1.5 Theorem of Van Kampen

The theorem of Van Kampen[28] allows the computation of the fundamental group of a space in terms of the fundamental groups of the open subsets of a suitable cover.

Let X be an arcwise connected topological space, $(\Lambda, \leq)$ an ordered set, $\{U_\lambda : \lambda \in \Lambda\}$ an open cover of X such that:

(a) all U_λ are arcwise connected;

(b) $\lambda \leq \mu$ if and only if $U_\lambda \subseteq U_\mu$;

(c) the family $\{U_\lambda : \lambda \in \Lambda\}$ is stable under finite intersections;

(d) there exists $x_0 \in \bigcap_{\lambda \in \Lambda} U_\lambda$.

Let us denote by $\iota_\lambda : U_\lambda \to X$ and $\iota_{\lambda,\mu} : U_\lambda \to U_\mu$ the canonical inclusions (where $\lambda \leq \mu$), and write for short $\pi_1(U_\lambda)$ instead of $\pi_1(U_\lambda, x_0)$. We then have morphisms $\iota_{\lambda\#} : \pi_1(U_\lambda) \to \pi_1(X)$ and $\iota_{\lambda,\mu\#} : \pi_1(U_\lambda) \to \pi_1(U_\mu)$ (where $\lambda \leq \mu$), which simply associate to the class of a loop the class of the same loop viewed in the larger space. In particular, one has an inductive system $\{\pi_1(U_\lambda), \iota_{\lambda,\mu} : \lambda, \mu \in \Lambda\}$ (see Appendix A.1).

Let $M \subset \Lambda$ such that $X = \bigcup_{\lambda \in M} U_\lambda$.

Proposition 1.5.1. *$\pi_1(X)$ is generated by $\{\iota_{\lambda\#}(\pi_1(U_\lambda)) : \lambda \in M\}$.*

Proof. Let $\gamma : I \to X$ be a loop at x_0, $\delta > 0$ the Lebesgue number relative to the cover $\{\gamma^{-1}(U_\lambda) : \lambda \in M\}$ of I, $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ with $t_j - t_{j-1} < \delta$ and $\lambda_j \in M$ (with $j = 1, \dots, k$) such that $\gamma([t_{j-1}, t_j]) \subset U_{\lambda_j}$. Since $\gamma(t_j) \in U_{\lambda_j} \cap U_{\lambda_{j+1}}$ (arcwise connected), let σ_j (for $j = 1, \dots, k-1$) be an arc into $U_{\lambda_j} \cap U_{\lambda_{j+1}}$ from x_0 to $\gamma(t_j)$. Setting $\gamma_j = \gamma|_{[t_{j-1}, t_j]}$ (reparametrized by sending t_{j-1} to 0 and t_j to 1) it clearly holds $[\gamma] = [\gamma_1] \cdot \cdots \cdot [\gamma_k] = [\gamma_1 \cdot \sigma_1^{-1}] \cdot [\sigma_1 \cdot \gamma_2 \cdot \sigma_2^{-1}] \cdot \cdots \cdot [\sigma_{k-2} \cdot \gamma_{k-1} \cdot \sigma_{k-1}^{-1}][\sigma_{k-1} \cdot \gamma_k]$. □

An obvious consequence of Proposition 1.5.1 is:

Corollary 1.5.2. *If the open subsets U_λ $(\lambda \in M)$ are simply connected, such is also X.*

Example. $\mathbb{S}^n$ is simply connected for $n \geq 2$. Namely, let $N = e_{n+1}$ be the North pole, $S = -N$ the South pole, and set $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$: noting that $U \cap V$ is arcwise connected and that both U and V are simply connected, just apply Corollary 1.5.2. Note that this argument does not apply for $\mathbb{S}^1$ (in that case $U \cap V$ is not arcwise connected).

In general one has the following result (see Appendix A.1 for the notion of "inductive limit" of an inductive system).

Theorem 1.5.3. (Van Kampen) *In the category $\mathfrak{Groups}$ it holds*

$$\pi_1(X) = \varinjlim_{\lambda \in \Lambda} \pi_1(U_\lambda).$$

[28] The result has been proved independently also by Karl Seifert in the 30s of last century; in fact, it is often referred to as "Seifert - Van Kampen theorem".

Proof. Consider a group L and a family of morphisms $\psi_\lambda : \pi_1(U_\lambda) \to L$ such that $\psi_\lambda = \psi_\mu \circ \iota_{\lambda,\mu\#}$ for $\lambda \leq \mu$, and let us see if there exists a unique morphism $\psi : \pi_1(X) \to L$ such that $\psi_\lambda = \psi \circ \iota_{\lambda\#}$ for any $\lambda \in \Lambda$. So let $[\gamma] \in \pi_1(X)$: by Proposition 1.5.1 we may write $[\gamma] = \iota_{\lambda_1\#}([\gamma_1]) \cdot \dots \cdot \iota_{\lambda_k\#}([\gamma_k])$ with $[\gamma_j] \in \pi_1(U_{\lambda_j})$. If ψ exists, it must be necessarily unique because $\psi([\gamma]) = \psi(\iota_{\lambda_1\#}[\gamma_1]) \cdots \psi(\iota_{\lambda_k\#}[\gamma_k]) = \psi_{\lambda_1}([\gamma_1]) \cdots \psi_{\lambda_k}([\gamma_k])$ (the products are in L). We are left with showing that this is actually a well-posed definition for ψ, i.e. that if $[c_{x_0}] = \iota_{\lambda_1\#}([\sigma_1]) \cdot \dots \cdot \iota_{\lambda_k\#}([\sigma_k])$ then also $\psi_{\lambda_1}([\sigma_1]) \cdots \psi_{\lambda_k}([\sigma_k]) = e$ (where e is the identity element of L). Let $\sigma = \sigma_1 \cdot \dots \cdot \sigma_k$ be in X (hence, if $t \in [\frac{j-1}{k}, \frac{j}{k}]$ it holds $\sigma(t) = \sigma_j(kt-(j-1))$), and let $h : I \times I \to X$ be a homotopy rel ∂I between σ and c_{x_0}. Let $\varepsilon > 0$ the Lebesgue number relative to the cover $\{h^{-1}(U_\lambda) : \lambda \in \Lambda\}$ of $I \times I$, and let $r \in \mathbb{N}$ be such that $\frac{\sqrt{2}}{k^r} < \varepsilon$. Hence, setting $R_{i,j} = \left[\frac{i-1}{k^r}, \frac{i}{k^r}\right] \times \left[\frac{j-1}{k^r}, \frac{j}{k^r}\right] \subset I \times I$ (for $i, j = 1, \dots, k^r$), there exists $\lambda_{i,j} \in \Lambda$ such that $h(R_{i,j}) \subset U_{\lambda_{i,j}}$. Let $v_{i,j} = \left(\frac{i}{k^r}, \frac{j}{k^r}\right)$ (hence $R_{i,j}$ is the square with side k^{-r} and opposed vertices $v_{i-1,j-1}$ and $v_{i,j}$), $U_{\mu(i,j)}$ the intersection of the (one, two or four) $U_{\lambda_{l,m}}$ such that $v_{i,j} \in R_{l,m}$, and $\gamma_{i,j}$ a path in $U_{\mu(i,j)}$ from x_0 to $h(v_{i,j})$. Let $\alpha_{i,j}(t) = h\left(\frac{t+(i-1)}{k^r}, \frac{j}{k^r}\right)$ (path from $h(v_{i-1,j})$ to $h(v_{i,j})$) and $\beta_{i,j}(t) = h\left(\frac{i}{k^r}, \frac{t+(j-1)}{k^r}\right)$ (path from $h(v_{i,j-1})$ to $h(v_{i,j})$): note that $\left[\alpha_{(m-1)k^{r-1}+1,0} \cdot \dots \cdot \alpha_{mk^{r-1},0}\right] = [\sigma_m]$ (for $m = 1, \dots, k$) and that $\alpha_{i,k^r}(t) = \beta_{0,j}(t) = \beta_{k^r,j}(t) \equiv x_0$ (for $t \in I$ and $i, j = 1, \dots, k^r$). From the equality $[\alpha_{i,j-1} \cdot \beta_{i,j}] = [\beta_{i-1,j} \cdot \alpha_{i,j}]$ one gets (by inserting the paths $\gamma_{i,j}$ and their inverses to base at x_0) the relations $\left[\left((\gamma_{i-1,j-1} \cdot \alpha_{i,j-1}) \cdot \gamma_{i,j-1}^{-1}\right) \cdot \left((\gamma_{i,j-1} \cdot \beta_{i,j}) \cdot \gamma_{i,j}^{-1}\right)\right] = \left[\left((\gamma_{i-1,j-1} \cdot \beta_{i-1,j}) \cdot \gamma_{i-1,j}^{-1}\right) \cdot \left((\gamma_{i-1,j} \cdot \alpha_{i,j}) \cdot \gamma_{i,j}^{-1}\right)\right]$ in the group $\pi_1(U_{\lambda_{i,j}})$. Applying $\psi_{\lambda_{i,j}}$ and setting $a_{i,j} = \psi_{\lambda_{i,j}}\left(\left[(\gamma_{i-1,j} \cdot \alpha_{i,j}) \cdot \gamma_{i,j}^{-1}\right]\right)$ and $b_{i,j} = \psi_{\mu(i,j)}\left(\left[(\gamma_{i,j-1} \cdot \beta_{i,j}) \cdot \gamma_{i,j}^{-1}\right]\right)$, one then has the equality $a_{i,j-1}\, b_{i,j} = b_{i-1,j}\, a_{i,j}$ in L. Knowing that $a_{1,k^r} \cdots a_{k^r,k^r} = e$ and that $b_{0,j} = b_{k^r,j} = e$ (for any $j = 1, \dots, k^r$), one has $e = a_{1,k^r} \cdots a_{k^r,k^r} = (b_{0,k^r}\, a_{1,k^r}) a_{2,k^r} \cdots a_{k^r,k^r} = a_{1,k^r-1}(b_{1,k^r}\, a_{2,k^r}) \cdots a_{k^r,k^r} = \dots = a_{1,k^r-1} \cdots a_{k^r,k^r-1}$; by repeating the procedure one obtains $a_{1,0} \cdots a_{k^r,0} = e$, as required. □

Now the problem is to understand what $\varinjlim_{\lambda\in\Lambda} \pi_1(U_\lambda)$ seems like.

In general, the *free product* $*_{\lambda\in\Lambda} G_\lambda$ of a family of groups $\{G_\lambda : \lambda \in \Lambda\}$ is the group formed by finite "words" $a_1 \cdots a_k$ constructed with "letters" $a_j \in G_{\lambda_j}$ ($j = 1, \dots, k$; $k \geq 1$), where any letter is different from the identity element in the respective group and where two adjacent letters must belong to different groups (one often says "reduced letters"); also the "empty word" is considered to be an element. The operation is given by the natural juxtaposition $(a_1 \cdots a_k) \cdot (b_1 \cdots b_h) = a_1 \cdots a_k b_1 \cdots b_h$ where, in the case a_k and b_1 belong to a same group, the expression "$a_k b_1$" should be replaced by their product in that group (and possibly removed if $a_k b_1$ is the identity, causing then the same procedure for $a_{k-1} b_2$, and so on); the identity element is clearly the empty word.

Example. The free product of any number of copies of $\mathbb{Z}$ is called *free group*, in the sense that there is one generator for each copy of $\mathbb{Z}$ and the elements of the group are words formed by powers of these generators. For example, $\mathbb{Z} * \mathbb{Z}$ is formed by the words $r_1 s_1 \cdots r_k s_k$ where all r_j's and s_j's are integer (the r_j's are meant to belong to the first copy of $\mathbb{Z}$, and the s_j's to the second); or also, in abstract notation, by $a^{k_1} b^{h_1} \cdots a^{k_r} b^{h_r}$ where a and b denote the two generators and the exponents are integers.

Note that for any $\mu \in \Lambda$ there is a natural monomorphism $G_\mu \to *_{\lambda\in\Lambda} G_\lambda$; in fact one sees that the free product $*_{\lambda\in\Lambda} G_\lambda$ is the inductive limit in $\mathfrak{Groups}$ of the system $\{G_\lambda : \lambda \in \Lambda\}$ with trivial preorder, i.e. without considering morphisms[(29)]. More generally, if morphisms $f_{\lambda,\mu} : G_\lambda \to G_\mu$ are given for some pairs (λ, μ) with $f_{\lambda,\mu} \circ f_{\mu,\nu} = f_{\lambda,\nu}$ whenever defined, then to obtain the inductive limit of the system $\{G_\lambda,\ f_{\lambda,\mu} : \lambda, \mu \in \Lambda\}$ one must quotient out the previous free product $*_{\lambda\in\Lambda} G_\lambda$ by its normal subgroup N generated by all the elements of type $f_{\lambda,\mu}(a) f_{\lambda,\nu}(a)^{-1}$ for $a \in G_\lambda$ whenever the morphisms $f_{\lambda,\mu}$ and $f_{\lambda,\nu}$ are

[(29)]Namely, given a family of morphisms $\psi_\lambda : G_\lambda \to L$, the definition (necessary, hence unique) $\psi(a_1 \cdots a_k) = \psi_{\lambda_1}(a_1) \cdots \psi_{\lambda_k}(a_k)$ is a morphism.

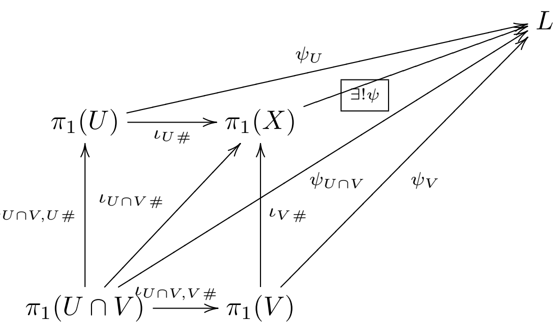
defined[30]: this procedure is said to be an *amalgamation* of the free product with respect to the given morphisms $f_{\lambda,\mu}$.

When applying the above notions to the framework of Van Kampen, the groups are $\pi_1(U_\lambda)$ and the morphisms are the maps $\iota_{\lambda,\mu\#} : \pi_1(U_\lambda) \to \pi_1(U_\mu)$, which send the class of a loop in the small open subset U_λ to the class of same loop viewed in the larger open subset U_μ: by Proposition 1.5.1 and the subsequent discussion it is then enough to consider the free product of the groups $\pi_1(U_\lambda)$'s of a selected family of open subsets U_λ with indices $\lambda \in M$ which cover X and are not contained in each other, and then to amalgamate only with respect to the double intersections $U_\lambda \cap U_\mu$ for $\lambda, \mu \in M$.[31]

Example. *(Wedge sums)* Consider a family of pointed spaces (X_λ, x_λ) for $\lambda \in M$, and let $X = \bigvee_{\lambda \in M} X_\lambda$ be their wedge sum (see p. 9). For each $\lambda \in M$ let V_λ be a open subset of x_λ in X_λ which has x_λ as deformation retract, and set $U_\lambda = X_\lambda \vee (\bigvee_{\mu \neq \lambda} V_\mu)$. It is clear that the U_λ's cover X; moreover, any intersection of two or more of them is always $\bigvee_{\lambda \in M} V_\lambda$, which is arcwise connected and, since it deformation-retracts to the base point, has trivial fundamental group and hence causes no effective amalgamation. Finally, since each U_λ deformation-retracts to the corresponding X_λ, by Van Kampen theorem it follows that $\pi_1(X) \simeq \ast_{\lambda \in M} \pi_1(X_\lambda)$.

The Theorem of Van Kampen is used mainly in the case of two open subsets $X = U \cup V$, with U, V and $U \cap V$ arcwise connected: in that case $\pi_1(X)$ will be the inductive limit of the system $\{\pi_1(U \cap V), \pi_1(U), \pi_1(V); , \iota_{U\cap V,U}, , \iota_{U\cap V,V}\}$, and its universal property is expressed by the following diagram:

(1.4)



One usually denotes the free product of two groups G and H by $G * H$ and, given another group K and morphisms $f : K \to G$ and $g : K \to H$, the free product of G and H amalgamated on K by $G *_K H$.[32] By what has been said we get:

[30] Namely, if in the previous notation $\psi_\lambda \circ \alpha_\lambda = \psi_\mu \circ \alpha_\mu$, for any $\lambda, \mu \in \Lambda$ then $N \subset \ker(\psi)$ and hence ψ factorizes uniquely through the quotient $\ast_{\lambda\in\Lambda} G_\lambda/N$.

