often simply write the short expression a_B to recall the open term where it comes from.

Sets are generated as follows:

Empty set

F-Em) $\mathsf{N}_0\ set$ E-Em) $\dfrac{a \in \mathsf{N}_0 \quad A(x)\ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$

Singleton

S) $\mathsf{N}_1\ set$ I-S) $\star \in \mathsf{N}_1$ E-S) $\dfrac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$ C-S) $\dfrac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$

Strong Indexed Sum set

F-Σ_s) $\dfrac{C(x)\ set\ \ [x \in B] \qquad B\ set}{\Sigma_{x \in B} C(x)\ set}$

List set

F-list) $\dfrac{C\ set}{List(C)\ set}$ I_1-list) $\dfrac{List(C)\ set}{\epsilon \in List(C)}$ I_2-list) $\dfrac{s \in List(C) \quad c \in C}{\mathsf{cons}(s, c) \in List(C)}$

E-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) \in L(s)}$

C_1-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\epsilon, a, l) = a \in L(\epsilon)}$

C_2-list) $\dfrac{\begin{array}{c} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\mathsf{cons}(s, c), a, l) = l(s, c, El_{List}(s, a, l)) \in L(\mathsf{cons}(s, c))}$

Disjoint Sum set

F-+) $\dfrac{B\ set \quad C\ set}{B + C\ set}$ I_1-+) $\dfrac{b \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) \in B + C}$ I_2-+) $\dfrac{c \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) \in B + C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ w \in B + C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(w, a_B, a_C) \in A(w)}$

C_1-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$

C_2-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$

Dependent Product set

F-Π) $\dfrac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x \in B} C(x)\ set}$ I-Π) $\dfrac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in \Pi_{x \in B} C(x)}$

E-Π) $\dfrac{b \in B \quad f \in \Pi_{x \in B} C(x)}{\mathsf{Ap}(f, b) \in C(b)}$

βC-Π) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$

Propositions are generated as follows:

Falsum

F-Fs) $\perp\ prop$ E-Fs) $\dfrac{a \in \perp \quad A\ prop}{\mathsf{r_o}(a) \in A}$

Disjunction

F-∨) $\dfrac{B\ prop \quad C\ prop}{B \vee C\ prop}$ $\mathrm{I_1}$-∨) $\dfrac{b \in B \qquad B\ prop \qquad C\ prop}{\mathsf{inl}_\vee(b) \in B \vee C}$ $\mathrm{I_2}$-∨) $\dfrac{c \in C \qquad B\ prop \qquad C\ prop}{\mathsf{inr}_\vee(c) \in B \vee C}$

E-∨) $\dfrac{\begin{array}{l} A\ prop \\ w \in B \vee C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(w, a_B, a_C) \in A}$

$\mathrm{C_1}$-∨) $\dfrac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ b \in B \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(\mathsf{inl}_\vee(b), a_B, a_C) = a_B(b) \in A}$

$\mathrm{C_2}$-∨) $\dfrac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ c \in C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_\vee(\mathsf{inr}_\vee(c), a_B, a_C) = a_C(c) \in A}$

Conjunction

F-∧) $\dfrac{B\ prop \qquad C\ prop}{B \wedge C\ prop}$ I-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\langle b,_\wedge c \rangle \in B \wedge C}$

$\mathrm{E_1}$-∧) $\dfrac{d \in B \wedge C}{\pi_1^B(d) \in B}$ $\mathrm{E_2}$-∧) $\dfrac{d \in B \wedge C}{\pi_2^C(d) \in C}$

β_1 C-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_1^B(\langle b,_\wedge c \rangle) = b \in B}$ β_2 C-∧) $\dfrac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_2^C(\langle b,_\wedge c \rangle) = c \in C}$

Implication

F-→) $\dfrac{B\ prop \qquad C\ prop}{B \rightarrow C\ prop}$

I-→) $\dfrac{c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\lambda_\rightarrow x^B.c(x) \in B \rightarrow C}$ E-→) $\dfrac{b \in B \quad f \in B \rightarrow C}{\mathsf{Ap}_\rightarrow(f, b) \in C}$

βC-→) $\dfrac{b \in B \quad c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\mathsf{Ap}_\rightarrow(\lambda_\rightarrow x^B.c(x), b) = c(b) \in C}$

Existential quantification

F-∃) $\dfrac{C(x)\ prop\ \ [x \in B]}{\exists_{x\in B} C(x)\ prop}$ I-∃) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ [x \in B]}{\langle b,_{\exists} c\rangle \in \exists_{x\in B} C(x)}$

E-∃) $\dfrac{\begin{array}{l} M\ prop \\ d \in \exists_{x\in B} C(x) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(d,m) \in M}$

C-∃) $\dfrac{\begin{array}{l} M\ prop \qquad C(x)\ prop\ [x \in B] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(\langle b,_{\exists} c\rangle, m) = m(b,c) \in M}$

Universal quantification

F-∀) $\dfrac{C(x)\ prop\ [x \in B]}{\forall_{x\in B} C(x)\ prop}$ I-∀) $\dfrac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\lambda_{\forall} x^{B}.c(x) \in \forall_{x\in B} C(x)}$

E-∀) $\dfrac{b \in B \quad f \in \forall_{x\in B} C(x)}{\mathsf{Ap}_{\forall}(f,b) \in C(b)}$ βC-∀) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\mathsf{Ap}_{\forall}(\lambda_{\forall} x^{B}.c(x), b) = c(b) \in C(b)}$

Propositional Equality

F-Id) $\dfrac{A\ col \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop}$ I-Id) $\dfrac{a \in A}{\mathsf{id_A}(a) \in \mathsf{Id}(A,a,a)}$

E-Id) $\dfrac{\begin{array}{l} C(x,y)\ prop\ [x \in A, y \in A] \\ a \in A \quad b \in A \quad p \in \mathsf{Id}(A,a,b) \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(p,c) \in C(a,b)}$

C-Id) $\dfrac{\begin{array}{l} C(x,y)\ prop\ [x \in A, y \in A] \\ a \in A \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(\mathsf{id}_A(a), c) = c(a) \in C(a,a)}$

Then, small propositions are generated as follows:

$\bot\ prop_s$ $\dfrac{B\ prop_s \quad C\ prop_s}{B \vee C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \to C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \wedge C\ prop_s}$

