Lezione 10

martedì 5 aprile 2016 12:28

Tutorato Lu + Giov. 14.30 - 16.15 B

Did. Supporto Merc. 16.30 - 18 (B)

[Materiale on line ?

Filo della dimostrazione

$$\int_0^1 |\gamma'(s)|\, ds = \dots \le \varepsilon + \sum_{i=1}^{k} \left| \int_{t_{i-1}}^{t_i} \gamma'(t_{i-1})\, ds - \gamma'(s) + \gamma'(s) \right| \le$$

$$\le \varepsilon + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \underbrace{|\gamma'(t_{i-1}) - \gamma'(s)|}_{\le \varepsilon} ds + \sum_{i=1}^{k} \left| \int_{t_{i-1}}^{t_i} \gamma'(s)\, ds \right|$$

$$\le 2\varepsilon + \boxed{\sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|}$$

]

ES 1

$$\gamma(t) = \left(t^2, \frac{2t^3}{3} - t^2\right) \qquad t \in \mathbb{R}$$

Caso 1 : $t \geq 0$. Poniamo

$$t^2 = x \iff t = \sqrt{x}$$
$$x \geq 0$$

$$\gamma(t) = \Big(x, \underbrace{\tfrac{2}{3} x^{3/2} - x}\Big)$$

$$\underbrace{\qquad\qquad}_{\displaystyle \| \atop \displaystyle f(x)} \quad , \quad x \geq 0$$

Per $t \geq 0$ avremo

$$\operatorname{Supt}(\gamma) = \operatorname{gr}(f)$$

Studio f:

Calcolo la derivata

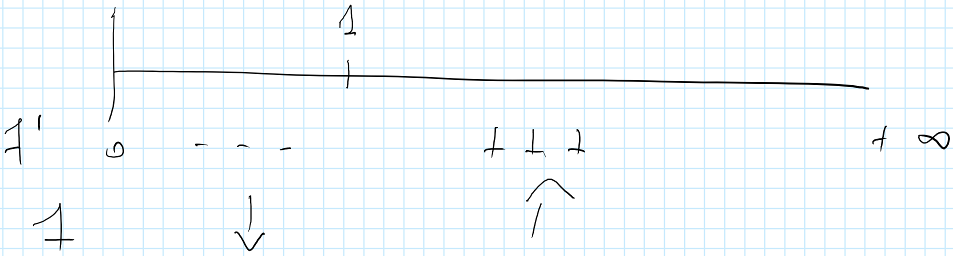
$$f'(x) = \sqrt{x} - 1$$

Intervalli di monot.