[31] Namely, a typical element of $\ast_{\lambda\in\Lambda} \pi_1(U_\lambda)$ is $[\gamma_1]\cdots[\gamma_k]$ where γ_j is a loop in U_{λ_j} but the class $[\gamma_j]$ is taken as loop in X (in fact, we should have written more precisely $i_{\lambda_j\#}([\gamma_j]_{U_{\lambda_j}})$): hence, by the compatibility of the various morphisms of type $i_\#$, the class $[\gamma_j]$ can be thought as coming from some U_λ with $\lambda \in M$, and this shows that $\ast_{\lambda\in\Lambda} \pi_1(U_\lambda) \simeq \ast_{\lambda\in M} \pi_1(U_\lambda)$. Similar considerations hold for the amalgamation: the objects $i_{\lambda,\mu\#}([\gamma]) \cdot i_{\lambda,\nu\#}([\gamma])^{-1}$ coming from intersections of three or more U_λ with $\lambda \in M$ can be thought as having already come from some double intersection.

[32] In the language of categories, the group $G *_K H$ is usually called the *pushout* of the morphisms $f : K \to G$ and $g : K \to H$. Note that the notation $G *_K H$ does not show explicitly what are f and g, but of course it is important to take them into account.

Corollary 1.5.4. *Let X be an arcwise connected topological space, $X = U \cup V$ an open cover with U, V and $U \cap V$ arcwise connected open subsets. Then*

$$\pi_1(X) \simeq \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V).$$

In particular, let us emphasize the following two cases.

(i) *If $U \cap V$ is simply connected, then $\pi_1(X) \simeq \pi_1(U) * \pi_1(V)$;*

(ii) *If V is simply connected, and N is the normal subgroup generated by the image* (by $\iota_{U\cap V,U\#}$) *of $\pi_1(U \cap V)$ in $\pi_1(U)$, then $\pi_1(X) \simeq \pi_1(U)/N$.*

Examples. **(1)** *(Plane with k holes)* Let $\{x_1,\dots,x_k\}$ be a family of k distinct points of the plane $\mathbb{R}^2$, and let $X = \mathbb{R}^2 \setminus \{x_1,\dots,x_k\}$. Then $\pi_1(X)$ is the free group with k generators. Namely for $k = 1$ we already know that $\pi_1(X) \simeq \mathbb{Z}$; given $k > 1$, let s_j be closed half lines with origin x_j and with empty intersection[(33)], set $U = X \setminus \{s_1,\dots,s_{k-1}\}$ and $V = X \setminus \{s_k\}$. Now, $X = U \cup V$, all the sets are arcwise connected and $U \cap V$ is simply connected (even contractible), and so $\pi_1(X) \simeq \pi_1(U) * \pi_1(V)$; but U is homotopically equivalent to $\mathbb{S}^1$, while $\pi_1(V)$ is free with $k-1$ generators by inductive hypothesis. The same holds for $Y = \mathbb{S}^2 \setminus \{y_1,\dots,y_{k+1}\}$ (with $\{y_1,\dots,y_{k+1}\}$ distinct points of $\mathbb{S}^2$).[(34)] **(2)** *(Bouquet of k circles)* The fundamental group of a "bouquet" of k circles (i.e. the wedge sum of k circles) is again a free group with k generators: this follows immediately from what has been said in general for wedge sums. Alternatively one could also note that the bouquet is in fact a strong deformation retract of the plane with k holes; another proof is to use induction and Van Kampen, by choosing for any circle C_j a point x_j different from the center of the bouquet ($j = 1,\dots,k$), and then taking $U = X \setminus \{x_1\}$ and $V = (C_1 \cup C_2) \setminus \{x_2\}$: then $U \cap V$ is contractible, V has the homotopy of a circle and U of a bouquet of $(k-1)$ circles. **(3)** *(Removing an annulus from $\mathbb{R}^3$)* Let $\mathbb{S}^1_{(x,y)} = \{(x,y,z) \in \mathbb{R}^3 : z = 0,\, x^2+y^2 = 1\}$, and let us compute $\pi_1(X)$ where $X = \mathbb{R}^3 \setminus \mathbb{S}^1_{(x,y)}$. Let $\mathbb{R}_z = \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}$ (the z-axis), and set $U = X \setminus \mathbb{R}_z$ and $V = \{(x,y,z) \in X : x^2+y^2 < 1\}$. Obviously $X = U \cup V$, all are arcwise connected and V is simply connected (even contractible). On the other hand, $U \cap V$ is homotopically equivalent to $\mathbb{S}^1$ and hence $\pi_1(U\cap V) \simeq \mathbb{Z}$, while, setting $T = \{(x,y,z) \in \mathbb{R}^3 : x > 0, y = 0\} \setminus \{(1,0,0)\}$, U is homeomorphic to $T \times \mathbb{S}^1$ (exercise) and hence $\pi_1(U) \simeq \pi_1(T) \times \pi_1(\mathbb{S}^1) \simeq \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$; moreover, one may identify the morphism $\pi_1(U\cap V) \to \pi_1(U) \simeq \pi_1(T) \times \pi_1(\mathbb{S}^1)$ with the morphism $\mathbb{Z} \to \mathbb{Z}^2$, $1 \mapsto (0,1)$ (the generator of $\pi_1(U \cap V)$ goes into the generator of the second factor). One therefore has $\pi_1(X) \simeq \mathbb{Z}^2/\mathbb{Z} \simeq \mathbb{Z}$. **(4)** *(Torus)* On the surface of $X = \mathbb{T}^2 = (\mathbb{S}^1)^2$ (the 2-dimensional torus viewed as a doughnut in $\mathbb{R}^3$, see Example 1.4) make a small circular hole F, and let $U = X \setminus F$ (open); let V be an open neighborhood of F in X (a "patch" above F). We are in fact in the hypotheses of Van Kampen's theorem; it is evident that V is contractible and that $U \cap V$ is homotopically equivalent to a circle (hence $\pi_1(V)$ is trivial and $\pi_1(U\cap V) \simeq \mathbb{Z}$). On the other hand U is homotopically equivalent to two tangent circles (this can be easily understood in the interpretation of $\mathbb{T}^2$ as a square modulo identifications, as recalled in the cited Example 1.4: making a hole in the interior of the square, the latter deformation-retracts radially on its boundary; as an useful exercise, we suggest to interpret this retraction on the doughnut), and then to a plane with two holes: hence $\pi_1(U)$ is free on two generators. Now, the normal subgroup of $\pi_1(U) \simeq \mathbb{Z} * \mathbb{Z}$ generated by the image of $\pi_1(U \cap V)$ is the subgroup of commutators[(35)] of $\mathbb{Z} * \mathbb{Z}$ (in the interpretation of the square, a generator of $\pi_1(U \cap V, x_0)$ is a loop based at a vertex which surrounds the hole: such loop is clearly homotopic rel ∂I to the boundary of the square run twice forth and back, which is exactly the commutator of the two

---

[(33)] For example, draw the lines $r_{l,m} = \{x_l + t(x_m - x_l) : t \in \mathbb{R}\}$ (for $1 \le l < m \le k$), then choose $y \in \mathbb{R}^2 \setminus \bigcup_{1\le l<m\le k} r_{l,m}$ and set $s_j = \{y + t(x_j - y) : t \ge 1\}$ (with $j = 1,\dots,k$).

[(34)] Y is homeomorphic to X by the stereographic projection from one of the y_j (recall that, considering in $\mathbb{R}^3$ the sphere $\mathbb{S} = \{x^2+y^2+(z-1)^2 = 1\} \simeq \mathbb{S}^2$ and the plane $\Pi = \{z = 0\} \simeq \mathbb{R}^2$, the "stereographic projection" from the North pole $N = (0,0,2) \in \mathbb{S}$ identifies diffeomorphically $\mathbb{S} \setminus \{N\}$ with Π by associating to $\underline{x} = (x,y,z) \in \mathbb{S} \setminus \{N\}$ the intersection point between Π and the half line coming from N and passing through $\underline{x}$).

[(35)] If G is a group, the subgroup of commutators of G is denoted by $[G,G]$ and it is the normal subgroup generated by the elements of the form $xyx^{-1}y^{-1}$ for $x,y \in G$. Obviously, G is abelian if and only if $[G,G]$ is trivial. Moreover, if G_A is the free group generated by a set A, then $G_A/[G_A,G_A] \simeq \mathbb{Z}^{(A)}$ (for the notation $\mathbb{Z}^{(A)}$ see Appendix A.1).

generators γ_1 and γ_2 of $\pi_1(U, x_0)$), so one has $\pi_1(X) \simeq \mathbb{Z}*\mathbb{Z}/[\mathbb{Z}*\mathbb{Z}, \mathbb{Z}*\mathbb{Z}] \simeq \mathbb{Z}^2$, as we have already seen. **(5)** *(Real projective line)* Let $\mathbb{P}^1$ be the real projective line, endowed with the quotient topology with respect to natural map $p : \mathbb{R}^2_\times \to \mathbb{P}^1$. Let $q = p|_{\mathbb{S}^1}$ (Hopf map): since $q : \mathbb{S}^1 \to \mathbb{P}^1$ is continuous, surjective and closed, then q is still quotient. The map $(\,\cdot\,)^2 : \mathbb{S}^1 \to \mathbb{S}^1$ is also quotient for the same reason, and has exactly the same fibers of q: it follows that $\mathbb{S}^1$ and $\mathbb{P}^1$ are canonically homeomorphic. hence $\pi_1(\mathbb{P}^1) \simeq \mathbb{Z}$. **(6)** *(Real projective plane)* Let $\mathbb{P}^2$ be the projective plane, endowed with the quotient topology with respect to the map $p : \mathbb{B}^2 \to \mathbb{P}^2$ obtained by identifying in $\mathbb{P}^2$ the pairs of antipodal points on $\mathbb{S}^1 \simeq \partial\mathbb{B}^2$ (if $\underline{x} = (x, y) \in \mathbb{B}^2$ and $[x_0, x_1, x_2]$ are homogeneous coordinates in $\mathbb{P}^2$ one can set $p(\underline{x}) = [1 - |\underline{x}|, x, y]$): note that $p(\mathbb{S}^1) \simeq \mathbb{P}^1$ (one can also identify $p|_{\mathbb{S}^1}$ with the Hopf map). The quotient map p is closed but not open; nevertheless, if $\mathbb{B}^2_\times = \mathbb{B}^2 \setminus \{0\}$ and $\dot{\mathbb{B}}^2 = \mathbb{B}^2 \setminus \mathbb{S}^1$, then $U = p(\mathbb{B}^2_\times)$ and $V = p(\dot{\mathbb{B}}^2)$ are open in $\mathbb{P}^2$.[36] From now on let us choose $\frac{1}{2} \in \dot{\mathbb{B}}^2_\times = U \cap V \subset \mathbb{C}$ as base point for the computation of fundamental groups. The set V (homeomorphic image of $\dot{\mathbb{B}}^2$) is clearly contractible, while $U \cap V$ (homeomorphic image of $\dot{\mathbb{B}}^2_\times$) has fundamental group $\simeq \mathbb{Z}$ generated by $[\gamma]$ obtained by (the image by p of) $t \mapsto \frac{e^{2\pi i t}}{2}$. As for $\mathbb{B}^2_\times$, it strong deformation-retracts to $\mathbb{S}^1$ by the affine homotopy $h(\underline{x}, t) = (1-t)\underline{x} + t\frac{\underline{x}}{|\underline{x}|}$, homotopy which descends via p to a strong deformation retraction $\tilde{h}$ of U a $p(\mathbb{S}^1) \simeq \mathbb{P}^1$.[37] Let $r = \tilde{h}(\,\cdot\,, 1) : U \to p(\mathbb{S}^1)$: by $r_\# : \pi_1(U, \frac{1}{2}) \xrightarrow{\sim} \pi_1(p(\mathbb{S}^1), 1) \simeq \mathbb{Z}$, the canonical generator of the second member comes from the generator $[\psi]$ of $\pi_1(U, \frac{1}{2})$ obtained by $t \mapsto p(e^{\pi i t}/2)$: hence $\iota_{U\cap V, U\,\#}$ sends the generator $[\gamma]$ in $[\psi]^2$, and hence $\pi_1(\mathbb{P}^2) \simeq \mathbb{Z}/2\mathbb{Z}$ (analogously to $\mathbb{P}^3$ and, as we shall show, to any $\mathbb{P}^n$ with $n \geq 2$). **(7)** *(Removing one or two annuli from $\mathbb{R}^3$)* If A is an annulus in $\mathbb{R}^3$ and $X = \mathbb{R}^3 \setminus A$, we already computed above that $\pi_1(X) \simeq \mathbb{Z}$: another method is to observe that X can be deformation-retracted first to a 2-sphere $\mathbb{S}^2$ plus a diameter, then to the a wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2$ (by slowly approaching the endpoints of the diameter along an equator), hence $\pi_1(X) \simeq \pi_1(\mathbb{S}^1) * \pi_1(\mathbb{S}^2) \simeq \mathbb{Z}$.

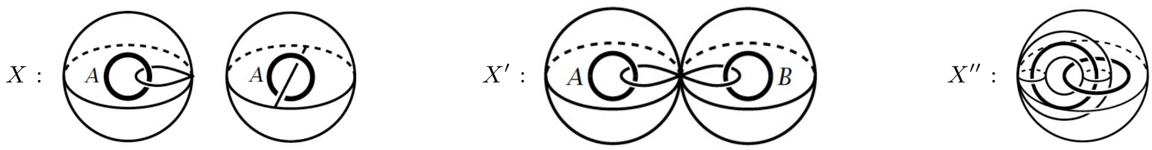


Figure 5: Deforming $\mathbb{R}^3$ minus one annulus; minus two unlinked annuli; minus two linked annuli.

Let us use the same approach for two other similar situations. • If B is another annulus of $\mathbb{R}^3$ unlinked with A, then $X' = \mathbb{R}^3 \setminus (A \sqcup B)$ can be deformation-retracted to $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^2$ and hence $\pi_1(X')$ is free on two generators. • If C is a third annulus of $\mathbb{R}^3$ linked with A, then $X'' = \mathbb{R}^3 \setminus (A \sqcup C)$ can be deformation-retracted to $\mathbb{S}^2 \vee \mathbb{T}^2$ and hence $\pi_1(X'')$ is isomorphic to $\pi_1(\mathbb{T}^2)$, i.e. a free abelian group of rank two. **(8)** *(Klein bottle)* Let us compute the fundamental group of the *Klein bottle* K by using its description in terms of fundamental polygon (i.e. a quotient of a polygon); the argument will be suitable to compute again the fundamental group of the torus $\mathbb{T}^2$ (see Figure 6). In both fundamental polygons take

[36] The map p is closed since $\mathbb{B}^2$ is compact and $\mathbb{P}^2$ is Hausdorff (the finite points of $\mathbb{P}^2$ have the same neighborhoods of the points of $\dot{\mathbb{B}}^2$, while a basis of neighborhoods of a point at infinity $p(\underline{x})$ with $\underline{x} \in \mathbb{S}^1$ is given by $A \cup (-A)$ where $A = \mathbb{B}^2 \cap U$ with $U \subset \mathbb{C}$ a small open ball centered in $\underline{x}$; hence it is still possible to separate the points of $\mathbb{P}^2$), but p is not open (the above A is open in $\mathbb{B}^2$, but its p-saturated $p^{-1}(p(A)) = A \cup (-A \cap \mathbb{S}^1)$ is not open: hence $p(A)$ is not open in the (quotient) topology of $\mathbb{P}^2$). On the other hand the open subsets $\mathbb{B}^2_\times$ and $\dot{\mathbb{B}}^2$ are already p-saturated, hence their images by p are open in $\mathbb{P}^2$.

[37] Recall the factorization property of quotient functions (Proposition 1.1.14): given a quotient function $f : X \to Y$ and a continuous function $g : X \to Z$, there exists a unique continuous function $h : Y \to Z$ such that $f = h \circ g$ if and only if g is constant on the fibers of f. Here we mean $X = \mathbb{B}^2_\times \times I$, $Y = U \times I$, $Z = U$, $f = p \times \mathrm{id}_I$ and $g = p \circ h$, and the factorization hypothesis are satisfied. The situation would be different if we would instead consider the strong deformation retraction of $\mathbb{B}^2_\times$ to $\alpha\mathbb{S}^1$ for a $0 < \alpha < 1$ (e.g. $\alpha = \frac{1}{2}$), for example the affine one $h_\alpha(\underline{x}, t) = (1-t)\underline{x} + \alpha t\frac{\underline{x}}{|\underline{x}|}$: namely, note that $p \circ h_\alpha$ is not constant on the fibers of $p \times \mathrm{id}_I$, since $(p \times \mathrm{id}_I)(\underline{x}, t) = (p \times \mathrm{id}_I)(-\underline{x}, t)$ but $p(h_\alpha(\underline{x}, t)) = -p(h_\alpha(-\underline{x}, t)) \neq p(h_\alpha(-\underline{x}, t))$ for any $\underline{x} \in \mathbb{S}^1$ and $t \in I$.