$\dfrac{C(x)\ prop_s\ \ [x \in B] \qquad B\ set}{\exists_{x\in B} C(x)\ prop_s}$ $\dfrac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\forall_{x\in B} C(x)\ prop_s}$ $\dfrac{A\ set \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop_s}$

And we add rules saying that a small proposition is a proposition and that a small proposition is a set:

prop$_s$-into-prop) $\dfrac{A\ prop_s}{A\ prop}$ **prop$_s$-into-set)** $\dfrac{A\ prop_s}{A\ set}$

Then, we also have the collection of small propositions and function collections from a set toward it:

Collection of small propositions

F-Pr) $\mathsf{prop_s}\ col$ I-Pr) $\dfrac{B\ prop_s}{B \in \mathsf{prop_s}}$ E-Pr) $\dfrac{B \in \mathsf{prop_s}}{B\ prop_s}$

Function collection to $\mathsf{props_s}$

$$\text{F-Fun)}\ \frac{B\ set}{B \to \mathsf{props_s}\ col} \qquad \text{I-Fun)}\ \frac{c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathsf{props_s}}$$

$$\text{E-Fun)}\ \frac{b \in B \quad f \in B \to \mathsf{props_s}}{\mathsf{Ap}(f,b) \in \mathsf{props_s}} \qquad \beta\text{C-Fun)}\ \frac{b \in B \quad c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathsf{props_s}}$$

Equality rules include those saying that type equality is an equivalence relation and substitution of equal terms in a type:

$$\text{ref)}\ \frac{A\ type}{A = A\ type} \qquad \text{sym)}\ \frac{A = B\ type}{B = A\ type} \qquad \text{tra)}\ \frac{A = B\ type \quad B = C\ type}{A = C\ type}$$

$$\text{subT)}\ \frac{\begin{array}{l} C(x_1, \ldots, x_n)\ type\ [\,x_1 \in A_1,\ \ldots,\ x_n \in A_n(x_1, \ldots, x_{n-1})\,] \\ a_1 = b_1 \in A_1\ \ \ldots\ \ a_n = b_n \in A_n(a_1, \ldots, a_{n-1}) \end{array}}{C(a_1, \ldots, a_n) = C(b_1, \ldots, b_n)\ type}$$

where $type \in \{col, set, prop, props_s\}$ with the same choice both in the premise and in the conclusion.

For terms into sets we add the following equality rules:

$$\text{ref)}\ \frac{a \in A}{a = a \in A} \qquad \text{sym)}\ \frac{a = b \in A}{b = a \in A} \qquad \text{tra)}\ \frac{a = b \in A \qquad b = c \in A}{a = c \in A}$$

$$\text{sub)}\ \frac{\begin{array}{l} c(x_1, \ldots, x_n) \in C(x_1, \ldots, x_n)\ \ [\,x_1 \in A_1,\ \ldots,\ x_n \in A_n(x_1, \ldots, x_{n-1})\,] \\ a_1 = b_1 \in A_1\ \ \ldots\ \ a_n = b_n \in A_n(a_1, \ldots, a_{n-1}) \end{array}}{c(a_1, \ldots, a_n) = c(b_1, \ldots, b_n) \in C(a_1, \ldots, a_n)}$$

$$\text{conv)}\ \frac{a \in A \qquad A = B\ type}{a \in B} \qquad \text{conv-eq)}\ \frac{a = b \in A \qquad A = B\ type}{a = b \in B}$$

We call mTT_{set} the fragment of mTT consisting only of judgements forming sets, small propositions with their elements and corresponding equalities.

Finally, the calculus mTT^{dp} is obtained by extending mTT with generic dependent collections:

Dependent Product collection

$$\text{F-}\Pi_c)\ \frac{C(x)\ col\ [x \in B]}{\Pi_{x \in B} C(x)\ col} \qquad \text{I-}\Pi_c)\ \frac{c(x) \in C(x)\ [x \in B]}{\lambda x^B.c(x) \in \Pi_{x \in B} C(x)}$$

$$\text{E-}\Pi_c)\ \frac{b \in B \quad f \in \Pi_{x \in B} C(x)}{\mathsf{Ap}(f,b) \in C(b)} \qquad \beta\text{C-}\Pi_c)\ \frac{b \in B \quad c(x) \in C(x)\ [x \in B]}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$$

Note that in mTT^{dp} function collections toward $\mathsf{props_s}$ are clearly a special instance of dependent product collections.

7 Appendix: The typed calculus emTT

Here we present the calculus emTT. To build its types and terms we use the same kinds of judgements used in mTT, namely

$$A\ type\ [\Gamma] \qquad A = B\ type\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad a = b \in A\ [\Gamma]$$

where types include collections, sets, propositions and small propositions: namely

$$type \in \{col, set, prop, props_s\}$$

Contexts are generated by the same context rules of mTT. Also here note that the piece of context common to all judgements involved in a rule is omitted and that typed variables appearing in a context are meant to be added to the implicit context as the last one.
Among types there are the following embeddings: sets are collections and propositions are collections

set-into-col) $\dfrac{A\ set}{A\ col}$ **prop-into-col)** $\dfrac{A\ prop}{A\ col}$

Collections are closed under strong indexed sums:

Strong Indexed Sum

F-Σ) $\dfrac{C(x)\ \ col\ [x \in B]}{\Sigma_{x\in B}C(x)\ col}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ col\ [x \in B]}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$

E-Σ) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ d \in \Sigma_{x\in B}C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(d, m) \in M(d)}$

C-Σ) $\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_\Sigma(\langle b, c\rangle, m) = m(b, c) \in M(\langle b, c\rangle)}$

Sets are generated as follows:

Empty set

F-Em) $\mathsf{N}_0\ set$ E-Em) $\dfrac{a \in \mathsf{N}_0 \quad A(x)\ \ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$

Singleton set

S) $\mathsf{N}_1\ set$ I-S) $\star \in \mathsf{N}_1$ E-S) $\dfrac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$ C-S) $\dfrac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$

Strong Indexed Sum set

F-Σ_s) $\dfrac{C(x)\ set\ [x \in B] \qquad B\ set}{\Sigma_{x\in B}C(x)\ set}$

List set

F-list) $\dfrac{C\ set}{List(C)\ set}$ I_1-list) $\dfrac{List(C)\ set}{\epsilon \in List(C)}$ I_2-list) $\dfrac{s \in List(C) \quad c \in C}{\mathsf{cons}(s, c) \in List(C)}$

