Esercizio 2 $\alpha \in \mathbb{R}$, limite

$$\lim_{x\to 0} \frac{\cos(\alpha x - x^2) + \operatorname{tg} x - \log(1+x) - 1}{\log(1+\sin x) - \operatorname{arctg} x} = L$$

Svolgimento. Sviluppi infinitesimali:

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{1}{4!}x^4 + o(x^4)$$

Mi fermo all'ordine 3:

$$\cos(\alpha x - x^2) = 1 - \frac{1}{2}(\alpha x - x^2)^2 + o(x^3)$$

$$= 1 - \frac{1}{2}(\alpha^2 x^2 + x^4 - 2\alpha x^3) + o(x^3)$$

$$= 1 - \frac{\alpha^2}{2}x^2 + \alpha x^3 + o(x^3) \qquad x \to 0.$$

Mi ricavo lo sviluppo

$$\operatorname{tg} x = x + Ax^3 + o(x^3) \qquad x \to 0$$

con $A \in \mathbb{R}$ da determinare:

$$A = \lim_{x\to 0} \frac{\operatorname{tg} x - x}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{x^3} \frac{\sin x - x\cos x}{\cos x}$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left(x - \frac{x^3}{6} + o(x^3) - x\left(1 - \frac{x^2}{2} + o(x^2)\right)\right)$$

$$= -\frac{1}{6} + \frac{1}{2} = \frac{3-1}{6} = \frac{1}{3}$$

Dunque $\operatorname{tg} x = x + \frac{1}{3}x^3 + o(x^3) \qquad x \to 0.$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

Numeratore:

$$N(x) = \cancel{1} - \frac{\alpha^2}{2}x^2 + \alpha x^3 + o(x^3) \; \boxed{\cancel{+x}} + \frac{1}{3}x^3 + o(x^3)$$

$$\boxed{\cancel{-x}} + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3) \; \cancel{-1}$$

$$= x^2\left(-\frac{\alpha^2}{2} + \frac{1}{2}\right) + x^3\left(\alpha + \frac{1}{3} - \frac{1}{3}\right) + o(x^3)$$

$$= \frac{1}{2}(-\alpha^2+1)\,x^2 + \alpha x^3 + o(x^3)$$

L'ordine è sempre 2 quando $\alpha^2 \neq 1$, se $\alpha^2 = 1$ l'ordine è 3.

$$\log(1+\sin x) = \sin x - \frac{1}{2}\sin^2 x + o(\sin^2 x)$$

$$= x - \frac{1}{2}x^2 + o(x^2)$$

$$\operatorname{arctg} x = x - \frac{1}{3}x^3 + o(x^3)$$

$$= x + o(x^2)$$

Denominatore:

$$D(x) = -\frac{1}{2}x^2 + o(x^2)$$

$$L = \lim_{x\to 0}\frac{N(x)}{D(x)} = \lim_{x\to 0}\frac{\frac{1}{2}(-\alpha^2+1)x^2 + o(x^2)}{-\frac{1}{2}x^2 + o(x^2)}$$

Distinguiamo i seguenti casi:

1° CASO: $1-\alpha^2 \neq 0$

$$L = \lim_{x \to 0} \frac{\frac{1}{2}(1-\alpha^2)\,x^2 + o(x^2)}{-\frac{1}{2}x^2 + o(x^2)} = \lim_{x \to 0} \frac{\frac{1}{2}(1-\alpha^2) + o(1)}{-\frac{1}{2} + o(1)} = \alpha^2 - 1$$

2° CASO: $1-\alpha^2 = 0$

$$L = \lim_{x \to 0} \frac{\frac{1}{2}\overbrace{(1-\alpha^2)}^{=0}x^2 + \alpha x^3 + o(x^3)}{-\frac{1}{2}x^2 + o(x^2)}$$

$$= \lim_{x \to 0} \frac{x^3}{x^2} \, \frac{\alpha + o(1)}{-\frac{1}{2} + o(1)} = 0 \cdot \frac{\alpha}{-1/2} = 0.$$

$\square$

Esercizio 3 $\alpha \in \mathbb{R}$ integrale ; (2) Calcolare per $\alpha = 1/4$

(1) Convergenza

$$\int_0^1 \frac{x\, e^{x/3}}{(e^{x/3}-1)^{2\alpha}}\, dx$$

Soluzione.