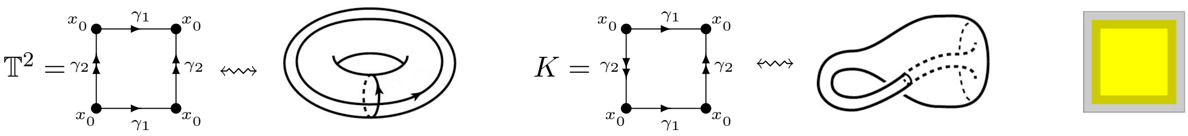


Figure 6: The torus, the Klein bottle and a suitable open cover for both.

U to be a central open square (yellow) whose edges are at some small distance $\delta > 0$ from the boundary; and V to be the open square crown (grey) of the points of the polygon whose distance from the boundary is $< 2\delta$. It is clear that U is contractible, and that $U \cap V$ (the overlapped yellow-grey zone) is homotopically equivalent to a circle; on the other hand, V can be deformation-retracted to the boundary, which can be identified to a "figure eight" (a bouquet of two circles) and hence has fundamental group free on two generators, i.e. $\mathbb{Z} * \mathbb{Z}$. So, by Corollary 1.5.4, the fundamental group is in both cases $(\mathbb{Z} * \mathbb{Z})/N$ where N is the normal subgroup generated by the image of $\pi_1(U \cap V) \simeq \mathbb{Z}$: hence, what makes the difference between $\mathbb{T}^2$ and K will be the different images of $\pi_1(U \cap V)$ into $\pi_1(V)$. Namely, a generator of $\pi_1(U \cap V)$ is a square-shaped loop, e.g. run counterclockwise: when deformed on the boundary, this loop becomes $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ in the case of $\mathbb{T}^2$, and $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2$ in the case of K. Hence for $\mathbb{T}^2$ the subgroup N is generated by the commutator of $[\gamma_1]$ and $[\gamma_2]$, and hence $\pi_1(\mathbb{T}^2)$ is the abelianization of $\mathbb{Z} * \mathbb{Z}$, i.e. $\mathbb{Z}^2$ (as we saw above); while $\pi_1(K)$ is the group with generators $a = [\gamma_1]$ and $b = [\gamma_2]$ with relation $aba^{-1}b = \text{id}$, i.e. $bab = a$. **(9)** *(g-fold torus)* A g-fold torus is a orientable closed surface of genus g; its fundamental polygon is a $4g$-gon with pairwise identifications of edges allowing g junctions naturally generalizing the one of the (1-)torus (the Figure 7 shows the case $n = 2$). To compute the fundamental group of the 2-fold torus from

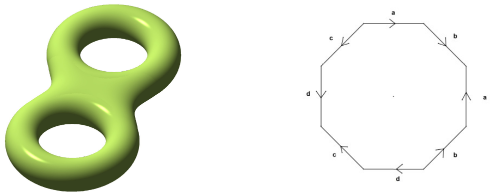


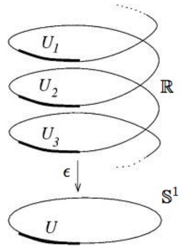
Figure 7: The double torus.

its fundamental polygon we can proceed exactly as we did above for the Klein bottle: U is contractible, $U \cap V$ is homotopically equivalent to a circle, while V can be deformation-retracted to the boundary, which in this case can be identified to a bouquet of four circles and hence has fundamental group free on four generators. Since a counterclockwise loop generating $\pi_1(U \cap V)$, when deformed on the boundary, becomes $bab^{-1}a^{-1}cdc^{-1}d^{-1}$, we get that the fundamental group of the 2-fold torus is the free group generated by a, b, c, d modulo the normal subgroup generated by $bab^{-1}a^{-1}cdc^{-1}d^{-1}$. More generally, the fundamental group of the g-fold torus is the free group generated by $a_1, b_1, \cdots, a_g, b_g$ modulo the normal subgroup generated by $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$. **(10)** *(Spaces with fundamental group $\mathbb{Z}/n\mathbb{Z}$)* Given any $n \in \mathbb{N}$, using the above technique of fundamental polygons it is then easy to construct a space whose fundamental group is $\mathbb{Z}/n\mathbb{Z}$: just consider a regular n-gon and identify all its edges with a chosen direction (e.g. conterclockwise). Namely, here we have $\pi_1(V) \simeq \mathbb{Z}$ (say with generator a) and the image of a generator of $\pi_1(U \cap V) \simeq \mathbb{Z}$ into $\pi_1(V)$ is a^n, hence the quotient $\pi_1(V)/N$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. **(11)** *(Graphs)* In a connected graph, a *tree* is a contractible subgraph; a tree is called *maximal* if it contains all vertices of X. If T is a maximal tree in a connected graph X, let $\{d_\lambda : \lambda \in \Lambda\}$ be the family of edges of $X - T$: then $\pi_1(X)$ is a free group with generators $[\gamma_\lambda]$ corresponding to each edge d_λ. This can be proved by

Figure 8: Graphs and maximal trees.

considering, for any $\lambda \in \Lambda$, an open neighborhood U_λ of $T + d_\lambda$ which deformation-retracts to $T + d_\lambda$: then each U_λ deformation-retracts into a circle, and the intersections of two or more U_λ's is contractible since it deformation-retracts to T. For example, the fundamental group of the graph X on the left of Figure 8 is free on four generators, each one corresponding to a loop containing only one of the edges not in any chosen maximal tree (whose edges are represented in black). Similarly, the graph Y on the right of Figure 8 — which can also be interpreted as the suspension of the three red vertices P_j with $j = 1, 2, 3$ — has fundamental group free on two generators (a maximal tree is depicted in black). As for this last example note that, setting $U_j = Y \setminus \{P_j\}$ (for $j = 1, 2, 3$), then the open cover $\{U_1, U_2\}$ is suitable for applying Van Kampen and confirms that $\pi_1(Y) \simeq \mathbb{Z} * \mathbb{Z}$, while the open cover $\{U_1, U_2, U_3\}$ is not suitable since $U_1 \cap U_2 \cap U_3 = Y \setminus \{P_1, P_2, P_3\}$ is not arcwise connected (hence one cannot conclude that $\pi_1(Y) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, a statement that would be false).

1.6 Covering spaces



The prototype of a covering space is the exponential map $\epsilon : \mathbb{R} \to \mathbb{S}^1$ given by $\epsilon(t) = e^{2\pi i t}$: the key property is that any small open interval of $\mathbb{S}^1$ is "well-covered" by this map, i.e. its inverse image is a family of pairwise disjoint homeomorphic copies of itself. We have already used this map to prove (see Proposition 1.4.2) that the fundamental group of $\mathbb{S}^1$ is free with one generator: in fact, we shall see that there is a deep relation between the classification of the covering spaces of some topological space X and the structure of the fundamental group of X.

1.6.1 Fiber bundles and covering spaces

Definition 1.6.1. Let X be a topological space. A *space on* X is a pair (Y, π) where Y is a topological space and $\pi : Y \to X$ is a surjective continuous function. A morphism from (Y_1, π_1) to (Y_2, π_2) is a continuous function $f : Y_1 \to Y_2$ such that $\pi_1 = \pi_2 \circ f$.

Given $x \in X$ and a space (Y, π) on X, we denote by $Y_x = \pi^{-1}(x)$ the *fiber* on x. Note that a morphism of spaces on X respects the fibers, in the sense that $f(Y_{1,x}) \subset Y_{2,x}$; in particular, if f is a isomorphism, for any $x \in X$ it is induced a homeomorphism $f_x : Y_{1,x} \xrightarrow{\sim} Y_{2,x}$. The simplest case of space on X is the one of type $(X \times F, p_X)$ where F is a topological space and p_X the projection on X. More generally:

Definition 1.6.2. A space (Y, π) on X is called *trivial* if there exists a topological space F and an isomorphism $f : (Y, \pi) \xrightarrow{\sim} (X \times F, p_X)$: in this case, such an isomorphism of spaces on X is called a *trivialization* of (Y, π).

Anyway, the most important notion is the one of "locally trivial space", or "fiber bundle".

Definition 1.6.3. Given a space (Y, π) on X and an open subset $U \subset X$, the *restriction* of (Y, π) to U (sometimes denoted by $Y|_U$) is the space on U given by $(\pi^{-1}(U), \pi|_{\pi^{-1}(U)})$. The space (Y, π) on X is called *locally trivial* (or also *fiber bundle*, or *bundle*) on X if there exists an open cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of X such that $Y|_{U_\lambda}$ is trivial for any $\lambda \in \Lambda$; i.e., for any $x \in X$ there exists an open neighborhood $U \subset X$ of x such that $Y|_U$ is trivial. A *local trivialization* of (Y, π) on U_λ is a trivialization of $Y|_{U_\lambda}$.

If the space (Y, π) on X is trivial, then obviously the map π is open and all fibers of (Y, π) on X are homeomorphic.[38] This is still true for any bundle on an arcwise connected space:

[38] If $f : (Y, \pi) \xrightarrow{\sim} (X \times F, p_X)$ is a trivialization, then all fibers of Y are homeomorphic to F (the fiber of $X \times F$). As for the openness, since f is a homeomorphism we are left with proving that $p_X : X \times F \to X$ is open. Let V be an open subset of $X \times F$, and $(x, f) \in V$: then there exist open subsets $U \subset X$ and $W \subset F$ such that $(x, f) \in U \times W \subset V$, and hence $U = p_X(U \times W) \subset p_X(V)$. Therefore $p_X(V)$ is a neighborhood of $p_X(x, f) = x$ because it contains U (an open neighborhood of x), and this proves that $p_X(V)$ is open.

Proposition 1.6.4. *If (Y, π) is a bundle on X, then π is open. Moreover, if X is arcwise connected, then all fibers of (Y, π) on X are homeomorphic.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X such that (Y, π) is trivial on every $U \in \mathcal{U}$. Let $V \subset Y$ be open, $y \in V$ and $U \in \mathcal{U}$ such that $x = \pi(y) \in U$: then $W = \pi^{-1}(U) \cap V$ is an open neighborhood of y, and since the map $\pi|_{\pi^{-1}(U)}$ is open and $W \subset \pi^{-1}(U)$, one has that $\pi(W) \subset U \cap \pi(V) \subset \pi(V)$ is an open neighborhood of x. Hence $\pi(V)$ is open in X. Now let X be arcwise connected; given $x_0, x_1 \in X$, let $\gamma : I \to X$ be a path between them and let $\lambda_0, \dots, \lambda_k \in \Lambda$ be such that $\gamma(I) \subset \bigcup_{j=0}^{k} U_{\lambda_j}$, $x_0 \in U_{\lambda_0}$, $x_1 \in U_{\lambda_k}$ and $U_{\lambda_j} \cap U_{\lambda_{j+1}} \neq \varnothing$. We are left with proving that $Y|_{U_{\lambda_j}}$ and $Y|_{U_{\lambda_{j+1}}}$ are isomorphic if further restricted to $U_{\lambda_j} \cap U_{\lambda_{j+1}}$: it is enough to observe that, given local trivializations $\psi_j : \pi^{-1}(U_{\lambda_j}) \xrightarrow{\sim} U_{\lambda_j} \times F_j$, the isomorphism $\psi_{j+1} \circ \psi_j^{-1}$ of trivial spaces on $U_{\lambda_j} \cap U_{\lambda_{j+1}}$ induces a homeomorphism $F_j \xrightarrow{\sim} F_{j+1}$. □

If (Y, π) is a bundle on X, then Y is usually referred to as the "total space" and X as the "base" of the bundle; moreover, if X is arcwise connected, thanks to Proposition 1.6.4 one directly talks about "bundle with fiber F" or "F-bundle", where F is a topological space homeomorphic to the fibers of π.

Remark 1.6.5. Since here we are interested only in topological matters, in the previous brief exposition of the notion of bundle we have not paid so much attention to the structure of the fiber, which has been required to be nothing more than a topological space. In other words: we saw that, if (Y, π) is a bundle on the arcwise connected space X with fiber F and U_1, U_2 are two open subsets of X with trivializations $\psi_j : \pi^{-1}(U_j) \xrightarrow{\sim} U_j \times F$, then for any $x \in U_1 \cap U_2$ there is an induced *homeomorphism* $(\psi_2 \circ \psi_1^{-1})(x, \cdot) : F \xrightarrow{\sim} F$, and we have not requested this homeomorphism to respect also possible further structures of F (hence, for example, that it should be a linear map if F is a vector space, or a orthogonal transformation if F is a sphere). In a more motivated exposition the structure of F has to be respected by these homeomorphisms, which are commonly called *transition functions.* In fact, the proper notion of bundle requires also a group structure operating effectively on the fiber.[39] More precisely, a *bundle of base X with total space Y, fiber F and group structure G* (where X, Y and F are topological spaces and G is a topological group) is the datum of:

(1) a space (Y, π) on X with fibers homeomorphic to F;

(2) an effective action of G as a group of homeomorphisms on F;

(3) an open cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of X with a family of local trivializations $\psi_\lambda : \pi^{-1}(U_\lambda) \xrightarrow{\sim} U_\lambda \times F$ (the map ψ_λ is usually called a *local chart* of Y over U_λ);

(4) for any $\lambda, \mu \in \Lambda$ such that $U_\lambda \cap U_\mu \neq \varnothing$, a continuous transition function $\alpha_{\lambda,\mu} : U_\lambda \cap U_\mu \to G$ such that $\psi_\lambda \psi_\mu^{-1}(x, t) = (x, \alpha_{\lambda,\mu}(x) \cdot t)$ for any $x \in U_\lambda \cap U_\mu$ and $t \in F$.

In particular, the bundle will be called: (a) *vector bundle* if F is a real or complex euclidean space (for example, real vector bundle *of rank n* if the fiber is $\mathbb{R}^n$) and G is the general linear group (or a subgroup of it) of the same euclidean space; (b) *sphere bundle* if F is a sphere in an euclidean space and G is the orthogonal group (or a subgroup of it) of the same euclidean space; (c) *principal bundle* if F is the same group G operating on itself

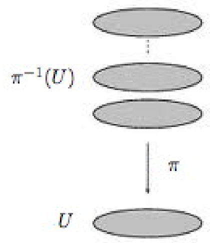
[39] Recall that a *(left) action* of a group G on a topological space F is a morphism from G to the group of autohomeomorphisms of F: in other words, the identity element of G acts as the identity of F, and $g_1(g_2(f)) = (g_1 g_2)(f)$ for any $g_1, g_2 \in G$ and $f \in F$. The action is called *effective* if the only element of G which operates trivially on F is the identity.

by right translation. In the case of particular structures on Y, X, F and G (e.g. if they are real or complex manifolds, or algebraic varieties) it is usual to require more regularity than continuity to the trivializations and to the transition functions, and so one can also talk about continuous, differentiable, holomorphic, algebraic ... bundles.

Examples. **(1)** If $H \subset G$ are closed subgroups of $GL(n;\mathbb{C})$ and $\pi : G \to G/H$ is the canonical projection, then (G, π) is a bundle with fiber H and structure group $\mathcal{N}(H)/H$, where $\mathcal{N}(H)$ is the normalizer of H in G (see for example Bredon [2, II.14, pp. 110-111]). **(2)** Given a real manifold M of class $\mathcal{C}^1$ and dimension n, the tangent bundle $TM = \{(x,v) : x \in M,\, v \in T_xM\}$ and the cotangent bundle $T^*M = \{(x,\alpha) : x \in M,\, \alpha \in T_x^*M\}$ are real vector bundles on M of rank n; if $N \subset M$ is a submanifold of dimension k, there are real vector bundles on N of rank $n-k$ by considering the normal bundle $T_NM = \{(x,v) : x \in N,\, v \in T_xM/T_xN\}$ and the conormal bundle $T_N^*M = \{(x,\alpha) : x \in N,\, \alpha \in (T_xN)^\perp \subset T_x^*M\}$. Note that the tangent bundle $T\mathbb{S}^1$ is trivial, being homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ by $\mathbb{S}^1 \times \mathbb{R} \xrightarrow{\sim} T\mathbb{S}^1$, $((x,y),t) \mapsto ((x,y),(-ty,tx))$.

From now on we shall assume that the topological space X is arcwise connected.

Definition 1.6.6. A *covering space* of X is a bundle (Y, π) on X with discrete fiber. In this case one also says that $\pi : Y \to X$ is a *covering map.* The cardinality of the fibers (well-defined thanks to Proposition 1.6.4) is called *multiplicity* of the covering (if such multiplicity is finite, say n, one also talks about a "n-sheet covering"). A *morphism* of covering spaces on X is a morphism as spaces on X.



