Lezione 19

lunedì 10 novembre 2014
10:12

ES 1 (i) Sviluppare la funzione $[\log(1+x)]^2$ per $x \to 0$ in modo preciso fino al terzo ordine

(ii) Al variare di $\alpha \in \mathbb{R}$ calcolare il limite

$$\lim_{x \to 0^+} \frac{e^{\alpha x^2} - \cos x + [\log(1+x)]^2}{x^3} = ?$$

(i) Parto dallo sviluppo

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \qquad x \to 0$$

Faccio il quadrato

$$[\log(1+x)]^2 = \left[x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \right]^2$$

$$= x^2 - x^3 + o(x^3) \qquad x \to 0$$

(ii) Parto da

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3) \qquad x \to 0$$

Regola di sostituzione:

$$e^{\alpha x^2} = 1 + \alpha x^2 + \underbrace{\frac{\alpha^2 x^4}{2} + o(x^4)}_{= o(x^3)}$$

$$= 1 + \alpha x^2 + o(x^3) \qquad x \to 0$$

Sviluppo coseno

$$\cos x = 1 - \frac{x^2}{2!} + o(x^3)$$

$$-\cos x = -1 + \frac{x^2}{2} + o(x^3)$$

Posso sviluppare tutto il numeratore:

$$N(x) = e^{\alpha x^2} - \cos x + [\log(1+x)]^2$$

$$N(x) = e^{\alpha x^2} - \cos x + [\log(1+x)]^2$$

$$= \cancel{1} + \alpha x^2 + o(x^3) \cancel{-1} + \frac{x^2}{2} + o(x^3) + x^2 - x^3 + o(x^3)$$

$$= x^2\left(\alpha + \frac{1}{2} + 1\right) - x^3 + o(x^3)$$

$$= \left(\frac{3}{2} + \alpha\right) x^2 - x^3 + o(x^3) \qquad x \to 0$$

Calcolo il limite

$$L = \lim_{x \to 0^+} \frac{N(x)}{x^3} = \lim_{x \to 0^+} \frac{\boxed{\left(\frac{3}{2} + \alpha\right)} x^2 - x^3 + o(x^3)}{x^3} =$$

Distinguo i casi

$1°$: $\frac{3}{2} + \alpha = 0$

$$L = \lim_{x \to 0^+} \frac{-x^3 + o(x^3)}{x^3} = \lim_{x \to 0^+} - \frac{\cancel{x^3}(1 + o(1))}{\cancel{x^3}}$$

$$= -1$$

$2°$ caso $\frac{3}{2} + \alpha \neq 0$

$$L = \lim_{x \to 0^+} \frac{\left(\frac{3}{2} + \alpha\right) x^2 \overbrace{- x^3 + o(x^3)}^{= o(x^2)}}{x^3}$$

$$= \lim_{x \to 0^+} \underbrace{\frac{1}{x}}_{\to +\infty} \left[\left(\frac{3}{2} + \alpha\right) + o(1)\right] \begin{cases} +\infty & \frac{3}{2} + \alpha > 0 \\ -\infty & \frac{3}{2} + \alpha < 0 \end{cases}$$

ES 2 Calcolare il seguente limite

$$L = \lim_{x \to 0^+} \frac{(\cosh x - 1)\log x + x^{x+1} - x}{\sin^2(x) \log x} = \left[\frac{0}{0}\right]$$

$x \to 0^+ \qquad \sin^2(x) \quad \log x$

Soluzione.

Attenzione: $\log x$ <u>NON</u> si sviluppo per $x \to 0^+$

Al denominatore:

$$\sin x = x + o(x^2) \qquad x \to 0$$

$$(\sin x)^2 = (x + o(x^2))^2 = x^2 + 2x\, o(x^2) + o(x^2)^2$$

$$= x^2 + o(x^3)$$

Il denominatore è:

$$D(x) = \sin^2 x \ \log x = \left(x^2 + o(x^3)\right) \log x$$

$$= x^2 \cdot \log x \ (1 + o(x))$$

Lavoro sul numeratore:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5) \qquad x \to 0$$

$$\cosh x - 1 = \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) \qquad x \to 0$$

Voglio sviluppare x^{x+1} per $x \to 0$

$$x^{x+1} = x^x \cdot x = x\, e^{\log x^x} = x\, e^{\overset{\overset{0}{\uparrow}\ x \to 0^+ \ \text{NOTO}}{x \log x}}$$

Sviluppo dell'esponenziale:

$$e^x = 1 + x + \frac{1}{2} x^2 + o(x^2) \qquad x \to 0$$

Siccome $x \log x \to 0$ per $x \to 0$ posso usare la regola di sost.

$$e^{x \log x} = 1 + x \log x + \frac{1}{2} (x \log x)^2 + o\left((x \log x)^2\right)$$