(1) Convergenza. Integrale improprio in $x = 0$

$$e^{x/3} = 1 + \frac{x}{3} + o(x) \quad \text{per} \quad x \to 0$$

Dunque $f(x) = \dfrac{x\, e^{x/3}}{(e^{x/3}-1)^{2\alpha}}$ verifica

$$f(x) = \frac{x}{x^{2\alpha}} \frac{e^{x/3}}{\left(\frac{1}{3}+o(1)\right)^{2\alpha}} = \frac{1}{x^{2\alpha-1}} \left(3 + o(1)\right)^{2\alpha}$$

Per il criterio del confronto Asintotico l'integrale converge se e solo se converge

$$\int_0^1 \frac{1}{x^{2\alpha-1}}\, dx < \infty \iff 2\alpha - 1 < 1$$
$$\iff \boxed{\alpha < 1}\,.$$

(2) Per $\alpha = 1/4$

$$I = \int_0^1 \frac{x\, e^{x/3}}{\sqrt{e^{x/3}-1}}\, dx = \int_0^{t_1} \frac{3\log(1+t^2) \cdot (1+t^2)}{t} \frac{6t}{1+t^2}\, dt$$

Sostituzione

$t = \sqrt{e^{x/3} - 1}$ $\qquad x = 0 \to t = 0$

$t^2 = e^{x/3} - 1$ $\qquad x = 1 \to t = \sqrt{e^{1/3}-1} = t_1$

$t^2 + 1 = e^{x/3}$

$x = 3\log(1+t^2)$ $\qquad dx = 3\, \dfrac{2t}{1+t^2}\, dt$

Quindi:

$$I = 18 \int_0^{t_1} \log(1+t^2)\, dt =$$

$$= 18 \left[\left[t \log(1+t^2) \right]_{t=0}^{t=t_1} - \int_0^{t_1} t \, \frac{2t}{1+t^2} \, dt \right]$$

$$= 18 \left[t_1 \log(1+t_1^2) - 2 \int_0^{t_1} \frac{t^2}{1+t^2} \, dt \right]$$

$$= \ldots$$

A parte:

$$\int_0^{t_1} \frac{t^2}{1+t^2} \, dt = \int_0^{t_1} \left(1 - \frac{1}{1+t^2}\right) dt = t_1 - \operatorname{arctg}(t_1)$$

Dunque

$$I = 18 \left[t_1 \log(1+t_1^2) - 2t_1 + 2 \operatorname{arctg}(t_1) \right]$$

ESERCIZIO 4 Disequazione in $\mathbb{C}$

$$\mathrm{Im}\left(i\,(\bar{z}+i-1)^2\right) \geq \mathrm{Re}\left((z-2-i)^2\right)$$

Soluzione. $z = x+iy$ con $x, y \in \mathbb{R}$:

$$i(\bar{z}+i-1)^2 = i(x-iy+i-1)^2 = i\left((x-1)+i(1-y)\right)^2$$

$$= i\left((x-1)^2-(1-y)^2+2i(x-1)(1-y)\right)$$

$$= i\left((x-1)^2-(1-y)^2\right) - 2(x-1)(1-y)$$

Dunque

$$\mathrm{Im}\left(i\,(\bar{z}+i-1)^2\right) = (x-1)^2-(1-y)^2$$

$$(z-2-i)^2 = (x+iy-2-i)^2 = (x-2+i(y-1))^2 =$$

$$= (x-2)^2-(y-1)^2+2i(x-2)(y-1)$$

$$\mathrm{Re}\left((z-2-i)^2\right) = (x-2)^2-(y-1)^2$$

La diseq. iniziale diventa:

$$(x-1)^2-(1-y)^2 \geq (x-2)^2-(y-1)^2$$

$$\Updownarrow$$

$$x^2-2x+1 \geq x^2-4x+4$$

$$\Updownarrow$$

$$2x \geq 3$$