In other words, the fact that $\pi : Y \to X$ is a covering map means that for any $x \in X$ there exists a neighborhood $U \subset X$ of x such that $\pi^{-1}(U)$ is the disjoint union of homeomorphic copies of U, i.e. $\pi^{-1}(U) = \bigsqcup_{\lambda\in\Lambda} V_\lambda$, with $\pi|_{V_\lambda} : V_\lambda \xrightarrow{\sim} U$: such an open subset U is said to be *evenly covered.* It is then clear that, if (Y, π) is a covering space of X and $U \subset X$ is open, then $Y|_U$ (see Definition 1.6.3) is a covering space of U.

Proposition 1.6.7. *Any covering space is a local homeomorphism.*[(40)] *Conversely, a local homeomorphism (Y, π) where Y is Hausdorff and whose fibers are finite sets with the same cardinality is a covering.*

Proof. The first statement follows immediately from the definitions and Proposition 1.6.4. For the second, let k be the cardinal of the fibers; fixed $x_0 \in X$, let $\pi^{-1}(x_0) = \{y_1, \dots, y_k\}$. Thanks to the hypotheses, we may choose by recurrence some neighborhoods $V_{y_j} \subset Y$ of y_j such that $\pi|_{V_{y_j}} : V_{y_j} \xrightarrow{\sim} U_{y_j} = \pi(V_{y_j})$ and $V_{y_j} \subset Y \setminus \overline{\bigcup_{i=1}^{j-1} V_{y_i}}$. So set $U = \bigcap_{j=1}^k U_{y_j} \subset X$ and $V_j = \pi^{-1}(U) \cap V_{y_j}$: it clearly holds $\pi^{-1}(U) = \bigsqcup_{j=1}^k V_j$, and $\pi|_{V_j} : V_j \to U$ is a homeomorphism. □

Examples. **(1)** Let $p(z) = \sum_{j=0}^n a_j z^j$ be any polynomial with complex coefficients and $a_n \neq 0$, and let $\Gamma = p(p'^{-1}(0))$ (the set of critical values of p). Then, setting $X = \mathbb{C}\setminus\Gamma$ and $Y = p^{-1}(X)$, the space (Y, p) is a n-sheet covering space of X. For example, if $p(z) = z^n$ the open subset which are evenly covered are those $U \subset X = \mathbb{C}^\times$ such that the inclusion map $j : U \hookrightarrow X$ is nullhomotopic, i.e. those which do not contain

---

(40) A space (Y, π) on X is called *local homeomorphism* if π is open and if for any $y \in Y$ there exists an open neighborhood $V \subset Y$ of y such that $\pi|_V : V \to \pi(V)$ is a homeomorphism.

loops of nonzero index in 0.[41] Among them we find all simply connected open subsets of X, for example also the "tickened spiral" $\{z = re^{i\theta} \in X : \theta \in \mathbb{R}_{>0},\ \theta - \phi(\theta) < r < \theta + \phi(\theta)\}$ where $\phi : \mathbb{R}_{>0} \to]0,1[$ is any strictly increasing continuous function with $\phi(\theta) < \frac{\theta}{2}$, ($\lim_{\theta\to 0^+} \phi(\theta) = 0^+$ and) $\lim_{\theta\to+\infty} \phi(\theta) = 1^-$, or also the open subsets U of X contained in $U_\alpha = \mathbb{C} \setminus \{re^{i\alpha} : r \geq 0\} \subset X$ for some $\alpha \in \mathbb{R}$. **(2)** The map $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a covering of X of countable multiplicity: for any $\alpha \in \mathbb{R}$, the already known open subsets $U_\alpha = \mathbb{C} \setminus \{re^{i\alpha} : r \geq 0\} \subset X$ are evenly covered, being $\exp^{-1}(U_\alpha) = \bigsqcup_{k\in\mathbb{Z}}\{z \in \mathbb{C} : \alpha + 2k\pi < \mathrm{Im}(z) < \alpha + 2(k+1)\pi\}$. If one then considers the open subset $U = \mathbb{C} \setminus \{0,1\} \subset X$, one has $\exp^{-1}(U) = \mathbb{C} \setminus 2\pi i\mathbb{Z}$: hence $\exp : \mathbb{C} \setminus 2\pi i\mathbb{Z} \to \mathbb{C} \setminus \{0,1\}$, as the restriction of a covering space, is itself a covering space. **(3)** The above covering spaces of $\mathbb{C}^\times$ induce covering spaces of $\mathbb{S}^1$, which are respectively the maps $z^n : \mathbb{S}^1 \to \mathbb{S}^1$ (n-fold) and $\epsilon : \mathbb{R} \to \mathbb{S}^1$, $\epsilon(t) = e^{2\pi it}$ (countable). In fact, we shall show that these examples exhaust (up to isomorphism) all connected covering spaces of $\mathbb{S}^1$. Given $z_0 \in \mathbb{S}^1$, the open subset $\mathbb{S}^1 \setminus \{z_0\}$ is evenly covered by these covering spaces: for example, if $z_0 = 1$ then $\epsilon^{-1}(\mathbb{S}^1 \setminus \{1\}) = \bigsqcup_{k\in\mathbb{Z}}]k, k+1[$, and $(z^n)^{-1}(\mathbb{S}^1 \setminus \{1\}) = \bigsqcup_{k=0}^{n-1}\{e^{i\theta} : \frac{2k\pi}{n} < \theta < \frac{2(k+1)\pi}{n}\}$.

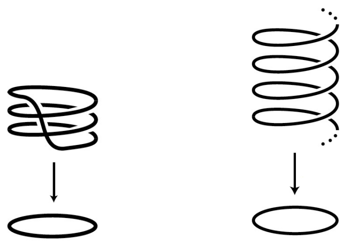


Figure 9: Connected covering spaces of the circle.

(4) Let H be a discrete topological group operating on the left on a topological space Y in a properly discontinuous way (i.e., for any $y \in Y$ there exists a open neighborhood $V \subset Y$ of y such that $g_1V \cap g_2V = \varnothing$ if $g_1 \neq g_2$). Let X be the space of orbits of H in Y, endowed with the quotient topology: then the canonical projection $\pi : Y \to X$ is a covering space. Namely, given $y \in Y$ let V be an open neighborhood of y with the properties of discontinuity just defined, and let $U = \pi(V)$ (an open neighborhood of $\pi(y)$, because π is open): one has $\pi^{-1}(U) = \bigsqcup_{g\in H} gV$, and $\pi|_{gV} : gV \to U$ is a homeomorphism.[42] For example, let $Y = G$ be a topological group and be H a discrete subgroup operating by multiplication on the left: such action is properly discontinuous, and the projection $\pi : Y = G \to X = G/H$ is a covering.[43] In the case (2), we had $Y = \mathbb{C}$ and $H = 2\pi i\mathbb{Z}$ operating by translation; in the case (3), we had $Y = \mathbb{R}$ and $H = \mathbb{Z}$. **(5)** Setting $Y =]0,2[$ and $X = \mathbb{S}^1$, the map $\pi = \epsilon|_{]0,2[} : Y \to X$, $\pi(t) = e^{2\pi it}$ is a local

---

[41] Suppose that $U \subset X$ is an open subset containing a loop γ (say based at a point z) of index nonzero in 0. We shall show that, for $w \in p^{-1}(z)$ (i.e. $w^n = z$), there exists a unique "lifting" of γ based at w, i.e. a path δ completely contained in the inverse image $V = p^{-1}(U)$ in $Y = \mathbb{C}$ such that $\delta(0) = w$ and $p \circ \delta = \gamma$ (here the computation can be performed also explicitly: if $\delta(t) = r(t)e^{i\theta(t)}$ and $\gamma(t) = \rho(t)e^{i\varphi(t)}$ with $\gamma(0) = \gamma(1) = z$, from $p \circ \delta = \gamma$ one gets $r(t) = \sqrt[n]{\rho(t)}$ and $\theta(t) = \frac{\varphi(t)+2k\pi}{n}$ for some $0 \leq k \leq n-1$, and the good k can be found by requiring that $\delta(0) = w$), whose extremity is another w' in the inverse image of z certainly different from w (namely, if $w' = w$ then δ should be nullhomotopic because $\mathbb{C}$ is simply connected, and hence also $\gamma = p \circ \delta$ would be nullhomotopic): but then V could not be a disjoint union of copies homeomorphic to U by p, hence U is not evenly covered. Conversely, if U is not evenly covered there exist a point z in U, two distinct points w and w' in the inverse image $p^{-1}(z)$ of z and a path α from w to w' completely contained in V. Now, $p \circ \alpha$ is surely a loop in U based at z; on the other hand, if ψ is the shortest path from w to w' along the circle containing both of them, it is clear that α and ψ are paths homotopic rel ∂I (because $\mathbb{C}$ is simply connected) hence also $p \circ \alpha$ and $p \circ \psi$ are loops homotopic rel ∂I: but $p \circ \psi$ is the loop based at z which describes the circle one or more times, hence its index in 0 is nonzero, and hence also the index in 0 of $p \circ \alpha$ is nonzero.

[42] It is clearly continuous, open and surjective; it is also injective, because from $\pi(gy_1) = \pi(gy_2)$ one gets $gy_2 = hgy_1$ for some $h \in H$, hence $gU \cap hgU \neq \varnothing$, hence $g = hg$, i.e. $h = e$ and $gy_1 = gy_2$.

[43] This fact will explain in a more general framework the properties of the maps of canonical projection (see §1.4) of lifting uniquely paths and homotopies. In that case we were considering the *right* classes (i.e. $G/H = \{gH : g \in G\}$), hence H was acting on G on the right instead than on the left.

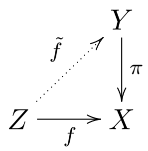
homeomorphism with discrete fibers, but it is not a covering (the cardinality of the fibers is not the same for any point). The same conclusion holds with $Y = \mathbb{R}_{>0}$ (no neighborhood of 1 is evenly covered). **(6)** Setting $Y = \{(z,w) \in \mathbb{C}^2 : z = w^2\}$ and $V = \{(z,w) \in \mathbb{C}^2 : z = w^2,\ z \neq 0\}$ (open subset of Y), the first projection $p_1 : Y \to \mathbb{C}$ ($(z,w) \mapsto z$) is not a covering space, while $p_1 : V \to \mathbb{C}^\times$ is: namely in the first case the map is even not a local homeomorphism, while in the second there is a local homeomorphism with fibers finite and of the same cardinality.[44]

1.6.2 Liftings and the Monodromy lemma

A crucial feature of covering maps $\pi : Y \to X$ is that they are able to *lift* maps from the base X to the full space Y in a *unique way*.

Let X be an arcwise connected topological space.

Definition 1.6.8. Let (Y,π) be a space on X, $f : Z \to X$ a continuous function. A *lifting* of f by π is a continuous function $\tilde{f} : Z \to Y$ such that $f = \pi \circ \tilde{f}$. In particular, if $Z = A \subset X$ and $f = \iota_A$ is the canonical inclusion, a lifting of ι_A is called a (continuous) *section* of π over A.



Example. If X is a differential manifold and $U \subset X$ is open, a section of the tangent bundle on U is a vector field in U.

Proposition 1.6.9. *If $\pi : Y \to X$ is a local homeomorphism, two liftings of $f : Z \to X$ which coincide in one point, coincide in a whole neighborhood of the point itself. If moreover Z is connected and π is a covering space, or if Z is connected and Y is Hausdorff, then they are equal.*

Proof. If $\tilde{f}_1(z_0) = \tilde{f}_2(z_0) = y_0$, and if $V \subset Y$ is a neighborhood of y_0 on which $\pi : V \xrightarrow{\sim} U = \pi(V)$ is a homeomorphism, then $\tilde{f}_1$ and $\tilde{f}_2$ must necessarily coincide on the neighborhood $W = f^{-1}(U) \cap \tilde{f}_1^{-1}(V) \cap \tilde{f}_2^{-1}(V)$ of z_0. This says that $Z' = \{z \in Z : \tilde{f}_1(z) = \tilde{f}_2(z)\}$ is an open subset of Z. If π is a covering space, or if Y is of Hausdorff, Z' is also a closed subset of Z (in the first case, if $z \in Z \setminus Z'$ let $U \subset X$ be an evenly covered neighborhood of $f(z)$: then $\pi^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} V_\lambda$ with $\tilde{f}_1(z) \in V_{\lambda_1}$ and $\tilde{f}_2(z) \in V_{\lambda_2}$ (where $\lambda_1 \neq \lambda_2$), so that $z \in \tilde{f}_1^{-1}(V_{\lambda_1}) \cap \tilde{f}_2^{-1}(V_{\lambda_2}) \subset Z \setminus Z'$, i.e. $Z \setminus Z'$ is open; in the second see Lemma 1.2.2), and this implies that $\tilde{f}_1 = \tilde{f}_2$ because Z is connected. $\square$

Definition 1.6.10. The space (Y,π) on X *has the property of lifting paths (uniquely)* if for any path $\gamma : I \to X$ and any initial point $y_0 \in \pi^{-1}(\gamma(0))$ there exists a (unique) path

[44] π_1 is not a local homeomorphism in $(0,0)$; while it is in $(z_0, w_0) \in V$, by taking as neighborhood a small open ball not containing $(0,0)$.

$\tilde{\gamma}_{y_0} : I \to Y$ such that $\gamma = \pi \circ \tilde{\gamma}_{y_0}$ and $\tilde{\gamma}_{y_0}(0) = y_0$; if Z is a topological space, we say that (Y, π) *has the property of lifting homotopies (uniquely) with respect to* Z if for any homotopy $h : Z \times I \to X$ and any lifting $\alpha_0 : Z \to Y$ of the base h_0 (i.e., $\pi \circ \alpha_0 = h_0$) there exists a (unique) homotopy $\tilde{h}_{\alpha_0} : Z \times I \to Y$ such that $h = \pi \circ \tilde{h}_{\alpha_0}$ and $(\tilde{h}_{\alpha_0})_0 = \alpha_0$.

Proposition 1.6.11. *Let (Y, π) be a space on X which lifts paths uniquely, and let $\gamma, \phi : I \to X$ be two paths with $x_0 = \gamma(0)$ and $x_1 = \gamma(1) = \phi(0)$. Then, given $y_0 \in Y_{x_0}$ and set $y_1 = \tilde{\gamma}_{y_0}(1) \in Y_{x_1}$, it holds $\widetilde{(\gamma \cdot \phi)}_{y_0} = \tilde{\gamma}_{y_0} \cdot \tilde{\phi}_{y_1}$.*

Proof. Obvious. □

Lemma 1.6.12. *The covering spaces have the property of lifting paths uniquely.*

Proof. Uniqueness is given by Proposition 1.6.9; as for the existence, let $\pi : Y \to X$ be the covering space, $\gamma : I \to X$ a path in X with $\gamma(0) = x_0$, and let $y_0 \in \pi^{-1}(x_0)$. Let us define the lifting $\tilde{\gamma}_{y_0}$ piecewise. There exist $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and evenly covered open subsets $U_j \subset X$ such that $\gamma([t_{j-1}, t_j]) \subset U_j$ (where $j = 1, \dots, m$). Let $s_1 : U_1 \xrightarrow{\sim} V_1$ be the section of π over U_1 with $y_0 \in V_1$ (i.e. $s_1 = (\pi|_{V_1})^{-1}$), and set $\tilde{\gamma}_{y_0}\big|_{[t_0,t_1]} = s_1 \circ \gamma|_{[t_0,t_1]}$; then, constructed $\tilde{\gamma}_{y_0}\big|_{[t_{j-1},t_j]}$, let $s_{j+1} : U_{j+1} \xrightarrow{\sim} V_{j+1}$ be the section of π over U_{j+1} with $\tilde{\gamma}_{y_0}(t_j) \in V_{j+1}$ and set $\tilde{\gamma}_{y_0}\big|_{[t_j,t_{j+1}]} = s_{j+1} \circ \gamma|_{[t_j,t_{j+1}]}$. The path $\tilde{\gamma}_{y_0}$ obtained joining the paths $\tilde{\gamma}_{y_0}\big|_{[t_{j-1},t_j]}$ for $j = 1, \dots, m$ will be continuous by the Gluing lemma. □

Actually, a local homeomorphism lifting paths uniquely does much more:

Lemma 1.6.13. *Any local homeomorphism which has the property of lifting paths uniquely has also the property of lifting homotopies uniquely.*