E-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) \in L(s)}$

C_1-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\epsilon, a, l) = a \in L(\epsilon)}$

C_2-list) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\mathsf{cons}(s, c), a, l) = l(s, c, El_{List}(s, a, l)) \in L(\mathsf{cons}(s, c))}$

Disjoint Sum set

F-+) $\dfrac{B\ set \quad C\ set}{B+C\ set}$ I$_1$-+) $\dfrac{b \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) \in B+C}$ I$_2$-+) $\dfrac{c \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) \in B+C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B+C] \\ w \in B+C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(w, a_B, a_C) \in A(w)}$

C$_1$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B+C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$

C$_2$-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B+C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$

Dependent Product set

F-Π) $\dfrac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x \in B} C(x)\ set}$ I-Π) $\dfrac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in \Pi_{x \in B} C(x)}$

E-Π) $\dfrac{b \in B \quad f \in \Pi_{x \in B} C(x)}{\mathsf{Ap}(f, b) \in C(b)}$

βC-Π) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$

ηC-Π $\dfrac{f \in \Pi_{x \in B} C(x)}{\lambda x^B.\mathsf{Ap}(f, x) = f \in \Pi_{x \in B} C(x)} (x\ not\ free\ in\ f)$

Quotient set

Q) $\dfrac{\begin{array}{l} A\ set \quad R(x,y)\ prop_s\ [x \in A, y \in A] \\ \qquad \mathsf{true} \in R(x,x)\ [x \in A] \\ \qquad \mathsf{true} \in R(y,x)\ [x \in A,\ y \in A,\ u \in R(x,y)] \\ \qquad \mathsf{true} \in R(x,z)\ [x \in A,\ y \in A,\ z \in A, \\ \qquad\qquad\qquad\qquad u \in R(x,y),\ v \in R(y,z)] \end{array}}{A/R\ set}$

I-Q) $\dfrac{a \in A \qquad A/R\ set}{[a] \in A/R}$ eq-Q) $\dfrac{a \in A \quad b \in A \quad \mathsf{true} \in R(a,b) \qquad A/R\ set}{[a] = [b] \in A/R}$

E-Q) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p \in A/R \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(p, l) \in L(p)}$

C-Q) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ a \in A \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q([a], l) = l(a) \in L([a])}$

Effectiveness

eff) $\dfrac{a \in A \quad b \in A \quad [a] = [b] \in A/R \qquad A/R\ set}{\mathsf{true} \in R(a,b)}$

emTT propositions are mono, namely they are inhabited by at most a canonical proof-term:

prop-mono) $\dfrac{A\ prop \qquad p \in A \quad q \in A}{p = q \in A}$ **prop-true**) $\dfrac{A\ prop \qquad p \in A}{\mathsf{true} \in A}$

Propositions are generated as follows:

Falsum

F-Fs) $\perp\ prop$ E-Fs) $\dfrac{\mathsf{true} \in \perp \quad A\ prop}{\mathsf{true} \in A}$

Extensional Propositional Equality

F-Eq) $\dfrac{C\ col \quad c \in C \quad d \in C}{\mathsf{Eq}(C, c, d)\ prop}$ I-Eq) $\dfrac{c \in C}{\mathsf{true} \in \mathsf{Eq}(C, c, c)}$ E-Eq) $\dfrac{\mathsf{true} \in \mathsf{Eq}(C, c, d)}{c = d \in C}$

Implication

F-Im) $\dfrac{B\ prop \quad C\ prop}{B \to C\ prop}$ I-Im) $\dfrac{\mathsf{true} \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \to C}$

E-Im) $\dfrac{\mathsf{true} \in B \quad \mathsf{true} \in B \to C}{\mathsf{true} \in C}$

Conjunction

F-$\wedge$) $\dfrac{B\ prop \quad C\ prop}{B \wedge C\ prop}$ I-$\wedge$) $\dfrac{\mathsf{true} \in B \quad \mathsf{true} \in C \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \wedge C}$

E_1-$\wedge$) $\dfrac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in B}$ E_2-$\wedge$) $\dfrac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in C}$

Disjunction

F-$\vee$) $\dfrac{B\ prop \quad C\ prop}{B \vee C\ prop}$ I_1-$\vee$) $\dfrac{\mathsf{true} \in B \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \vee C}$ I_2-$\vee$) $\dfrac{\mathsf{true} \in C \qquad B\ prop \qquad C\ prop}{\mathsf{true} \in B \vee C}$

E-$\vee$) $\dfrac{A\ prop \qquad \mathsf{true} \in B \vee C \quad \mathsf{true} \in A\ [x \in B] \quad \mathsf{true} \in A\ [y \in C]}{\mathsf{true} \in A}$

Existential quantification

F-$\exists$) $\dfrac{C(x)\ prop\ \ [x \in B]}{\exists_{x \in B} C(x)\ prop}$ I-$\exists$) $\dfrac{b \in B \quad \mathsf{true} \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\mathsf{true} \in \exists_{x \in B} C(x)}$

E-$\exists$) $\dfrac{M\ prop \qquad \mathsf{true} \in \exists_{x \in B} C(x) \quad \mathsf{true} \in M\ [x \in B, y \in C(x)]}{\mathsf{true} \in M}$

Universal quantification

F-$\forall$) $\dfrac{C(x)\ prop\ [x \in B]}{\forall_{x \in B} C(x)\ prop}$ I-$\forall$) $\dfrac{\mathsf{true} \in C(x)\ [x \in B] \qquad C(x)\ prop\ \ [x \in B]}{\mathsf{true} \in \forall_{x \in B} C(x)}$

E-$\forall$) $\dfrac{b \in B \quad \mathsf{true} \in \forall_{x \in B} C(x)}{\mathsf{true} \in C(b)}$

As in mTT, small propositions are generated as follows:

$\perp\ prop_s$ $\dfrac{B\ prop_s \quad C\ prop_s}{B \vee C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \to C\ prop_s}$ $\dfrac{B\ prop_s \qquad C\ prop_s}{B \wedge C\ prop_s}$