Dunque

$$x^{x+1} = x\, x^x = x \left(1 + x \log x + \frac{1}{2} x^2 \log^2 x + o(x^2 \log^2 x)\right)$$

Sviluppo il numeratore

$$N(x) = (\cosh x - 1)\log x + x^{x+1} - x$$

$$= \left(\underbrace{\frac{x^2}{2}} + o(x^3)\right)\log x + \cancel{x} + \underbrace{x^2\log x} + \frac{1}{2}x^3\log^2 x + o(x^3\log^2 x) - \cancel{x}$$

$$= x^2\log x\left(\frac{1}{2} + 1\right) + o(x^2\log x)$$

$$= x^2\log x\left[\frac{3}{2} + o(1)\right]$$

Il limite è

$$L = \lim_{x\to 0^+}\frac{N(x)}{D(x)} = \lim_{x\to 0^+}\frac{\cancel{x^2\log x}\left(\frac{3}{2} + o(1)\right)}{\cancel{x^2\log x}\,(1 + o(1))} = \frac{3}{2}.$$

□

DOMANDA: Quanto lunghi prendere gli sviluppi?

Risposta: se nei conti dopo le semplificazioni algebriche rimangono solo gli "o piccoli" allora uno degli sviluppi era troppo corto.

ES. 3 Calcolare il seguente limite:

$$L = \lim_{x\to 0}\frac{\log\left(\frac{1+\sin x}{1+x}\right)}{\sin x + \log(1-x) + 1 - \cos x}.$$

Soluzione

$$\sin x = x - \frac{x^3}{3!} + o(x^4) \qquad x\to 0$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \qquad x\to 0$$

$(-x)^2 \qquad (-x)^3$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \qquad x \to 0$$

$$\log(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + o((-x)^3)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) \qquad x \to 0$$

$$1 - \cos x = \frac{x^2}{2} - \frac{x^4}{4!} + o(x^5)$$

ok

$$D(x) = \cancel{x} - \underbrace{\frac{x^3}{3!}} + o(x^4) - \cancel{x} - \frac{x^2}{2} - \underbrace{\frac{x^3}{3}} + o(x^3) + \frac{x^2}{2} - \frac{x^4}{4!} + o(x^5)$$

$$= x^3\left(-\frac{1}{6} - \frac{1}{3}\right) + o(x^3)$$

$$= \boxed{-\frac{1}{2}x^3 + o(x^3)}$$

Numeratore

$$N(x) = \log\left(\frac{1+\sin x}{1+x}\right) = \log(1+\sin x) - \log(1+x)$$

Ora

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \qquad x \to 0$$

Siccome $\sin x \to 0$ $\quad x \to 0$ posso usare la Reg. di sost.:

$$\log(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} + o(\sin^3 x)$$

$$x \to 0$$

ORA

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + o(x^3)\right)^2 = x^2 + o(x^3)$$

$$(\sin x)^3 = \left(x - x^3 + o(x^3)\right)^3$$

$$(\sin x)^3 = \left(x - \frac{x^3}{3!} + o(x^3)\right)^3$$

$$= \left(x\left(1 - \frac{x^2}{3!} + o(x^2)\right)\right)^3$$

$$= x^3 \cdot \left(1 - \frac{x^2}{3!} + o(x^2)\right)^3$$

$$= x^3 \,(1 + o(1))$$

Infine:

$$\boxed{o(\sin^3 x) = o(x^3)} \qquad x \to 0$$

Infatti

$$\lim_{x\to 0} \frac{o(\sin^3 x)}{x^3} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^3 \cdot o(1) = 0$$

Numeratore

$$N(x) = \log(1+\sin x) - \log(1+x)$$

$$= \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} + o(\sin^3 x)$$

$$- \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)$$

$$= \cancel{x} - \frac{x^3}{3!} + o(x^3) - \frac{1}{2}\left(\cancel{x^2} + o(x^3)\right) + \frac{1}{3}\left(x^3 + o(x^3)\right)$$

$$+ o(x^3)$$

$$- \cancel{x} + \cancel{\frac{x^2}{2}} - \frac{x^3}{3} + o(x^3)$$

$$= x^3 \left(-\frac{1}{6} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}}\right) + o(x^3)$$

$$= -\frac{1}{6} x^3 + o(x^3)$$

Conclusione

$$L = \lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{-\frac{1}{6}x^3 + o(x^3)}{-\frac{1}{2}x^3 + o(x^3)}$$

$$= \lim_{x\to 0} \cancel{\frac{x^3}{x^3}} \frac{-\frac{1}{6} + o(1)}{-\frac{1}{2} + o(1)} = \frac{-1/6}{-1/2} = \frac{2}{6} = \frac{1}{3}.$$