Proof. Let $\pi : Y \to X$ be a local homeomorphism with the property of lifting paths uniquely, and let $h : Z \times I \to X$ be a homotopy and $\alpha_0 : Z \to Y$ with $\pi \circ \alpha_0 = h_0$. For any $z \in Z$, the path $\gamma^z : I \to X$, $\gamma^z(t) = h(z, t)$ lifts uniquely to $\tilde{\gamma}^z : I \to Y$ such that $\tilde{\gamma}^z(0) = \alpha_0(z)$: this leads necessarily to define $\tilde{h} : Z \times I \to Y$ as $\tilde{h}(z, t) = \tilde{\gamma}^z(t)$. We are left with showing the continuity of $\tilde{h}$. Now, since π is a local homeomorphism, any continuous function with values in X admits locally liftings around any point: given $(z_0, s) \in Z \times I$, let $V_s \subset Y$ be an open neighborhood of $\tilde{h}(z_0, s) = \tilde{\gamma}^{z_0}(s)$ such that $\pi|_{V_s} : V_s \xrightarrow{\sim} U_s = \pi(V_s)$, let $W_s \times J_s$ be a neighborhood of (z_0, s) such that $h(W_s \times J_s) \subset U_s$, and define $\tilde{h}_s = (\pi|_{V_s})^{-1} \circ h|_{W_s \times J_s} : W_s \times J_s \to V_s$. Observe that $\tilde{h}_s(z_0, s) = \tilde{\gamma}^{z_0}(s) = \tilde{h}(z_0, s)$: this implies that $\tilde{h}_s(z_0, t) = \tilde{h}(z_0, t)$ for any $t \in J_s$ (the paths $\tilde{h}_s(z_0, \cdot)$ and $\tilde{h}(z_0, \cdot)$ on J_s are both liftings of $h(z_0, \cdot)$ and coincide for $t = s$). Since $\{z_0\} \times I$ is compact, there exist $0 < s_1 < \cdots < s_r < 1$ such that $\{z_0\} \times I \subset \bigcup_{j=1}^r (W_{s_j} \times J_{s_j})$. As we saw above, the functions $\tilde{h}_{s_j}$ coincide on $\{z_0\} \times I$: hence the sections $(\pi|_{V_{s_j}})^{-1}$ of π must coincide on the connected compact subset $h(\{z_0\} \times I) \subset X$. But, since by Proposition 1.6.9 two sections of a local homeomorphism which coincide on a connected compact subspace coincide on a whole open neighborhood of the compact itself, there exists an open neighborhood $W_0 \subset \bigcap_{j=1}^r W_{s_j}$ of z_0 such that $\tilde{h}_{s_j}$ and $\tilde{h}_{s_{j+1}}$ coincide on $W_0 \times (J_{s_j} \cap J_{s_{j+1}})$ (where $j = 1, \dots, r-1$) giving rise in this way to a continuous function $\tilde{h}' : W_0 \times I \to Y$; but it will hold also $\tilde{h}' = \tilde{h}\Big|_{W_0 \times I}$ (again by the uniqueness of lifting of paths defined by fixing repeatedly a $z \in W_0$), and hence $\tilde{h}$ is continuous in all of $W_0 \times I$. □

Hence we get the fundamental property of covering spaces:

Proposition 1.6.14. *The covering spaces have the property of lifting homotopies uniquely.*

Proof. Follows from Lemmas 1.6.12 and 1.6.13. □

As a consequence, homotopic paths are lifted to paths ending at the same point, and even homotopic (see Figure 10):

Corollary 1.6.15. (Monodromy lemma) *Let $\pi : Y \to X$ be a covering space, α and β be two paths in X with $[\alpha] = [\beta]$. Then, if $\tilde{\alpha}$ and $\tilde{\beta}$ are liftings of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0)$, it holds also $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and, even, $[\tilde{\alpha}] = [\tilde{\beta}]$.*

Proof. Let $h : I \times I \to X$ be a homotopy rel ∂I between α and β: by Proposition 1.6.14 (with $Z = I$ in the second factor), there exists a unique homotopy $\tilde{h} : I \times I \to Y$ such that $h = \pi \circ \tilde{h}$ and $\tilde{h}(0, \tau) \equiv y_0 = \tilde{\alpha}(0) = \tilde{\beta}(0)$ for any $\tau \in I$. By the property of lifting paths uniquely, one gets $\tilde{\alpha}(t) = \tilde{h}(t, 0)$ and $\tilde{\beta}(t) = \tilde{h}(t, 1)$. Now, the map $\tilde{h}(1, \cdot) : I \to Y$ is continuous; since $h = \pi \circ \tilde{h}$, that map must take values in the (discrete) fiber of π on $\alpha(1) = \beta(1)$: hence it is constant, and in particular $\tilde{\alpha}(1) = \tilde{h}(1, 0) = \tilde{h}(1, 1) = \tilde{\beta}(1)$. □

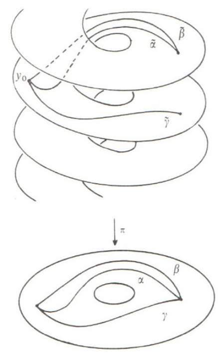


Figure 10: The Monodromy lemma.

From the Monodromy lemma it follows that the only connected covering of a simply connected space is, up to homeomorphisms, the space itself:

Corollary 1.6.16. *Let X be a simply connected topological space, $\pi : Y \to X$ a covering with Y arcwise connected. Then π is a homeomorphism.*

Proof. We already know that π is a local homeomorphism: it is enough to prove that π is injective. So let $y_1, y_2 \in Y_{x_0}$, and let $\gamma : I \to Y$ be a path from y_1 to y_2. The path $\pi \circ \gamma$ is a loop based at x_0, hence $[\pi \circ \gamma] = [c_{x_0}]$ by hypothesis. By the Monodromy lemma (with $\alpha = \pi \circ \gamma$, $\beta = c_{x_0}$, $\tilde{\alpha} = \gamma$ and $\tilde{\beta} = c_{y_1}$) we get $y_2 = \gamma(1) = c_{y_1}(1) = y_1$. □

1.6.3 Classification of covering spaces

Let X be an arcwise connected topological space and $x_0 \in X$. Let us see how the subgroups of $\pi_1(X, x_0)$ are in corrispondence with the covering spaces of X.

Proposition 1.6.17. *Let $\pi : Y \to X$ be a covering space, and $y_0 \in Y_{x_0}$. Then the morphism $\pi_\# : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Let $\pi_\#([\gamma]) = [\pi \circ \gamma] = [c_{x_0}]$: observing that $\widetilde{(\pi \circ \gamma)}_{y_0} = \gamma$ and $\widetilde{(c_{x_0})}_{y_0} = c_{y_0}$, by the Monodromy lemma one has $[\gamma] = [c_{y_0}]$. □

Definition 1.6.18. We denote by $G(Y, y_0)$ (*characteristic subgroup of the covering space*) the isomorphic image of $\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$ by $\pi_\#$:

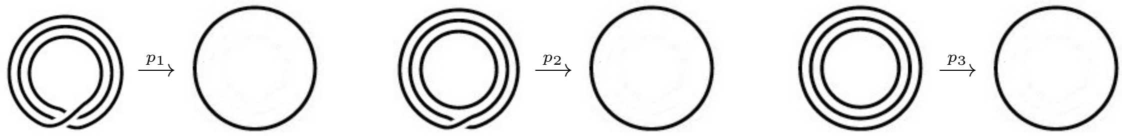
$$G(Y, y_0) = \pi_\#(\pi_1(Y, y_0)) = \{[\gamma] \in \pi_1(X, x_0) : \tilde{\gamma}_{y_0} \text{ is a loop based at } y_0\}.$$

Proposition 1.6.19. *Let $\pi : Y \to X$ be a covering space, and $y_0 \in Y_{x_0}$. Then the subgroups of $\pi_1(X, x_0)$ conjugated to $G(Y, y_0)$ are exactly the subgroups $G(Y, y_1)$ with $y_1 \in Y_{x_0}$ in the same arc-component of y_0 in Y.*

Proof. Exercise. □

Remark 1.6.20. *(Monodromy action)* The group $\pi_1(X, x_0)$ acts on the right on the fiber Y_{x_0}: in other words, there is a *monodromy* morphism $\mu : \pi_1(X, x_0) \to \mathfrak{S}_{Y_{x_0}}$, where $\mathfrak{S}_{Y_{x_0}}$ denotes the group of permutations of the fiber Y_{x_0}. This action is described as follows: given $y \in Y_{x_0}$ and γ a loop in X based at x_0, let $\tilde{\gamma}_y$ be the lifting of γ with starting point y, and define $\mu([\gamma])(y) = y \cdot [\gamma] := \tilde{\gamma}_y(1)$. Hence the stabilizer of some $y \in Y$ is precisely $G(Y, y)$. Moreover, if the covering space Y is arcwise connected then by Proposition 1.6.19 the subgroup acting trivially on Y_{x_0} is the *heart*[(45)] of $G(Y, y_0)$ in $\pi_1(X, x_0)$, for any $y_0 \in Y_{x_0}$.

Examples. **(1)** The fiber of the covering space $\exp : \mathbb{C} \to \mathbb{C}^\times$ over $z_0 = re^{i\theta}$ is the set of complex logarithms $w_k = \log r + i(\theta + 2k\pi)$ for $k \in \mathbb{Z}$, and the generator $re^{2\pi i t}$ of $\pi_1(\mathbb{C}^\times, z_0)$ sends w_k to w_{k+1}. **(2)** Let us consider the following 3-sheet covering spaces of $\mathbb{S}^1$:



where $p_1 : \mathbb{S}^1 \to \mathbb{S}^1$, $p_1(z) = z^3$; $p_2 : \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$, $p_2(z) = z^2$ or $p_2(z) = z$ according to the fact that z belongs to the first or to the second copy of $\mathbb{S}^1$; and $p_3 : \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$, $p_3(z) = z$. Then, denoting the fiber of p_j always by $\{y_1, y_2, y_3\}$ (where y_1 is the external one, y_2 the middle one and y_3 the internal one), the action of a generator of $\pi_1(\mathbb{S}^1)$ on the fiber is (in the standard notation of $\mathfrak{S}_3$) the cyclic permutation (1 2 3) for p_1, the transposition (2 3) for p_2, and the identity for p_3.

Lemma 1.6.21. *Let $\pi : Y \to X$ be a covering space, α and β two paths in X from x_0 to x_1, and let $y_0 \in Y_{x_0}$. Then $\tilde{\alpha}_{y_0}$ and $\tilde{\beta}_{y_0}$ have the same ending point if and only if $[\alpha \cdot \beta^{-1}] \in G(Y, y_0)$.*

Proof. Exercise (apply Proposition 1.6.11). □

We saw that every covering space $\pi : Y \to X$ has the property of lifting paths uniquely: given a path (continuous function) $f : I \to X$ and a starting point in the covering space (i.e. a point $y_0 \in Y$ in the fiber of $x_0 = f(0)$), there exists a unique path (continuous function) $\tilde{f} : I \to Y$ such that $\pi \circ \tilde{f} = f$ and $\tilde{f}(0) = y_0$. If we aim to replace $(I, 0)$

[(45)] If G is a group and H is a subgroup of G, the *heart* of H is the largest normal subgroup of G contained in H: hence it is $\bigcap_{g \in G} gHg^{-1}$. Dually, the smallest normal subgroup of G containing H is the normal subgroup generated by the subset $\bigcup_{g \in G} gHg^{-1}$. The latter shoul not be confused with the *normalizer* $\mathcal{N}H = \{g \in G : gHg^{-1} = H\}$, which is the largest subgroup of G containing H as a normal subgroup. Hence, H is normal in G if and only if the heart of H and the normal subgroup generated by H coincide with H, and $\mathcal{N}H = G$.

by any pointed topological space (Z, z_0) the solution of the same problem depends on the properties of the function f to be lifted and also of the space Z, for which we must introduce a topological notion.

Definition 1.6.22. A topological space is said to be *locally (arcwise) connected* if any point has a basis of open and (arcwise) connected neighborhoods.

Example. The "comb space" (Figure 2(a)) is arcwise connected but not locally arcwise connected.

Proposition 1.6.23. (Lifting criterion) *Let X, Y and Z be topological spaces with Z arcwise connected and locally arcwise connected, $\pi : (Y, y_0) \to (X, x_0)$ a covering space and $f : (Z, z_0) \to (X, x_0)$ a continuous function. Then f admits a unique lifting $\tilde{f} : (Z, z_0) \to (Y, y_0)$ if and only if $f_\#(\pi_1(Z, z_0)) \subset G(Y, y_0)$.*

Proof. Necessity is an immediate consequence of the functoriality of π_1 (exercise); let us see now the sufficience. The uniqueness of $\tilde{f}$ comes from Proposition 1.6.9. As for the existence, given any $z \in Z$ let us choose a path $\alpha : I \to Z$ from z_0 to z: its image $f \circ \alpha : I \to X$ is a path from x_0 to $f(z)$, which lifts to a unique path $\widetilde{(f \circ \alpha)}_{y_0}$. The definition $\tilde{f}(z) = \widetilde{(f \circ \alpha)}_{y_0}(1)$ is well-posed: if β is another path in Z from z_0 to z, then $f \circ \alpha$ and $f \circ \beta$ are two paths in X from x_0 to $f(z)$, and their liftings from y_0 have the same endpoint if and only if (by Lemma 1.6.21) $[(f \circ \alpha) \cdot (f \circ \beta)^{-1}] = [f \circ (\alpha \cdot \beta^{-1})] = f_\#([\alpha \cdot \beta^{-1}]) \in G(Y, y_0)$, a fact ensured by the hypotheses. We are left with showing the continuity of $\tilde{f}$. Let $z \in Z$ and $V \subset Y$ be an open neighborhood of $\tilde{f}(z)$: we may assume that the open $U = \pi(V) \subset X$ is evenly covered. Let $W \subset f^{-1}(U)$ be open, arcwise connected and containing z (such a W exists since Z is locally arcwise connected): let us prove that $\tilde{f}(W) \subset V$. Let $\zeta \in W$, and let β_ζ be a path in W from z to ζ. We have $\tilde{f}(\zeta) = \widetilde{((f \circ (\alpha \cdot \beta_\zeta))}_{y_0}(1) = \widetilde{(f \circ \alpha)}_{y_0} \cdot \widetilde{(f \circ \beta_\zeta)}_{\tilde{f}(z)}(1)$: now, $\widetilde{(f \circ \beta_\zeta)}_{\tilde{f}(z)}$ is a path from $\tilde{f}(z) \in V$ which lifts $f \circ \beta_\zeta$ (path in U), and hence also the endpoint $\widetilde{(f \circ \beta_\zeta)}_{\tilde{f}(z)}(1)$ is in V (recall that $U = \pi(V)$ is evenly covered, and V is one of the disjoint sheets above U). $\square$

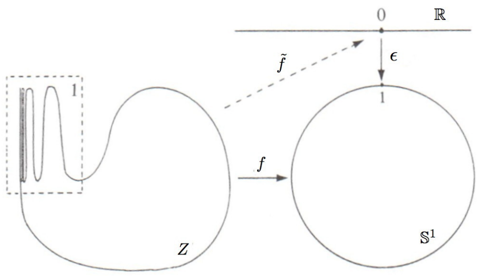


Figure 11: A non locally arcwise connected space for which the Lifting criterion does not work.

Remark 1.6.24. The hypothesis of locally arcwise connectedness cannot be dropped in the proof of the Lifting criterion (Proposition 1.6.23). For example, let Z be the "quasi-circle" of Figure 11 (starting with a vertical straight segment which will be later approached by a part of type "$\sin \frac{1}{x}$"), and let $f : Z \to \mathbb{S}^1$ be a quotient map which collapses all the points of Z contained in the dashed box into the point 1 of $\mathbb{S}^1$. If we consider the usual exponential covering $\epsilon : \mathbb{R} \to \mathbb{S}^1$ given by $\epsilon(t) = e^{2\pi i t}$, since the fundamental group of Z is trivial[(46)] the hypothesis on the characteristic subgroup is satisfied; however, if we lift f by ϵ to $\tilde{f} : Z \to \mathbb{R}$ with base point 0 in $\mathbb{R}$, then the upper point of the straight

[(46)] Namely, given a path $\alpha : I \to Z$, the support $\alpha(I) \subset Z$ is compact and hence it cannot collapse on the vertical straight segment: then α is nullhomotopic, in other words Z is simply connected.

segment goes to 0 while the part of type "$\sin\frac{1}{x}$" is sent to 1, but this shows that $\tilde{f}$ is not continuous.

We now show that, in the hypothesis of local arcwise connectedness, a covering space of X is covered by any other covering space of X having a smaller characteristic subgroup.