$\dfrac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\exists_{x \in B} C(x) \in prop_s}$ $\dfrac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\forall_{x \in B} C(x)\ prop_s}$ $\dfrac{A\ set \quad a \in A \quad b \in A}{\mathsf{Eq}(A, a, b)\ prop_s}$

And we add rules saying that a small proposition is a proposition and that a small proposition is a set:

prop$_s$-into-prop) $\dfrac{A\ prop_s}{A\ prop}$ **prop$_s$-into-set)** $\dfrac{A\ prop_s}{A\ set}$

Contrary to mTT, in emTT we do not have the intensional collection of small propositions but the quotient of the collection of small propositions under equiprovability representing the power collection of the singleton:

Power collection of the singleton

F-P) $\mathcal{P}(1)\ col$ I-P) $\dfrac{B\ prop_s}{[B] \in \mathcal{P}(1)}$ eq-P) $\dfrac{\mathsf{true} \in B \leftrightarrow C}{[B] = [C] \in \mathcal{P}(1)}$ eff-P) $\dfrac{[B] = [C] \in \mathcal{P}(1)}{\mathsf{true} \in B \leftrightarrow C}$

$\dfrac{U \in \mathcal{P}(1) \qquad V \in \mathcal{P}(1)}{\mathsf{Eq}(\mathcal{P}(1), U, V)\ prop_s}$ η-P) $\dfrac{U \in \mathcal{P}(1)}{U = [\,\mathsf{Eq}(\mathcal{P}(1), U, [\mathsf{tt}]\,)\,]}$

where $\mathsf{tt} \equiv \perp \to \perp$ represents the truth constant.

Then, we have also function collections from a set toward $\mathcal{P}(1)$:

Function collection to $\mathcal{P}(1)$

F-Fc) $\dfrac{B\ set}{B \to \mathcal{P}(1)\ col}$ I-Fc) $\dfrac{c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathcal{P}(1)}$

E-Fc) $\dfrac{b \in B \quad f \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f, b) \in \mathcal{P}(1)}$ βC-Fc) $\dfrac{b \in B \quad c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathcal{P}(1)}$

ηC-Fc) $\dfrac{f \in B \to \mathcal{P}(1)}{\lambda x^B.\mathsf{Ap}(f, x) = f \in B \to \mathcal{P}(1)}(x\ not\ free\ in\ f)$

Then, as in mTT we add the embedding rules of sets into collections **set-into-col**, of propositions into collections **prop-into-col**, of small propositions into sets **prop$_s$-into-set** and of small propositions into propositions **prop$_s$-into-prop**.
Moreover, we also add the equality rules ref), sym), tra) both for types and for terms saying that type and term equalities are equivalence relations, and the rules conv), conv-eq).
Contrary to mTT, we add all the equality rules about collections and sets saying that their constructors preserve type equality as follows:

Strong Indexed Sum-eq

eq-Σ) $\dfrac{C(x) = D(x)\ col\ \ [x \in B] \qquad B = E\ col}{\Sigma_{x \in B} C(x) = \Sigma_{x \in E} D(x)\ col}$

Function collection-eq

eq-Fc) $\dfrac{B = E\ set}{B \to \mathcal{P}(1) = E \to \mathcal{P}(1)\ col}$

Lists-eq

eq-list) $\dfrac{C = D\ set}{List(C) = List(D)\ set}$

Strong Indexed Sum set-eq

eq-Σ_s) $\dfrac{C(x) = D(x)\ set\ \ [x \in B] \qquad B = E\ set}{\Sigma_{x \in B} C(x) = \Sigma_{x \in E} D(x)\ set}$

Disjoint Sum-eq

eq-+) $\dfrac{B = D\ set \quad C = E\ set}{B + C = D + E\ set}$

Dependent Product-eq

eq-Π) $\dfrac{C(x) = D(x)\ set\ [x \in B] \qquad B = E\ set}{\Pi_{x \in B} C(x) = \Pi_{x \in E} D(x)\ set}$

Quotient set-eq

eq-Q) $\dfrac{A = B\ set \qquad R(x, y) = S(x, y)\ \ prop_s\ [x \in A, y \in A] \qquad \mathsf{Equiv}(R) \qquad \mathsf{Equiv}(S)}{A/R = B/S\ \ set}$

Then, emTT includes the following equality rules about propositions:

Disjunction-eq

$$\text{eq-}\vee)\ \frac{B = D\ prop \quad C = E\ prop}{B \vee C = D \vee E\ prop}$$

Implication-eq

$$\text{eq-}\rightarrow)\ \frac{B = D\ prop \qquad C = E\ prop}{B \rightarrow C = D \rightarrow E\ prop}$$

Conjunction-eq

$$\text{eq-}\wedge)\ \frac{B = D\ prop \qquad C = E\ prop}{B \wedge C = D \wedge E\ prop}$$

Propositional equality-eq

$$\text{eq-Eq})\ \frac{A = E\ col \quad a = e \in A \quad b = c \in A}{\mathsf{Eq}(A, a, b) = \mathsf{Eq}(E, e, c)\ prop}$$

Existential quantification-eq

$$\text{eq-}\exists)\ \frac{C(x) = D(x)\ prop\ \ [x \in B] \qquad B = E\ col}{\exists_{x \in B} C(x) = \exists_{x \in E} D(x)\ prop}$$

Universal quantification-eq

$$\text{eq-}\forall)\ \frac{C(x) = D(x)\ prop\ [x \in B] \qquad B = E\ col}{\forall_{x \in B} C(x) = \forall_{x \in E} D(x)\ prop}$$

Analogously, we add eq-$\vee$), eq-$\rightarrow$), eq-$\wedge$), eq-Eq), eq-$\exists$), eq-$\forall$) restricted to small propositions. Moreover, equality of propositions is that of collections, that of small propositions coincides with that of propositions and that of sets:

$$\textbf{prop-into-col eq)}\ \frac{A = B\ prop}{A = B\ col}$$

$$\textbf{prop}_s\textbf{-into-prop eq)}\ \frac{A = B\ prop_s}{A = B\ prop} \qquad \textbf{prop}_s\textbf{-into-set eq)}\ \frac{A = B\ prop_s}{A = B\ set}$$