Proposition 1.6.25. *Let $\pi : (Y, y_0) \to (X, x_0)$ and $p : (Z, z_0) \to (X, x_0)$ be two covering spaces of X with Z arcwise connected and locally arcwise connected. Then there exists a morphism of covering spaces $\varphi : ((Z, z_0), p) \to ((Y, y_0), \pi)$ if and only if $G(Z, z_0) \subset G(Y, y_0)$, and in such a case φ itself is a covering space.*

Proof. The first statement follows from the Lifting criterion (Proposition 1.6.23); we are left with proving that φ itself is a covering space. Given $y \in Y$, let β be a path in Y from y_0 to y, and consider the path $\pi \circ \beta$ in X from x_0 to $\pi(y)$, which lifts uniquely to $\alpha = \widetilde{(\pi \circ \beta)}_{z_0}$ in Z: since $\pi \circ (\varphi \circ \alpha) = p \circ \alpha = \pi \circ \beta$, using the Monodromy lemma one has $\varphi(\alpha(1)) = \beta(1) = y$. Hence φ is surjective. Let $V_{y_0} \subset Y$ be an open neighborhood of y_0 such that $U = \pi(V_{y_0})$ is an arcwise connected open neighborhood of x_0 evenly covered both for p and for π: one then has $\pi^{-1}(U) = \bigsqcup_{y \in \pi^{-1}(x_0)} V_y$ (with $\pi|_{V_y} : V_y \xrightarrow{\sim} U$ for any $y \in \pi^{-1}(x_0)$) and $p^{-1}(U) = \bigsqcup_{z \in p^{-1}(x_0)} W_z$ (with $p|_{W_z} : W_z \xrightarrow{\sim} U$ for any $z \in p^{-1}(x_0)$). One then shows that $\varphi^{-1}(V_{y_0}) = \bigsqcup_{z \in \varphi^{-1}(y_0)} W_z$ (note that $\varphi^{-1}(y_0) \subset p^{-1}(x_0)$): namely, if $z \in \varphi^{-1}(y_0)$ then $\varphi|_{W_z} : W_z \xrightarrow{\sim} V_{y_0}$,[47] and hence $\zeta \in W_z$ for a certain $z \in \varphi^{-1}(y_0)$ if and only if $\varphi(\zeta) \in V_{y_0}$, i.e. if and only if $\zeta \in \varphi^{-1}(V_{y_0})$. □

Corollary 1.6.26. (Uniqueness theorem) *Two arcwise connected and locally arcwise connected covering spaces of a (connected and locally arcwise connected) topological space are isomorphic if and only if they have the same characteristic subgroup.*[48]

The statement of a Existence theorem for a covering space with prescribed characteristic subgroup requires a slightly stronger topological hypothesis.

Definition 1.6.27. A topological space X is said *locally simply connected* if any $x \in X$ has a basis of open simply connected neighborhoods; more generally, X is said *semi-locally simply connected* if any $x \in X$ admits a open neighborhood $U \subset X$ such that any loop in U based at x is nullhomotopic *in* X (i.e., with homotopies not necessarily with values only in U).

Examples. **(1)** Obviously, manifolds are locally (hence also semi-locally) simply connected. **(2)** *(Shrinking wedge of circles)* Let $C = \bigcup_{n \in \mathbb{N}} C_n$, where C_n is the circle of center $(-1/n, 0)$ and radius $1/n$: then C is locally arcwise connected but neither locally nor semi-locally simply connected. The fundamental group of C turns out to be very complicated. In fact, the topology of C is the one induced from $\mathbb{R}^2$, so that a neighborhood of $(0,0)$ must contain all but a finite number of the C_n: hence this topology is much weaker than the wedge sum topology of $\bigvee_{\mathbb{N}} \mathbb{S}^1$. In particular, since the C_n collapse to $(0,0)$, this allows also infinite junctions of loops on different C_n to be continuous as loops in C based at 0, and hence to contribute to the complication of $\pi_1(C)$. In particular, for each sequence (r_n) of integers one can construct

[47] From $U = p(W_z) = \pi(\varphi(W_z))$ one has that the connected subset $\varphi(W_z)$ is in $p^{-1}(U) = \bigsqcup_{y \in \pi^{-1}(x_0)} V_y$ and contains y_0, hence $\varphi(W_z) \subset V_{y_0}$ and it holds even $\varphi(W_z) = V_{y_0}$ because otherwise $p|_{W_z} = \pi|_{V_{y_0}} \circ \varphi|_{W_z}$ would not be an isomorphism.

[48] It must be noted that we are always working in the framework of *pointed topological spaces*, so we are keeping track of a base point both in the base space (i.e. a x_0), and in the covering (i.e. a point in the fiber of x_0). In the case we do not keep track of base points, the result should be stated as follows: *Two arcwise connected and locally arcwise connected covering spaces of a (connected and locally arcwise connected) topological space are isomorphic if and only if they have conjugated characteristic subgroups.*

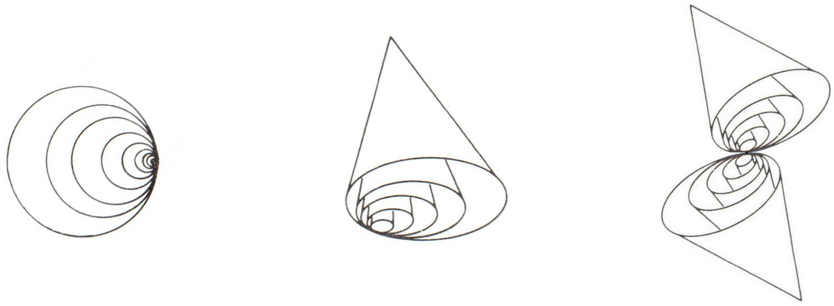


Figure 12: The sets C (shrinking wedge of circles), T (a cone of C) and X (union of two copies of T).

a loop $\gamma_{(r_n)}$ in C winding r_k times at each C_k, and these loops are mutually nonhomotopic: this fact provides a a surjective morphisms $\pi_1(C) \to \prod_{\mathbb{N}} \mathbb{Z}$ and so, the direct product $\prod_{\mathbb{N}} \mathbb{Z}$ being uncountable, also $\pi_1(C)$ is uncountable and hence deeply different from $\pi_1(\bigvee_{\mathbb{N}} \mathbb{S}^1) \simeq *_{\mathbb{N}} \mathbb{Z}$ (which has countably many generators, and hence is countable). **(3)** Let T be a cone in $\mathbb{R}^3$ with base C (where all the C_n are meant to be e.g. in the plane (x, y) with center $(-1/n, 0, 0)$): then T is locally arcwise connected and clearly contractible, hence simply connected; in particular T is semi-locally simply connected, but not locally simply connected. **(4)** Let X be the union of two copies of T at the base point, e.g. $X = T \cup (-T)$ where $-T = \{(x, y, z) \in \mathbb{R}^3 : (-x, -y, -z) \in T\}$ is the opposite to T (hence T and $-T$ have only the point $(0, 0, 0)$ in common): then X is connected and locally arcwise connected, but neither simply connected nor semi-locally (hence, nor locally) simply connected. The argument is as follows: using the notation introduced above for C, the loops $\gamma_{(r_n)}$ with all but a finite number of the r_n equal to zero are nullhomotopic in X (namely, if $N \in \mathbb{N}$ is the largest number such that $r_N \neq 0$, then all extremities $(\pm\frac{2}{n}, 0, 0)$ of the loops which constitute $\gamma_{(r_n)}$ keep being at "security distance" $\frac{1}{N} > 0$ from $(0, 0, 0)$, hence they can be sent to the vertices of $\pm T$ for $0 \leq t \leq \frac{1}{2}$, and then down to $(0, 0, 0)$ $\frac{1}{2} \leq t \leq 1$ without breaking the continuity of the homotopy), while the $\gamma_{(r_n)}$ with infinitely many r_n different from zero are not.

Proposition 1.6.28. (Existence theorem) *Given an arcwise connected, locally arcwise connected and semi-locally simply connected topological space (X, x_0) and a subgroup $H \subset \pi_1(X; x_0)$, there exists a unique (up to a canonical isomorphism) covering space $\pi : Y \to X$ such that $G(Y, y_0) = H$.*

Proof. (Sketch) The idea is to consider on the set $\Omega_{x_0,x}$ of paths from x_0 to $x \in X$ the equivalence relation given by $\alpha \sim \beta$ if $[\alpha \cdot \beta^{-1}] \in H$, then to define $Y = \bigsqcup_{x \in X} (\Omega_{x_0,x}/ \sim)$, y_0 as the class of c_{x_0} in Ω_{x_0,x_0} and $\pi : Y \to X$ given by $[\gamma] \to \gamma(1)$, then finally to endow Y with a suitable topology using the hypotheses on X. For more details we refer for example to Jänich [9, from p. 144]. □

Examples. (1) The subgroups of $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ are $\mathbb{Z}$ itself, $n\mathbb{Z}$ (for $n \in \mathbb{N}$) and $\{0\}$: they correspond to the coverings $(\mathbb{S}^1, \mathrm{id})$, $(\mathbb{S}^1, z^n)$ (for $n \in \mathbb{N}$) and $(\mathbb{R}, \epsilon)$ (recall that $\epsilon(t) = \exp(2\pi i t)$), which therefore represent —up to isomorphism— all arcwise connected and locally acwise connected covering spaces of $\mathbb{S}^1$. **(2)** As for the bouquet $X = \mathbb{S}^1 \vee \mathbb{S}^1$, the family of subgroups of $\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$ is much richer than the one of $\mathbb{Z}$, and hence the classification of arcwise connected and locally arcwise connected covering spaces of X is much more interesting (see e.g. [8, §1.3], or the example at p. 41).

Remark 1.6.29. The hypothesis of semi-local simple connectedness is necessary for the proof of Proposition 1.6.28. For example, the above double cone of shrinking wedge of circles $X = T \cup (-T)$ has been proved to have nontrivial fundamental group, but it is possible to prove that any arcwise connected covering space of X is necessarily trivial[(49)]: hence the proper subgroups of $\pi_1(X; x_0)$ do not correspond to any covering space of X.

1.6.4 Covering automorphisms

Let X and Y be topological spaces, $\pi : Y \to X$ a covering space, and consider the set of endomorphisms of (Y, π), i.e. $\text{End}(Y|X) = \{\varphi : Y \to Y : \pi \circ \varphi = \pi\}$. The subset

$$\text{Aut}(Y|X) = \{\varphi : Y \to Y : \varphi \text{ homeomorphism}, \pi \circ \varphi = \pi\}$$

(the "covering automorphisms", or *deck transformations*) has a natural structure of group, given by the composition.

Examples. **(1)** The deck transformations of $\epsilon : \mathbb{R} \to \mathbb{S}^1$ (where $\epsilon(t) = e^{2\pi i t}$) are the translations $\tau_k : \mathbb{R} \to \mathbb{R}$ given by $\tau_k(t) = t + k$ for $k \in \mathbb{Z}$, hence $\text{Aut}(\mathbb{R}|\mathbb{S}^1) \simeq \mathbb{Z}$. **(2)** The deck transformations of $z^n : \mathbb{S}^1 \to \mathbb{S}^1$ are the rotations of multiples of $2\pi/n$, hence $\text{Aut}((\mathbb{S}^1, z^n)|\mathbb{S}^1) \simeq \mathbb{Z}/n\mathbb{Z}$.

An immediate consequence of Corollary 1.6.26 is the following

Proposition 1.6.30. *Let $\pi : Y \to X$ be a covering space with X and Y arcwise connected and locally arcwise connected topological spaces, $x_0 \in X$, $y_0, y_1 \in Y_{x_0}$. Then there exists $\varphi \in \text{Aut}(Y|X)$ with $\varphi(y_0) = y_1$ if and only if $G(Y, y_0) = G(Y, y_1)$.*

What does this condition mean? By Proposition 1.6.19 we know that $G(Y, y_1)$ is conjugated to $G(Y, y_0)$ in $\pi_1(X, x_0)$: if γ is a path in Y from y_0 to y_1, setting $\alpha = \pi \circ \gamma$ it holds $G(Y, y_1) = [\alpha^{-1}] \cdot G(Y, y_0) \cdot [\alpha]$. Hence, the condition $G(Y, y_0) = G(Y, y_1)$ is equivalent to $[\alpha] \in \mathcal{N}\big(G(Y, y_0)\big)$ (the *normalizer* of $G(Y, y_0)$ in $\pi_1(X, x_0)$).

Theorem 1.6.31. *Let $\pi : (Y, y_0) \to (X, x_0)$ be a covering space with X and Y arcwise connected and locally arcwise connected topological spaces. Then for any $[\alpha] \in \mathcal{N}\big(G(Y, y_0)\big)$ there exists one and only one covering automorphism $\varphi_{[\alpha]}$ such that $\varphi_{[\alpha]}(y_0) = \tilde{\alpha}_{y_0}(1)$. The application $\mathcal{N}\big(G(Y, y_0)\big) \to \text{Aut}(Y|X)$ obtained in this way is a surjective morphism of groups with kernel $G(Y, y_0)$, and provides an isomorphism of groups*

$$\frac{\mathcal{N}\big(G(Y, y_0)\big)}{G(Y, y_0)} \quad \xrightarrow{\sim} \quad \text{Aut}(Y|X).$$

Proof. Let $y_1 = \tilde{\alpha}_{y_0}(1)$: then the hypothesis $[\alpha] \in \mathcal{N}\big(G(Y, y_0)\big)$ is equivalent to the fact that $G(Y, y_0) = G(Y, y_1)$, and the result follows from the Lifting criterion (Proposition 1.6.23). In particular, one constructs explicitly $\varphi_{[\alpha]} \in \text{Aut}(Y|X)$ as follows: given $y \in Y$ and a path β_y from y_0 to y, one sets $\varphi_{[\alpha]}(y) = \widetilde{(\pi \circ \beta_y)}_{y_1}(1)$ (note that $\varphi_{[\alpha]}(y_0) = c_{y_1}(1) = y_1$, as required). $\square$

(49) The idea is that a arcwise connected covering space $\pi : Y \to X$ induces on T and $-T$ (which are simply connected) trivial covering spaces $(\pm T) \times F$; if F would not be a point (i.e., if such induced covering spaces would not be homeomorphisms), two points of the same fiber would stay in different arcwise connected components, and that would contradict the fact that π is arcwise connected (just think that $X = T \cup (-T)$, and that $x_0 = (0, 0, 0)$ is the only point in common between T and $-T$...); hence $F = \{\text{pt}\}$, and since $X = T \cup (-T)$ this implies that π is a homeomorphism.

Definition 1.6.32. A connected and locally arcwise connected covering space $\pi : (Y, y_0) \to (X, x_0)$ is called *normal*, or *Galois*, if $G(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$.

The following proposition shows the properties of normal covering spaces. In particular they act transitively on the fibers, and hence these covering spaces can be viewed as those having a complete symmetry among different sheets.

Proposition 1.6.33. *A connected and locally arcwise connected normal covering space* $\pi : (Y, y_0) \to (X, x_0)$ *has the following properties.*

(i) $\pi_1(X, x_0)/G(Y, y_0) \xrightarrow{\sim} \mathrm{Aut}(Y|X)$.

(ii) $\mathrm{Aut}(Y|X)$ *operates on the left on* Y *in a properly discontinuous way* (hence freely)[(50)], *and the orbits are the fibers of* π*: in particular, the multiplicity of the covering space is equal to the index of* $G(Y, y_0)$ *in* $\pi_1(X, x_0)$.

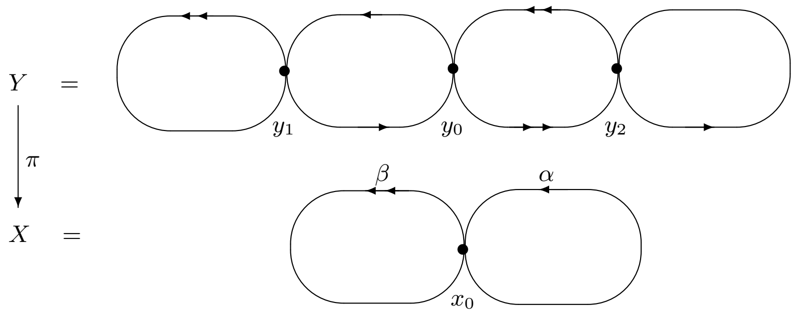
(iii) *Denoted by* $Y/\mathrm{Aut}(Y|X)$ *the space of orbits of* $\mathrm{Aut}(Y|X)$ *in* Y (i.e., the space of fibers of π) *with the quotient topology, the natural bijection* $Y/\mathrm{Aut}(Y|X) \to X$ *is a homeomorphism.*

(iv) *Either all liftings of loops in* X *based at* x_0 *are loops in* Y*, or no one of them is.* (Such condition is also sufficient in order that a covering space be Galois.)

Proof. (i) follows from Theorem 1.6.31, as well as the fact that the action on the left of $\mathrm{Aut}(Y|X)$ has the fibers of π as orbits. Let $U \subset X$ be a evenly covered open neighborhood of x_0: hence one has $\pi^{-1}(U) = \bigsqcup_{\lambda\in\Lambda} V_\lambda$, and let $y_0 \in V_{\lambda_0}$. If $\varphi_1(V_{\lambda_0}) \cap \varphi_2(V_{\lambda_0}) \neq \varnothing$, let $y_1, y_2 \in V_{\lambda_0}$ be such that $\varphi_1(y_1) = \varphi_2(y_2) \in U$; in particular y_1 and y_2 belong to the same fiber of π, and also to the same sheet V_{λ_0}: hence $y_1 = y_2 \equiv y$, which implies $\varphi_1 = \varphi_2$ because they coincide in the point y (uniqueness of lifting). This shows (ii). The bijection in (iii) is continuous by definition of quotient topology, and is open because such is also π. Finally, to be Galois is equivalent to the fact that $G(Y, y_1) = G(Y, y_2)$ for any y_1 and y_2 in the same fiber of π, and this is equivalent to (iv). □

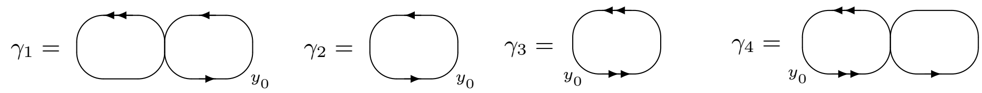
Remark 1.6.34. From Remark 1.6.20 it follows immediately that, if the covering space $\pi : (Y, y_0) \to (X, x_0)$ is Galois, the subgroup of $\pi_1(X, x_0)$ acting trivially on the fiber Y_{x_0} is $G(Y, y_0)$.

Example. If $\pi_1(X, x_0)$ is commutative, then obviously all arcwise connected and locally arcwise connected covering spaces of X are normal. On the other hand, let us show an example of connected and locally arcwise connected covering space which is not normal. Consider the function



(50) Recall that the action of a group G on a set Z si said *properly discontinuous* if any $z \in Z$ has a neighborhood $U \subset Z$ such that $g_1U \cap g_2U = \varnothing$ if $g_1 \neq g_2$, and *free* (a weaker notion) if any point has trivial stabilizer, i.e. $G_z = \{1\}$ for any $z \in Z$.

(We mean that $\pi(y_j) = x_0$ ($j = 0, 1, 2$) and that the arcs of Y denoted by $\to$ (resp. by $\twoheadrightarrow$) are sent into α (resp. into β) in the specified direction. Note that π is a surjective local homeomorphism with fiber of cardinality 3: by Proposition 1.6.7, π is a 3-sheet covering space. Consider the morphism of pointed spaces $\pi : (Y, y_0) \to (X, x_0)$, and the injective morphism $\pi_\# : \pi_1(Y, y_0) \to \pi_1(X, x_0)$. The space X is the bouquet of two circles and hence $\pi_1(X, x_0)$ is free on two generators $[\alpha]$ and $[\beta]$; on the other hand, $\pi_1(Y, y_0)$ is generated by the classes of loops



and one has $\pi_\#([\gamma_1]) = [\alpha \cdot \beta \cdot \alpha]$, $\pi_\#([\gamma_2]) = [\alpha^2]$, $\pi_\#([\gamma_3]) = [\beta^2]$, $\pi_\#([\gamma_4]) = [\beta \cdot \alpha \cdot \beta]$: hence the characteristic subgroup $G(Y, y_0)$ is generated by $[\alpha \cdot \beta \cdot \alpha]$, $[\alpha^2]$, $[\beta^2]$ and $[\beta \cdot \alpha \cdot \beta]$. Would there exist a covering automorphism sending y_0 for example into y_1, the liftings from y_0 and y_1 of the same loop based at x_0 should be either both loops or no one of them: actually $\tilde{\alpha}_{y_0}$ and $\tilde{\alpha}_{y_1}$ are not loops (the first one goes from y_0 to y_1, the second from y_1 to y_0), but $\tilde{\beta}_{y_0}$ is not a loop (goes from y_0 to y_2) while $\tilde{\beta}_{y_1}$ is. Therefore one has $\mathrm{Aut}(Y|X) = \{\mathrm{id}_Y\}$, which implies $\mathcal{N}\big(G(Y, y_0)\big) = G(Y, y_0)$ (hence the covering space is not normal). Hence $G(Y, y_1) = [\alpha^{-1}] \cdot G(Y, y_0) \cdot [\alpha] \neq G(Y, y_0)$ and $G(Y, y_2) = [\beta^{-1}] \cdot G(Y, y_0) \cdot [\beta] \neq G(Y, y_0)$. The heart $\bigcap_{j=0,1,2} G(Y, y_j)$ is the subgroup of $\pi_1(X, x_0)$ formed by the loops whose liftings from the y_j's are all loops: it is generated by $[\alpha^2]$ and $[\beta^2]$. The action of $\pi_1(X, x_0)$ on $\pi^{-1}(x_0) = \{y_0, y_1, y_2\}$ is given by $[\alpha] = (0\ 1)$ and $[\beta] = (0\ 2)$: hence the morphism $\pi_1(X, x_0) \to \mathfrak{S}_3$ is surjective.

A particular case of Galois covering space is, if it exists, to one with $G(Y, y_0) = \{1\}$.

Definition 1.6.35. A connected and locally arcwise connected covering space $\pi : Y \to X$ is called *universal cover* of X if Y is simply connected.

Theorem 1.6.36. *If X is a connected, locally arcwise connected and semi-locally simply connected topological space, there exists a unique —up to canonical isomorphisms— universal cover $\tilde{\pi} : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, with the following properties.*

(i) $\pi_1(X, x_0) \simeq \mathrm{Aut}(\tilde{X}|X)$.

(ii) $\mathrm{Aut}(\tilde{X}|X)$ *operates on the left on $\tilde{X}$ in a properly discontinuous way* (hence freely), *and the orbits are the fibers of $\tilde{\pi}$.*

(iii) *if $\pi : (Y, y_0) \to (X, x_0)$ is another connected and locally arcwise connected covering space, there exists one and only one covering space $\tilde{\pi}_Y : (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ such that $\tilde{\pi} = \pi \circ \tilde{\pi}_Y$.* (Hence, the universal cover of X is also the universal cover of any other connected and locally arcwise connected covering space of X.)

(iv) *The universal cover determines all other arcwise connected and locally arcwise connected covering spaces of X, in the following sense: if $\Gamma \subset \mathrm{Aut}(\tilde{X}|X)$ is a subgroup, denoted by $\tilde{X}/\Gamma$ the space of orbits of Γ in $\tilde{X}$ endowed with the quotient topology, the natural map $\tilde{\pi}_\Gamma : (\tilde{X}/\Gamma, [\tilde{x}_0]) \to (X, x_0)$ is a connected and locally arcwise connected covering of X, and moreover all arcwise connected and locally arcwise connected covering spaces of X are obtained in this way, up to a canonical isomorphism.*

Proof. Existence and uniqueness follow from Proposition 1.6.28 and Corollary 1.6.26. (i) and (ii) follow from Proposition 1.6.33, (iii) from Proposition 1.6.25, (iv) follows from (iii) and Proposition 1.6.33 (which says that $Y \simeq \tilde{X}/\mathrm{Aut}(\tilde{X}|Y)$, and $\Gamma = \mathrm{Aut}(\tilde{X}|Y) \subset \mathrm{Aut}(\tilde{X}|X)$). □

In particular, it is useful to remark that (i) provides a method —often the preferable one— for computing the fundamental group of X:

$$\pi_1(X, x_0) \simeq \mathrm{Aut}(\tilde{X}|X).$$

Some examples of this method will be shown in the next Section.

1.7 Exercises and complements

(1) The universal covering of $\mathbb{S}^1$ is the exponential map $\epsilon : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C}$, $\epsilon(t) = e^{2\pi i t}$; more generally, the universal cover of the torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ is the map $\epsilon : \mathbb{R}^n \to \mathbb{T}^n$, $\epsilon(t_1, \dots, t_n) = (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$, and it holds $\mathrm{Aut}(\mathbb{R}^n|\mathbb{T}^n) = \{\tau : \mathbb{R}^n \to \mathbb{R}^n,\ \tau(t_1, \dots, t_n) = (t_1 + k_1, \dots, t_n + k_n)$ for some $k_j \in \mathbb{Z}\} \simeq \mathbb{Z}^n$, which yields again $\pi_1(\mathbb{T}^n) \simeq \mathbb{Z}^n$.

(2) The real space projective $\mathbb{P}^n$ (with $n \geq 1$) is defined as the space of orbits of the multiplicative group $\mathbb{R}^\times$ in $(\mathbb{R}^{n+1})^\times$ (i.e., the family of homogeneous lines of $\mathbb{R}^{n+1}$), endowed with the quotient topology. Consider the Hopf map $q : \mathbb{S}^n \to \mathbb{P}^n$ (the restriction to $\mathbb{S}^n$ of the projection $(\mathbb{R}^{n+1})^\times \to \mathbb{P}^n$), which is a 2-sheet covering of $\mathbb{P}^n$. If $n \geq 2$ this map is the universal cover; since the covering automorphisms of q are $\pm\,\mathrm{id}_{\mathbb{S}^n}$, we get $\pi_1(\mathbb{P}^n) \simeq \mathrm{Aut}(\mathbb{S}^n|\mathbb{P}^n) = \mathbb{Z}/2\mathbb{Z}$. On the other hand, for $n = 1$ we saw that the map $(\,\cdot\,)^2 : \mathbb{S}^1 \to \mathbb{S}^1$ has the same fibers of q, hence there exists a homeomorphism $\gamma : \mathbb{S}^1 \xrightarrow{\sim} \mathbb{P}^1$ such that $q = \gamma \circ (\,\cdot\,)^2$. Therefore $\pi_1(\mathbb{P}^1) \simeq \mathbb{Z}$.

An interesting application is a particular case of the Borsuk-Ulam theorem (*if $n \geq 2$, there does not exist any continuous functions of $\mathbb{S}^n$ into itself which is odd and nullhomotopic*):

Corollary 1.7.1. (Borsuk-Ulam, particular case) *If $n \geq 2$, there does not exist odd continuous functions of $\mathbb{S}^n$ with values in $\mathbb{S}^1$.*

Proof. By absurd let $f : \mathbb{S}^n \to \mathbb{S}^1$ be a odd continuous function, and denote by $q_j : \mathbb{S}^j \to \mathbb{P}^j$ the Hopf map. By Proposition 1.1.14, there exists $g : \mathbb{P}^n \to \mathbb{P}^1$ continuous such that $g \circ q_n = q_1 \circ f$ (note that $q_1 \circ f$ is constant on the fibers of q_n). Now, fixed $x_0 \in \mathbb{S}^n$, the morphism $g_\# : \pi_1(\mathbb{P}^n, q_n(x_0)) = \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathbb{P}^1, g(q_n(x_0))) = \mathbb{Z}$ must be zero, hence by the Lifting criterion (Proposition 1.6.23) there exists a unique lifting $\tilde{g} : \mathbb{P}^n \to \mathbb{S}^1$ such that $g = q_1 \circ \tilde{g}$ and $\tilde{g}(q_n(x_0)) = f(x_0)$. But one has also $f = \tilde{g} \circ q_n$ (namely f and $\tilde{g} \circ q_n$ are two liftings of $g \circ q_n$ which coincide in x_0): this is a contradiction because $f(-x) = -f(x)$ while $\tilde{g}(q_n(-x)) = \tilde{g}(q_n(x)) = f(x)$. □

A consequence of Corollary 1.7.1 is, for example, that *at any particular time there are two antipodal places on the Earth with the same temperature and the same pressure.* Namely let, at a certain fixed moment, $t : \mathbb{S}^2 \to \mathbb{R}$ be the temperature and $p : \mathbb{S}^2 \to \mathbb{R}$ be the pressure: if the statement would be false, the function $\varphi : \mathbb{S}^2 \to \mathbb{R}^2$ given by $\varphi(x) = (t(x) - t(-x), p(x) - p(-x))$ would never take the value $(0, 0)$ and hence the function $f : \mathbb{S}^2 \to \mathbb{S}^1$ given by $f(x) = \varphi(x)/|\varphi(x)|$ would be continuous and odd.

(3) We now deal with the fundamental group of topological groups.

Proposition 1.7.2. *Let G a topological group with identity element* 1. *Then:*

(i) *the fundamental group $\pi_1(G, 1)$ is commutative;*

(ii) *if G is a connected and locally arcwise connected topological group, and $\pi : (E, e) \to (G, 1)$ is a connected and locally arcwise connected covering, then there exists one and only one multiplication on E for which* (a) *E is a topological group with identity element e,* (b) *π is a morphism.*

Proof. (The student should verify by exercise the unproven statements.) (i) Let $\mu : G \times G \to G$ be the multiplication. This map induces a pointwise product between loops based at 1, by defining $(\alpha * \beta)(t) = \mu(\alpha(t), \beta(t))$ for any $t \in I$; and this product induces a product $*$ in $\pi_1(G, 1)$. We then have two operations

($*$ and $\cdot$) in $\pi_1(G,1)$. Now, by a elementary algebraic fact, if a set S is endowed with two binary operations $*$ and $\cdot$ with a same identity element and such that $(a*b)\cdot(c*d) = (a\cdot c)*(b\cdot d)$ for any $a,b,c,d \in S$, then $*$ and $\cdot$ coincide and are associative and commutative: such condition is verified in our case. (ii) The map $\mu\circ(\pi\times\pi) : (E\times E,(e,e)) \to (G,1)$ (given n covering spaces $\pi_j : Y_j \to X_j$, and set $\pi = \pi_1 \times \cdots \times \pi_n$, $Y = Y_1 \times \cdots \times Y_n$ and $X = X_1 \times \cdots \times X_n$, also $\pi : Y \to X$ is a covering) lifts uniquely to a map $\tilde{\mu} : (E\times E,(e,e)) \to (E,e)$: this follows from Proposition 1.6.23, since $(\mu\circ(\pi\times\pi))_\#([\alpha],[\beta]) = [\mu(\pi\circ\alpha,\pi\circ\beta)] = [\pi\circ\alpha]*[\pi\circ\beta] = [\pi\circ\alpha]\cdot[\pi\circ\beta] = \pi_\#([\alpha\cdot\beta]) \in G(E,e)$. The uniqueness of lifting shows all the properties which are required to $\tilde{\mu}$ to be the desired operation in E: for example, to show that e is the identity element of $\tilde{\mu}$ we define $\tilde{\mu}_e : (E,e) \to (E,e)$ by $\tilde{\mu}_e(y) = \tilde{\mu}(y,e)$: since $\pi\circ\tilde{\mu}_e = \pi$, we get $\tilde{\mu}_e = \mathrm{id}_E$ (because id_E and $\tilde{\mu}_e$ are two morphisms of the covering space π which coincide in e). □

(4) Let us study the covering spaces of manifolds and, in particular, what happens to the fundamental group when we remove, from a given manifold, a closed submanifold which is "small enough".

Let M be a (arcwise) connected C^0 manifold of dimension m, $N \subset M$ a C^0 submanifold of dimension $n \le m$, and let $\iota : M \setminus N \to M$ be the open embedding.

Proposition 1.7.3. *If $\pi : P \to M$ is a local homeomorphism, then on P is naturally induced a structure of C^0 manifold of dimension m, and on $\pi^{-1}(N)$ a structure of submanifold of dimension n.*

Proof. The local charts on P and $\pi^{-1}(N)$ are just the local pullbacks of local charts on M and N. □

Proposition 1.7.4. *The following statements hold.*

(i) *If $m - n \ge 2$, then $M \setminus N$ is connected.*

(ii) *If $m - n \ge 3$, any covering space of $M \setminus N$ extends to one of M* (i.e., given a covering space $q : P \to M \setminus N$ there exist a covering space $\tilde{q} : \widetilde{P} \to M$ and a injective morphism of manifolds $\iota_P : P \to \widetilde{P}$ such that $\tilde{q}\circ\iota_P = \iota\circ q$).