Equality of sets is that of collections:

$$\textbf{set-into-col eq)}\ \frac{A = B\ set}{A = B\ col}$$

Contrary to mTT, also for terms we add equality rules saying that all the constructors preserve equality as in [NPS90]:

$$\text{I-eq }\Sigma)\ \frac{b = b' \in B \quad c = c' \in C(b) \qquad C(x)\ col\ [x \in B]}{\langle b, c \rangle = \langle b', c' \rangle \in \Sigma_{x \in B} C(x)}$$

$$\text{E-eq }\Sigma)\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ d = d' \in \Sigma_{x \in B} C(x) \quad m(x, y) = m'(x, y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) = El_{\Sigma}(d', m') \in M(d)}$$

$$\text{E-eq Em})\ \frac{a = a' \in \mathsf{N}_0 \quad A(x)\ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) = \mathsf{emp_o}(a') \in A(a)} \qquad \text{E-eq S})\ \frac{t = t' \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c = c' \in M(\star)}{El_{\mathsf{N}_1}(t, c) = El_{\mathsf{N}_1}(t', c') \in M(t)}$$

$$\text{I}_2\text{-eq list})\ \frac{s = s' \in List(C) \quad c = c' \in C}{\mathsf{cons}(s, c) = \mathsf{cons}(s', c') \in List(C)}$$

$$\text{E-eq list})\ \frac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s = s' \in List(C) \quad a = a' \in L(\epsilon) \\ l(x, y, z) = l'(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) = El_{List}(s', a', l') \in L(s)}$$

I-eq Q) $\dfrac{a = a' \in A \qquad A/R\ set}{[a] = [a'] \in A/R}$

E-eq Q) $\dfrac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p = p' \in A/R \quad l(x) = l'(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(p, l) = El_Q(p', l') \in L(p)}$

I$_1$-+) $\dfrac{b = b' \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) = \mathsf{inl}(b') \in B + C}$ I$_2$-+) $\dfrac{c = c' \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) = \mathsf{inr}(c') \in B + C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ d = d' \in B + C \quad a_B(x) = a'_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) = a'_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_+(d, a_B, a_C) = El_+(d', a'_B, a'_C) \in A(w)}$

I-eq Π) $\dfrac{c(x) = c'(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in \Pi_{x \in B} C(x)}$ E-eq Π) $\dfrac{b = b' \in B \quad f = f' \in \Pi_{x \in B} C(x)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in C(b)}$

I-eq Fc) $\dfrac{c(x) = c'(x) \in \mathcal{P}(1)\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in B \to \mathcal{P}(1)}$ E-eq Fc) $\dfrac{b = b' \in B \quad f = f' \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in \mathcal{P}(1)}$

Note that I-eq Π) is the so-called ξ-rule in [Mar75].

We call emTT$_{set}$ the fragment of emTT consisting only of judgements forming sets, small propositions with their elements and corresponding equalities.

Finally, the calculus emTT^{dp} is obtained by extending emTT with generic dependent collections and quotient collections:

Dependent Product Collection

F-Π$_c$) $\dfrac{C(x)\ col\ [x \in B]}{\Pi_{x \in B} C(x)\ col}$ I-Π$_c$) $\dfrac{c(x) \in C(x)\ [x \in B]}{\lambda x^B.c(x) \in \Pi_{x \in B} C(x)}$ E-Π$_c$) $\dfrac{b \in B \quad f \in \Pi_{x \in B} C(x)}{\mathsf{Ap}(f, b) \in C(b)}$

βC-Π$_c$) $\dfrac{b \in B \quad c(x) \in C(x)\ [x \in B]}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in C(b)}$ ηC-Π$_c$) $\dfrac{f \in \Pi_{x \in B} C(x)}{\lambda x^B.\mathsf{Ap}(f, x) = f \in \Pi_{x \in B} C(x)}(x\ not\ free\ in\ f)$

Quotient collection

$$\mathrm{Q}_c)\ \frac{R(x,y)\ \ prop\ [x\in A, y\in A]\quad \begin{array}{l}\mathsf{true}\in R(x,x)\ [x\in A]\\ \mathsf{true}\in R(y,x)\ [x\in A,\ y\in A,\ u\in R(x,y)]\\ \mathsf{true}\in R(x,z)\ [x\in A,\ y\in A,\ z\in A,\\ \qquad\qquad\qquad\qquad u\in R(x,y),\ v\in R(y,z)]\end{array}}{A/R\ \ col}$$

$$\text{I-Q}_c)\ \frac{a\in A \qquad A/R\ col}{[a]\in A/R}\qquad \text{eq-Q}_c)\ \frac{a\in A \quad b\in A \quad \mathsf{true}\in R(a,b) \qquad A/R\ col}{[a]=[b]\in A/R}$$

$$\text{E-Q}_c)\ \frac{\begin{array}{l}L(z)\ col\ [z\in A/R]\\ p\in A/R \quad l(x)\in L([x])\ [x\in A] \quad l(x)=l(y)\in L([x])\ [x\in A, y\in A, d\in R(x,y)]\end{array}}{El_Q(p,l)\in L(p)}$$

$$\text{C-Q}_c)\ \frac{\begin{array}{l}L(z)\ col\ [z\in A/R]\\ a\in A \quad l(x)\in L([x])\ [x\in A] \quad l(x)=l(y)\in L([x])\ [x\in A, y\in A, d\in R(x,y)]\end{array}}{El_Q([a],l)=l(a)\in L([a])}$$

Effectiveness

$$\text{eff}_c)\ \frac{a\in A \quad b\in A \quad [a]=[b]\in A/R \qquad A/R\ col}{\mathsf{true}\in R(a,b)}$$

Then, we also add the corresponding equality rules about dependent product collections and about quotient collections as eq-Π), I-eq Π), E-eq Π) eq-Q), I-eq Q), E-eq Q).

Note that in emTT^{dp} function collections toward $\mathcal{P}(1)$ are clearly a special instance of dependent product collections.

8 Appendix: Interpretation of emTT into mTT

Here we define the interpretation of emTT-type and term signatures as mTT-extensional dependent types and terms, respectively, by using canonical isomorphisms. This interprets the so-called "raw syntax" in [Mai05], namely the signatures of types and terms in emTT, in a partial way. Indeed, as it is well explained in [Str91], we can not define a total interpretation by induction on the derivation of types and typed terms in emTT, because term equalities are involved in the formation of types and typed terms, and hence the interpretation would depend on the validity of equality which should instead follow as a consequence of the chosen interpretation.

Def. 8.1 (Interpretation of emTT into mTT and of emTT^{dp} into mTT^{dp}) In the following we define supports and related equalities of mTT (mTT^{dp})-extensional dependent types interpreting emTT (emTT^{dp})-type signatures, and mTT (mTT^{dp})-extensional terms interpreting emTT (emTT^{dp})-term signatures, all by induction on their formation. They are both described under a context, since free variables are assumed to be typed. We also warn that, when we interpret a type or a term signatures depending on more than one term, we assumed to have matched the types of the already interpreted terms via canonical isomorphisms.
The **empty context** is interpreted as follows:
$[\ ]^I \equiv [x:\mathsf{N}_1]$ and $[\mathsf{N}_{1_=}] \equiv (\mathsf{N}_1, =_{\mathsf{N}_1})$
with $z =_{\mathsf{N}_1} z' \equiv \mathsf{Id}(\mathsf{N}_1, z, z')$ for $z, z' \in \mathsf{N}_1$

The **assumption of variable** is interpreted as follows:
$(\,x\in A\,)[\Gamma^I, x\in A, \Delta^I]\,) \equiv x\in A^I\ [\Gamma^I, x\in A^I, \Delta^I]$

Collection constructors are interpreted as follows:

Power collection of the singleton :
$\mathcal{P}(1)^I\ col\ [\Gamma^I] \equiv \mathsf{props_s}\ [\Gamma^I]$
and $z =_{\mathsf{props_s}^I} z' \equiv (z \to z') \wedge (z' \to z)$ for $z, z' \in \mathsf{props_s}$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv w$ for $\overline{x}, \overline{x'} \in \Gamma^I, w \in \mathsf{props_s}$.
$([A])^I \equiv A^I$ for A small proposition.
$(\mathsf{true} \in B \leftrightarrow C\ \ [\Gamma])^I \equiv r \in (B \leftrightarrow C)^I\ [\Gamma^I]$ provided that $r \in (B \leftrightarrow C)^I\ [\Gamma^I]$ is derivable in mTT.

Strong Indexed Sum :
$(\Sigma_{y\in B} C(y))^I\ col\ [\Gamma^I] \equiv \Sigma_{y\in B^I} C^I(y)\ col\ [\Gamma^I]$
and $z =_{\Sigma_{y\in B} C(y)^I} z' \equiv \exists_{d\in \pi_1(z) =_{B^I} \pi_1(z')}\ \sigma_{\pi_1(z)}^{\pi_1(z')}(\pi_2(z)) =_{C^I(\pi_1(z'))} \pi_2(z')$ for $z, z' \in (\Sigma_{y\in B} C(y))^I$.
$(\langle b, d\rangle)^I \equiv \langle b^{\tilde{I}}, d^{\tilde{I}}\rangle$ and $El_\Sigma(d, m)^I \equiv El_\Sigma(d^{\tilde{I}}, (w_1, w_2).m^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \mathsf{El}_\Sigma(w, (w_1, w_2).\langle\, \sigma_{\overline{x}}^{\overline{x'}}(w_1)\,,\, \sigma_{\overline{x}, w_1}^{\overline{x'}, \sigma_{\overline{x}}^{\overline{x'}}(w_1)}(w_2)\,\rangle\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\Sigma_{y\in B} C(y))^{\tilde{I}}(\overline{x})$.

Dependent Product collection :
$(\Pi_{y\in B} C(y))^I\ col\ [\Gamma^I] \equiv \Sigma_{h\in \Pi_{y\in B^I} C^I(y)}\ \ \forall_{y_1\in B^I}\ \forall_{y_2\in B^I}\ \ \forall_{d\in y_1 =_{B^I} y_2}\ \sigma_{\overline{x}, y_1}^{\overline{x}, y_2}(\mathsf{Ap}(h, y_1)) =_{C^I(y_2)} \mathsf{Ap}(h, y_2)$
and $z =_{\Pi_{y\in B^I} C(y)^I} z' \equiv \forall_{y\in B^I}\ \ \mathsf{Ap}(\pi_1(z), y) =_{C^I(y)} \mathsf{Ap}(\pi_1(z'), y)$ for $z, z' \in (\Pi_{y\in B} C(y))^I$.
$(\lambda y^B.c)^I \equiv \langle\, \lambda y^{B^I}.c^{\tilde{I}}\,,\, p\,\rangle$ where $p \in \forall_{y_1\in B^I}\ \forall_{y_2\in B^I}\ \forall_{d\in y_1 =_{B^I} y_2}\ \sigma_{\overline{x}, y_1}^{\overline{x}, y_2}(c^{\tilde{I}}(y_1)) =_{C^I(y_2)} c^{\tilde{I}}(y_2)$
$(\mathsf{Ap}(f, b))^I \equiv \mathsf{Ap}(\pi_1(f^{\tilde{I}}), b^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \langle\, \lambda y'^{B^I(\overline{x'})}.\ \sigma_{\overline{x}, \sigma_{\overline{x'}}^{\overline{x}}(y')}^{\overline{x'}, y'}(\mathsf{Ap}(\pi_1(w)\,,\, \sigma_{\overline{x'}}^{\overline{x}}(y')))\,,\, p\,\rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\Pi_{y\in B} C(y))^I(\overline{x})$
where p is the proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

Function collection to $\mathcal{P}(1)$:
$(B \to \mathcal{P}(1)\ col\ [\Gamma])^I \equiv \Sigma_{h\in B^I \to \mathsf{props_s}}\ \ \forall_{y_1\in B^I}\ \forall_{y_2\in B^I}\ \ y_1 =_{B^I} y_2 \to (\mathsf{Ap}(h, y_1) \leftrightarrow \mathsf{Ap}(h, y_2))$
and $z =_{\mathcal{P}} z' \equiv \forall_{y\in B^I}\ \ \mathsf{Ap}(\pi_1(z), y) \leftrightarrow \mathsf{Ap}(\pi_1(z'), y)$ for $z, z' \in (B \to \mathcal{P}(1))^I$
$(\lambda y^B.c)^I \equiv \langle\, \lambda y^{B^I}.c^{\tilde{I}}\,,\, p\,\rangle$ where $p \in \forall_{y_1\in B^I}\ \forall_{y_2\in B^I}\ \ y_1 =_{B^I} y_2 \to (c^{\tilde{I}}(y_1) \leftrightarrow c^{\tilde{I}}(y_2))$
$(\mathsf{Ap}(f, b))^I \equiv \mathsf{Ap}(\pi_1(f^{\tilde{I}}), b^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \langle\, \lambda y'^{B^I(\overline{x'})}.\, \sigma_{\overline{x}, \sigma_{\overline{x'}}^{\overline{x}}(y')}^{\overline{x'}, y'}(\mathsf{Ap}(\pi_1(w)\,,\, \sigma_{\overline{x'}}^{\overline{x}}(y'))),\, p\,\rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (B \to \mathcal{P}(1))^I(\overline{x})$
where p is the proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