Proof. (i) It is clear that $\mathbb{R}^m \setminus \mathbb{R}^n$ is connected if and only if $m-n \ge 2$. In general, it is enough to show that, given $x, y \in M \setminus N$, there exists a path in $M \setminus N$ from x to y. Let $\alpha : I \to M$ be a path from x to y, and $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ be an atlas of M such that $U_\lambda \cap N = \varnothing$ or $\varphi_\lambda(U_\lambda \cap N) = \mathbb{R}^n$. By compactness, there exist $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1$ and $\lambda_1, \dots, \lambda_r \in \Lambda$ such that $\alpha([t_{j-1}, t_j]) \subset U_{\lambda_j}$, where $j = 1, \dots, r$. Since any $U_{\lambda_j} \setminus (U_{\lambda_j} \cap N)$ is connected, we can construct a sequence of paths γ_j such that (1) $\gamma_j(I) \subset U_{\lambda_j} \setminus (U_{\lambda_j} \cap N)$; (2) $\gamma_1(0) = x$, $\gamma_{j+1}(0) = \gamma_j(1)$, $\gamma_r(1) = y$. (In particular, if $\alpha([t_{j-1}, t_j]) \subset U_{\lambda_j} \setminus (U_{\lambda_j} \cap N)$ we can choose $\gamma_j = \alpha|_{[t_{j-1},t_j]}$ with a reparametrization taking t_{j-1} into 0 and t_j into 1.) The path γ obtained by joining the paths γ_j has the required properties: hence $M \setminus N$ is connected. (ii) For the costruction of the covering space $\tilde{q}$, obtained through a gluing procedure of covering spaces on local charts, we refer e.g. to Godbillon [5, X.2]. □

Theorem 1.7.5. *For $x_0 \in M \setminus N$ consider the morphism of fundamental groups*

$$\iota_\# : \pi_1(M \setminus N, x_0) \to \pi_1(M, x_0).$$

Then:

(i) *if $m - n \ge 2$, then $\iota_\#$ is surjective;*

(ii) *if $m - n \ge 3$, then $\iota_\#$ is a isomorphism.*

Proof. (i) It is enough to show that, given $x, y \in M \setminus N$ and a path $\alpha : I \to M$ from x to y, there exists a path $\gamma : I \to M \setminus N$ from x to y with $[\alpha] = [\gamma]$. Following the proof of the first statement of Proposition 1.7.4, we may also require that (3) γ_j be homotopic (not necessarily rel ∂I) to $\alpha|_{[t_j, t_{j+1}]}$. The path γ obtained by joining the paths γ_j has the required properties. (ii) Let $H = \ker(\iota_\#)$, and let us prove that $H = \{1\}$. Consider the connected covering space $q : P \to M \setminus N$ having H as characteristic subgroup.[51] From Proposition 1.7.4 we know that there exist a covering $\tilde{q} : \widetilde{P} \to M$ and an injective morphism of manifolds $\iota_P : P \to \widetilde{P}$ such that $\tilde{q} \circ \iota_P = \iota \circ q$. By definition of H, one has $(\tilde{q}_\# \circ \iota_{P\#})(\pi_1(P)) = (\iota_\# \circ q_\#)(\pi_1(P)) = \iota_\#(H) = \{1\}$, hence ($\tilde{q}_\#$ being injective and $\iota_{P\#}$ surjective) it holds $\pi_1(\widetilde{P}) = \{1\}$: i.e., $\widetilde{P}$ is the universal cover of M. On the other hand, since $\widetilde{P} \setminus \iota_P(P)$ is a submanifold of codimension $m - n \geq 3$ of $\widetilde{P}$ (namely, from $\iota_P(P) = \tilde{q}^{-1}(M \setminus N)$ we get $\widetilde{P} \setminus \iota_P(P) = \tilde{q}^{-1}(N)$, and it is enough to recall Proposition 1.7.3), the universal cover $\pi : S \to P$ extends to $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$, which ($\widetilde{P}$ being simply connected) is a homeomorphism: hence also π is a homeomorphism, because the fiber has cardinality 1. Therefore P is simply connected, i.e. $H = \{1\}$. □

(5) We now deal with the fundamental group of some classical real groups. Given $n \in \mathbb{N}$, let $M(n, \mathbb{K})$ be the vector space of square matrices of order n with coefficients in $\mathbb{K} = \mathbb{R}, \mathbb{C}$. If $A \in M(n, \mathbb{C})$, we denote by $A^* = {}^t\overline{A}$ the adjoint matrix, and:

$$\begin{array}{rcl l}
GL(n,\mathbb{C}) = & \{A \in M(n,\mathbb{C}) : \det(A) \neq 0\} & \text{(complex general linear group)} \\
U(n,\mathbb{C}) = & \{A \in GL(n,\mathbb{C}) : A^{-1} = A^*\} & \text{(unitary group)} \\
SU(n,\mathbb{C}) = & \{A \in U(n,\mathbb{C}) : \det(A) = 1\} & \text{(special unitary group)} \\
GL(n,\mathbb{R}) = & M(n,\mathbb{R}) \cap GL(n,\mathbb{C}) & \text{(real general linear group)} \\
GL^{\pm}(n,\mathbb{R}) = & \{A \in GL(n,\mathbb{R}) : \det(A) \gtrless 0\} & \\
O(n,\mathbb{R}) = & M(n,\mathbb{R}) \cap U(n,\mathbb{C}) & \text{(real orthogonal group)} \\
SO(n,\mathbb{R}) = & M(n,\mathbb{R}) \cap SU(n,\mathbb{C}) & \text{(real special orthogonal group)}
\end{array}$$

and moreover

$$\begin{array}{rcl}
H(n,\mathbb{C}) = & \{A \in M(n,\mathbb{C}) : A = A^*\} & \text{(hermitian matrices)} \\
H^+(n,\mathbb{C}) = & \{A \in H(n,\mathbb{C}) : {}^t x A \overline{x} > 0 \ \forall\, x \in \mathbb{C}^n \setminus \{0\}\} & \text{(positive definite h. m.)} \\
S(n,\mathbb{R}) = & M(n,\mathbb{R}) \cap H(n,\mathbb{C}) & \text{(symmetric matrices)} \\
S^+(n,\mathbb{R}) = & M(n,\mathbb{R}) \cap H^+(n,\mathbb{C}) & \text{(positive definite s. m.).}
\end{array}$$

We briefly recall the following facts (for further details we refer e.g. to Godbillon [5, II.2]):

(1) $M(n, \mathbb{C})$ (resp. $M(n, \mathbb{R})$) is a vector space on $\mathbb{C}$ (resp. on $\mathbb{R}$) of dimension n^2, and $H(n, \mathbb{C})$ (resp. $S(n, \mathbb{R})$) is a real subspace of $M(n, \mathbb{C})$ (resp. $M(n, \mathbb{R})$) of dimension n^2 (resp. $n(n+1)/2$). The application $A \mapsto \|A\| = \sqrt{\operatorname{tr}(AA^*)}$ is a norm on $M(n, \mathbb{C})$, which induces the norm $\|A\| = \sqrt{\operatorname{tr}(A\,{}^tA)}$ on $M(n, \mathbb{R})$.

(2) $GL(n, \mathbb{C})$ is an open subset of $M(n, \mathbb{C})$ and a multiplicative topological group.

(3) The exponential $\exp : M(n, \mathbb{C}) \to GL(n, \mathbb{C})$ (where $\exp(A) = \sum_{n=0}^{\infty} A^n/n!$) satisfies $\exp(A + B) = \exp(A)\exp(B)$ if $AB = BA$, and is also a diffeomorphism between an open neighborhood of $0 \in M(n, \mathbb{C})$ and an open neighborhood of the identity

[51] Such a q exists, because the manifolds —and $M \setminus N$ is so— are locally simply connected, and in particular locally arcwise connected and semi-locally simply connected.

$1_n \in GL(n,\mathbb{C})$: this makes $GL(n,\mathbb{C})$ into a real Lie group[52] of dimension $2n^2$. Moreover, exp induces a homeomorphism of $H(n,\mathbb{C})$ on $H^+(n,\mathbb{C})$ and of $S(n,\mathbb{R})$ on $S^+(n,\mathbb{R})$.

(4) $U(n,\mathbb{C})$, $SU(n,\mathbb{C})$, $GL(n,\mathbb{R})$, $GL^+(n,\mathbb{R})$, $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are closed Lie subgroups of $GL(n,\mathbb{C})$ of dimensions n^2, n^2-1, n^2, n^2, $n(n-1)/2$ and $n(n-1)/2$. Local charts around the identity 1_n of these groups, seen as real submanifolds of $GL(n,\mathbb{C})$, are obtained by inverting the restriction of exp respectively to the real subspaces $\mathfrak{u}(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : A + A^* = 0\}$, $\mathfrak{su}(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : A + A^* = 0,\ \mathrm{tr}(A) = 0\}$, $\mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ (for $GL(n,\mathbb{R})$ and $GL^+(n,\mathbb{R})$) and $\mathfrak{so}(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) : A + {}^t\!A = 0\}$ (for $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$)[53]. Moreover, $U(n,\mathbb{C})$, $SU(n,\mathbb{C})$, $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are compact (if $A \in U(n,\mathbb{C})$ then $\|A\| = \sqrt{n}$).

(5) There are isomorphisms of Lie groups $U(n,\mathbb{C}) \simeq \mathbb{S}^1 \times SU(n,\mathbb{C})$ and $O(n,\mathbb{R}) \simeq \{\pm 1\} \times SO(n,\mathbb{R})$, given by $A \mapsto (\det(A), A/\det(A))$. Moreover, $SU(n,\mathbb{C})$ and $SO(n,\mathbb{R})$ are arcwise connected, and hence $U(n,\mathbb{C})$ is arcwise connected while $O(n,\mathbb{R})$ has two connected components.

(6) $GL(n,\mathbb{C})$ is homeomorphic to the product $H^+(n,\mathbb{C}) \times U(n,\mathbb{C})$ (*polar decomposition*) and in particular, by (3) and (5), $GL(n,\mathbb{C})$ is arcwise connected. A homeomorphism is induced between $GL(n,\mathbb{R})$ and $S^+(n,\mathbb{C}) \times O(n,\mathbb{R})$ and in particular between $GL^+(n,\mathbb{R})$ and $S^+(n,\mathbb{C}) \times SO(n,\mathbb{R})$, hence $GL(n,\mathbb{R})$ has two connected components $GL^\pm(n,\mathbb{R})$.

From (3) one has $\pi_1(H^+(n,\mathbb{C}), 1_n) \simeq \{1\}$ and $\pi_1(S^+(n,\mathbb{R}), 1_n) \simeq \{1\}$: hence from (5) and (6) one has

$$\begin{aligned} \pi_1(GL(n,\mathbb{C})) &\simeq \pi_1(U(n,\mathbb{C})) &&\simeq \mathbb{Z} \times \pi_1(SU(n,\mathbb{C})), \\ \pi_1(GL(n,\mathbb{R}), 1_n) &\simeq \pi_1(O(n,\mathbb{R}), 1_n) &&\simeq \pi_1(SO(n,\mathbb{R})). \end{aligned}$$

We are left with computing the fundamental groups of $SO(n;\mathbb{R})$ and $SU(n,\mathbb{C})$.

- It holds $SO(1;\mathbb{R}) = \{1\}$, and $SO(2;\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \simeq \mathbb{S}^1$. For $n = 3$, note that $\mathbb{S}^3$ (intended as the group of quaternions of unitary norm, see Example 1.4) operates on $\mathbb{R}^3$ by $\mathbb{S}^3 \times \mathbb{R}^3 \ni (q,u) \mapsto quq^{-1}$: such trasformation is linear and preserves the norms, i.e. it is in $SO(3;\mathbb{R})$, and one obtains in this way a morphism $\mathbb{S}^3 \to SO(3;\mathbb{R})$, which is surjective and has $\{\pm 1\}$ as kernel. This shows that $SO(3;\mathbb{R})$

[52] A *real Lie group of dimension* m is a topological group with a structure of real $\mathcal{C}^1$ manifold of dimension m which makes the multiplication and the inversion into differentiable maps. By the way, $GL(n,\mathbb{C})$ would be of course also a *complex* Lie group, but here we are interested only in its structure of real differentiable manifold.

[53] In the terminology of Lie theory, these vector subspaces are the *Lie algebras* associated to the Lie subgroups: in the ambient vector space $M(n,\mathbb{C})$, they are the tangent space to the Lie subgroups at the identity 1_n (hence, one uses also the notation $\mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$). In general, a Lie algebra on a field $\mathbb{K}$ is a vector space V on $\mathbb{K}$ endowed with a internal multiplication $[\,\cdot\,,\cdot\,]$ which is bilinear, antisymmetric and satisfying the Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for any $x,y,z \in V$: in our case, for $A, B \in M(n,\mathbb{C})$, it is $[A,B] = AB - BA$ (the commutator of A and B). Note that each one of the real subspaces of $M(n,\mathbb{C})$ considered here is stable with respect to such operation.

is homeomorphic to $\mathbb{P}^3$, and hence $\pi_1(SO(3;\mathbb{R})) \simeq \mathbb{Z}/2\mathbb{Z}$. In the general case, note that $SO(n;\mathbb{R})$ is a submanifold of the open subset $GL^+(n;\mathbb{R})$ of $M(n;\mathbb{R})$, of dimension $n(n-1)/2$. We embed $N = SO(n-1;\mathbb{R})$ into $M = SO(n;\mathbb{R})$ as those orthogonal transformations that fix the North pole e_n: note that $\dim M - \dim N = n-1$, hence for $n \geq 4$ we have $\pi_1(SO(n;\mathbb{R}) \setminus SO(n-1;\mathbb{R})) \simeq \pi_1(SO(n;\mathbb{R}))$. Now we aim to show that $SO(n;\mathbb{R}) \setminus SO(n-1;\mathbb{R})$ is an open subset of $SO(n;\mathbb{R})$ which deformation-retracts to a manifold homeomorphic to $SO(n-1;\mathbb{R})$, and this will imply that $\pi_1(SO(n;\mathbb{R})) \simeq \mathbb{Z}/2\mathbb{Z}$ for $n \geq 4$. Let $\rho : SO(n;\mathbb{R}) \to \mathbb{S}^{n-1}$ be the map $\rho(A) = Ae_n$: setting $V = \mathbb{S}^{n-1} \setminus \{e_n\}$, we have precisely $SO(n;\mathbb{R}) \setminus SO(n-1;\mathbb{R}) = \rho^{-1}(V)$. Now define a function $f : V \times SO(n-1;\mathbb{R}) \to \rho^{-1}(V)$. Let $s : \mathbb{S}^{n-1} \setminus \{-e_n\} \to SO(n)$ be the continuous application under which, given $x \neq -e_n$ (the South pole), $s(x)$ induces the identity on $\langle x, e_n\rangle^\perp$ and the rotation sending e_n into x in the plane $\langle x, e_n\rangle$. Let $a : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be the antipodal map: since $a(V) = \mathbb{S}^{n-1} \setminus \{-e_n\}$, for $x \in V$ the trasformation $s(a(x))$ is well defined. So let us set $f(x,\alpha) = a \circ s(a(x)) \circ \alpha$ (note that f is well defined because $f(x,\alpha)(e_n) = x \neq e_n$). Well, such f is a homeomorphism: namely it is continuous, and its inverse is given by $g(\beta) = (\beta(e_n), s(a(\beta(e_n)))^{-1} \circ a \circ \beta)$. Since V is contractible, we have proven the claim. Summarizing up, one has

$$\pi_1(GL(n,\mathbb{R}), 1_n) \simeq \pi_1(O(n,\mathbb{R}), 1_n) \simeq \pi_1(SO(n;\mathbb{R})) \simeq \begin{cases} \{1\} & (n=1) \\ \mathbb{Z} & (n=2) \\ \mathbb{Z}/2\mathbb{Z} & (n \geq 3) \end{cases}$$

- As for $SU(n,\mathbb{C})$, let us begin by observing that $SU(1,\mathbb{C}) = \{1\}$ is simply connected. Since $\dim_{\mathbb{R}}(SU(n,\mathbb{C})) = n^2 - 1$, if $n \geq 2$ one has $\dim_{\mathbb{R}}(SU(n,\mathbb{C})) - \dim_{\mathbb{R}}(SU(n-1,\mathbb{C})) = 2n-1 \geq 3$, and hence $\pi_1(SU(n,\mathbb{C}) \setminus SU(n-1,\mathbb{C})) \xrightarrow{\sim} \pi_1(SU(n,\mathbb{C}))$. On the other hand, arguing as before one shows that for $n \geq 2$ there exists a homeomorphism $f : SU(n,\mathbb{C}) \setminus SU(n-1,\mathbb{C}) \to V \times SU(n-1,\mathbb{C})$, where $V = \mathbb{S}^{2n-1} \setminus \{e_{2n}\} \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$, contractible. This implies that $SU(n,\mathbb{C})$ is simply connected and therefore

$$\pi_1(SU(n,\mathbb{C})) = \{1\}, \qquad \pi_1(GL(n,\mathbb{C})) \simeq \pi_1(U(n,\mathbb{C})) \simeq \mathbb{Z}.$$