Quotient collection :
$(A/R\ col\ [\Gamma])^I \equiv A^I\ col\ [\Gamma^I]$
and $z =_{A/R^I} z' \equiv R^I(z, z')$ for $z, z' \in A^I$.
$([a]) \equiv a^I$ and $El_Q(p, l)^I \equiv l^{\tilde{I}}(p^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w)$ is defined as the substitution isomorphism of $A^I_{=}\,[\Gamma^I_{=}]$.
$(\mathsf{true} \in R(a, b)\ [\Gamma])^I \equiv r \in R(a, b)^I\ [\Gamma^I]$ provided that $r \in R(a, b)^I\ [\Gamma^I]$ is derivable in mTT.

Now we give the interpretation of sets:

Empty set : $(\mathsf{N_0}\ set\ [\Gamma])^I \equiv \mathsf{N_0}\ set\ [\Gamma^I]$
and $z =_{\mathsf{N_0}^I} z' \equiv \mathsf{Id}(\mathsf{N_0}, z, z')$ for $z, z' \in \mathsf{N_0}$.
$(\mathsf{emp_o}(a))^I \equiv \mathsf{emp_o}(a^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv w$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in \mathsf{N_0}$.

Singleton set : $(\mathsf{N_1}\ set\ [\Gamma])^I \equiv \mathsf{N_1}\ set\ [\Gamma^I]$
and $z =_{\mathsf{N_1}^I} z' \equiv \mathsf{Id}(\mathsf{N_1}, z, z')$ for $z, z' \in \mathsf{N_1}$.
$(\star)^I \equiv \star$ and $(El_{\mathsf{N_1}}(t, c))^I \equiv El_{\mathsf{N_1}}(t^{\tilde{I}}, c^{\tilde{I}})$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv w$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in \mathsf{N_1}$.

List set : $(List(C)\ set\ [\Gamma])^I \equiv List(C^I)\ set\ [\Gamma^I]$
and $z =_{List(C^I)} z'$ defined as in theorem 4.20.
$(\epsilon)^I \equiv \epsilon$ and $(\mathsf{cons}(s, c))^I \equiv \mathsf{cons}(s^{\tilde{I}}, c^{\tilde{I}})$
$(El_{List}(s, a, l))^I \equiv El_{List}(s^{\tilde{I}}, a^{\tilde{I}}, l^{\tilde{I}})$

$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv El_{List}(\, w\, ,\, \epsilon,\, (y_1, y_2, z).\mathsf{cons}(\, z\, ,\, \sigma_{\overline{x}}^{\overline{x'}}(y_2)\,)\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\, List(C)\,)^I(\overline{x})$.

Disjoint Sum set : $(\, B + C\ set\ [\Gamma])^I \equiv B^I + C^I\ set\ [\Gamma^I]$
and $z =_{B^I + C^I} z'$ is defined as in theorem 4.20.
$(\,\mathsf{inl}(b)\,)^I \equiv \mathsf{inl}(b^I)$ and $(\,\mathsf{inl}(c)\,)^I \equiv \mathsf{inl}(c^I)$ and $(\, El_+(d, a_B, a_C)\,)^I \equiv El_+(\, d^{\tilde{I}},\, a_B^{\tilde{I}},\, a_C^{\tilde{I}})\,)$
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv El_+(w, (y_1).\sigma_{\overline{x}}^{\overline{x'}}(y_1)\, ,\, (y_2).\sigma_{\overline{x}}^{\overline{x'}}(y_2)\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\, B + C\,)^I(\overline{x})$.

Finally, **Strong Indexed Sum set**, **Dependent Product set** and **Quotient set** constructors with their terms are interpreted analogously to Strong Indexed Sums, Dependent Product collections and Quotient collections with their terms, respectively.

Lastly, emTT-propositions are interpreted as mTT-extensional propositions whose support is similar, except for extensional propositional equality, and whose equality is trivial, namely if $A\ prop\ [\Gamma]$ is a proposition then $z =_{A^I} z' \equiv \mathsf{tt}$ for all $\overline{x} \in \Gamma^I, z, z' \in A^I(\overline{x})$. Therefore in the following we just specify the support of their interpretation.

Falsum: $\bot^I\ prop\ [\Gamma^I]) \equiv \bot\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in A\,)^I \equiv \mathsf{r}_0(p) \in A^I$ provided that $p \in \bot$ is derived in mTT.
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv w$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in \bot$.

Extensional Propositional Equality: $\mathsf{Eq}(B, b_1, b_2)^I\ prop\ [\Gamma^I] \equiv b_1^{\tilde{I}} =_{B^I} b_2^{\tilde{I}}\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in \mathsf{Eq}(B, b, b)\ [\Gamma]\,)^I \equiv \mathsf{rfl}(b^{\tilde{I}}) \in b^{\tilde{I}} =_{B^I} b^{\tilde{I}}\ [\Gamma^I]$ provided that $b^{\tilde{I}} \in B^I\ [\Gamma^I]$ is derived in mTT.
Moreover, $\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv q$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in b_1^{\tilde{I}} =_{B^I} b_2^{\tilde{I}}$ where is q obtained by transitivity of equivalence relations from a proof of $\sigma_{\overline{x}}^{\overline{x'}}(b_1^{\tilde{I}}) =_{B^I[\overline{x}/\overline{x'}]} \sigma_{\overline{x}}^{\overline{x'}}(b_2^{\tilde{I}})$ by equality preservation of substitution morphisms of B^I in definition 4.24 and extensional equality of the term $b_1^{\tilde{I}}$ and $b_2^{\tilde{I}}$ in definition 4.11.

Implication: $(B \to C)^I\ prop\ [\Gamma^I] \equiv B^I \to C^I\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in B \to C\ [\Gamma]\,)^I \equiv \lambda_\to x^B.c \in B^I \to C^I\ [\Gamma^I]$ provided that $c \in C^I\ [\Gamma^I, y \in B^I]$ is derived in mTT.
$(\,\mathsf{true} \in C\ [\Gamma]\,)^I \equiv \mathsf{Ap}_\to(f, b) \in C^I\ [\Gamma^I]$ provided that $f \in B^I \to C^I\ [\Gamma^I]$ and $b \in B^I\ [\Gamma^I]$ are derived in mTT.
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \lambda_\to y \in B^I(\overline{x'}).\, \sigma_{\overline{x}}^{\overline{x'}}(\,\mathsf{Ap}_\to(w, \sigma_{\overline{x'}}^{\overline{x}}(y)\,)\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (B \to C)^I(\overline{x})$.

Conjunction $(B \wedge C)^I\ prop\ [\Gamma^I] \equiv B^I \wedge C^I\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in B \wedge C\ [\Gamma]\,)^I \equiv \langle b,_\wedge c\rangle \in B^I \wedge C^I\ [\Gamma^I]$ provided that $b \in B^I\ [\Gamma^I]$ and $c \in C^I\ [\Gamma^I]$ are derived in mTT.
$(\,\mathsf{true} \in B\ [\Gamma]\,)^I \equiv \pi_1^{B^I}(d) \in B^I\ [\Gamma^I]$ provided that $d \in B^I \wedge C^I\ [\Gamma^I]$ is derived in mTT.
$(\,\mathsf{true} \in C\ [\Gamma]\,)^I \equiv \pi_2^{C^I}(d) \in C^I\ [\Gamma^I]$ provided that $d \in B^I \wedge C^I\ [\Gamma^I]$ is derived in mTT.
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \langle \sigma_{\overline{x}}^{\overline{x'}}(\,\pi_1^{B^I}(w)\,)\,,_\wedge\, \sigma_{\overline{x}}^{\overline{x'}}(\,\pi_2^{C^I}(w)\,)\rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (B \wedge C)^I(\overline{x})$.

Disjunction $(B \vee C)^I\ prop\ [\Gamma^I] \equiv B^I \vee C^I\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in B \vee C\ [\Gamma]\,)^I \equiv \mathsf{inl}_\vee(b) \in B^I \vee C^I\ [\Gamma^I]$ provided that $b \in B^I\ [\Gamma^I]$ is derived in mTT.
$(\,\mathsf{true} \in B \vee C\ [\Gamma]\,)^I \equiv \mathsf{inr}_\vee(c) \in B^I \vee C^I\ [\Gamma^I]$ provided that $c \in C^I\ [\Gamma^I]$ is derived in mTT.
$(\,\mathsf{true} \in A\ [\Gamma]\,)^I \equiv El_\vee(d, a_B, a_C) \in A^I\ [\Gamma^I]$ provided that $d \in B^I \vee C^I\ [\Gamma^I]$, $a_B \in A^I\ [\Gamma^I, y \in B^I]$ and $a_C \in A^I\ [\Gamma^I, y \in C^I]$ are derived in mTT.
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv El_\vee(w, (y_1).\,\mathsf{inl}_\vee(\,\sigma_{\overline{x}}^{\overline{x'}}(y_1)\,)\, ,\, (y_2).\mathsf{inr}_\vee(\,\sigma_{\overline{x}}^{\overline{x'}}(y_2)\,)\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (B \vee C)^I(\overline{x})$.

Existential quantifier: $(\exists_{y \in B} C(y))^I\ prop\ [\Gamma^I]) \equiv \exists_{y \in B^I} C^I(y)\ prop\ [\Gamma^I]$
$(\,\mathsf{true} \in \exists_{y \in B} C(y)\ [\Gamma]\,)^I \equiv \langle b^{\tilde{I}},_\exists c\rangle \in \exists_{y \in B^I} C^I(y)\ [\Gamma^I]$ provided that $c \in C^I(b^{\tilde{I}})\ [\Gamma^I]$ is derived in mTT.
$(\,\mathsf{true} \in M\ [\Gamma]\,)^I \equiv El_\exists(d, m) \in M^I\ [\Gamma^I]$ provided that $d \in \exists_{y \in B^I} C^I(y)\ [\Gamma^I]$ and $m \in M^I\ [\Gamma^I, y \in B^I, z \in C^I(y)]$ are derived in mTT.
$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv El_\vee(w, (y, z).\,\langle\, \sigma_{\overline{x}}^{\overline{x'}}(y)\, ,\, \sigma_{\overline{x}, y}^{\overline{x'}, \sigma_{\overline{x}}^{\overline{x'}}(y)}(z)\,\rangle\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\, \exists_{y \in B}\, C(y)\,)^I(\overline{x})$.

Universal quantifier: $(\forall_{y \in B} C(y)^I\ prop\ [\Gamma^I]) \equiv \forall_{y \in B^I} C^I(y)\ prop\ [\Gamma^I])$
$(\,\mathsf{true} \in \forall_{y \in B} C(y)\ [\Gamma]\,)^I \equiv \lambda_\forall y^{B^I}.c \in \forall_{y \in B^I} C^I(y)$ provided that $c \in C^I(y)\ [\Gamma^I, y \in B^I]$ is derived in mTT.
$(\,\mathsf{true} \in C(b)\ [\Gamma]\,)^I \equiv \mathsf{Ap}_\forall(f, b^{\tilde{I}}) \in C^I(b^{\tilde{I}})\ [\Gamma^I]$ provided that $f \in \forall_{y \in B^I} C^I(y)\ [\Gamma^I]$ is derived in mTT.

$\sigma_{\overline{x}}^{\overline{x'}}(w) \equiv \lambda_{\forall} y^{B^I(\overline{x'})}.\, \sigma_{\overline{x}, \sigma_{\overline{x'}}^{\overline{x}}(y)}^{\overline{x'}, y}(\, \mathsf{Ap}_{\forall}(w,\, \sigma_{\overline{x'}}^{\overline{x}}(y))\,)$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\, \forall_{y \in B}\, C(y)\,)^I(\overline{x})$.