$$f'(x) \geq 0 \iff \sqrt{x} - 1 \geq 0$$

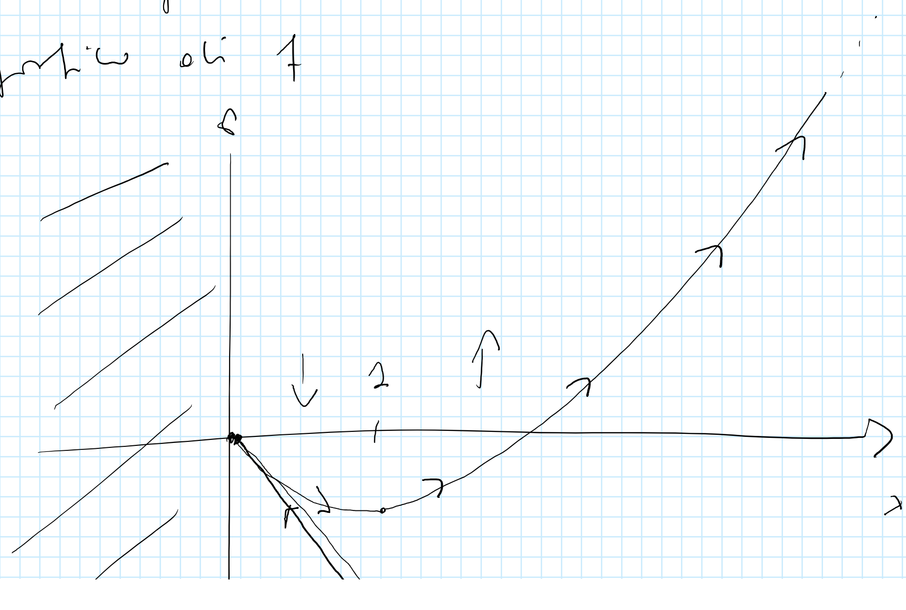
$$\iff \sqrt{x} \geq 1$$

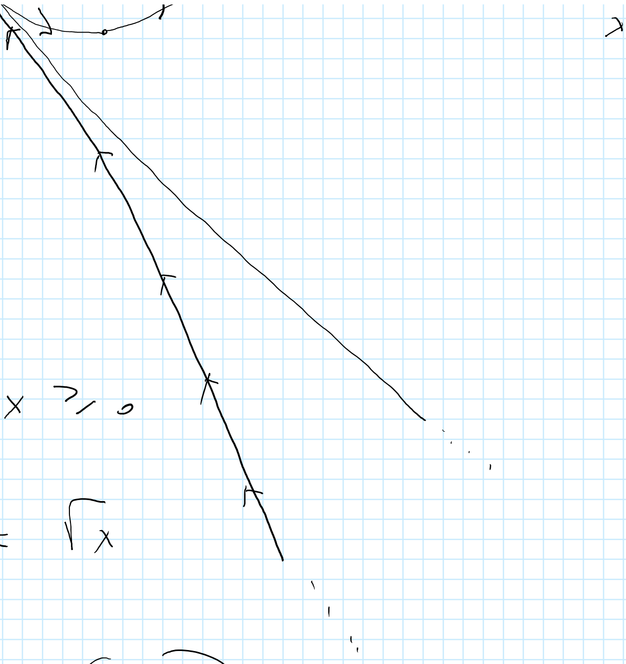
$$\iff x \geq 1$$



$x = 1$ pto di min assoluto

Grafico di f





Caso 2: $t \leq 0$

Poniamo $t^2 = x \geq 0$

$$-t = |t| = \sqrt{t^2} = \sqrt{x}$$

$$\Updownarrow$$

$$t = -\sqrt{x}$$

Dunque

$$r(t) = (x, \underbrace{-x^{\frac{3}{2}} - x}_{= g(x)}), \quad t \leq 0, \quad x \geq 0$$

- $g(0) = 0$, $\lim_{x \to \infty} g(x) = -\infty$
- decresce.

~~**ES. 2** Si consideri il tratto di cicloide $r: [0, 2\pi] \to \mathbb{R}^2$~~

~~$r(t) = (t - \sin t, 1 - \cos t)$,~~

~~$t \in [0, 2\pi]$~~

Posto $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y - 1 \geq 0\}$

$$\Updownarrow$$

$y \geq 1 - x^2$

si consideri $f : A \to \mathbb{R}$

$$f(x,y) = \sqrt{x^2 + y - 1} .$$

Al variare di $\alpha \in \mathbb{R}$ si consideri

$$\gamma_\alpha : [-1,1] \to \mathbb{R}^2$$

$$\gamma_\alpha(t) = (t, 1 - \alpha t^2)$$

$$t \in [-1, 1]$$

Determinare tutti gli $\alpha \in \mathbb{R}$ tali che $\gamma_\alpha(t) \in A \quad \forall t \in [-1,1]$.
Per tali α calcolare

$$I_\alpha = \int_{\gamma_\alpha} f \, ds .$$

Soluzione. Voglio che

$$\gamma_\alpha(t) \in A = \{ x^2 + y - 1 \geq 0 \} \quad \forall t \in [-1,1]$$

$$\{ (x,y) \in \mathbb{R}^2 : x^2 + y - 1 \geq 0 \}$$

$$= (\underbrace{t}_{x}, \underbrace{1 - \alpha t^2}_{y})$$

Deve essere

$$t^2 + \cancel{1} - \alpha t^2 \cancel{- 1} \geq 0 \quad \forall t \in [-1,1]$$

$$\Updownarrow$$

$$t^2 (1 - \alpha) \geq 0 \quad \forall t \in [-1,1]$$

$$\Updownarrow$$

$$\Updownarrow$$

$$1 - \alpha \geq 0 \iff \alpha \leq 1$$

Conto:

$$\int_{\gamma_\alpha} f \, ds = \int_{-1}^{1} f(\gamma_\alpha(t)) \, |\dot\gamma_\alpha(t)| \, dt$$

Abbiamo

$$\dot\gamma_\alpha(t) = (1, -2\alpha t)$$

$$|\dot\gamma_\alpha(t)| = \sqrt{1 + 4\alpha^2 t^2}$$

poi

$$f(\gamma_\alpha(t)) = \sqrt{t^2 + \cancel{1} - \alpha t^2 - \cancel{1}} =$$

$$f(x,y) = \sqrt{x^2 + y - 1}$$

$$= \sqrt{t^2(1-\alpha)}$$

$$= |t| \sqrt{1-\alpha}$$

Conclusione

$$I_\alpha = \int_{-1}^{1} |t| \sqrt{1-\alpha} \sqrt{1 + 4\alpha^2 t^2} \, dt$$

$$= 2\sqrt{1-\alpha} \int_0^1 t \sqrt{1 + 4\alpha^2 t^2} \, dt$$

$$t^2 = s$$

$$2t\,dt = ds$$

$$= \sqrt{1-\alpha} \int_0^1 \sqrt{1+4\alpha^2 s}\, ds$$

$$= \sqrt{1-\alpha} \left[\frac{(1+4\alpha^2 s)^{\frac{3}{2}}}{\frac{3}{2}\, 4\alpha^2} \right]_{s=0}^{s=1}$$

$= \dots$ fine.

<u>ES. 3</u> Sia $\alpha > 0$ un parametro e consider. la curva piana

$$\gamma : [0,1] \to \mathbb{R}^2$$

$$\circledast \quad \gamma(t) = \left(t^2 \cos\left(\frac{1}{t^\alpha}\right), t^2 \sin\left(\frac{1}{t^\alpha}\right) \right) t \in (0,1]$$

$$\gamma(0) = (0,0) \qquad t = 0$$

(1) Riparametrizz. γ in coord. polari e disegnare il supp. di γ

(2) Calcolare tutti gli $\alpha > 0$ tali che γ sia rettificabile.

Risoluzione

(1) Sostituzione $\frac{1}{t^\alpha} = \vartheta$

$$\Updownarrow$$

$$t^\alpha = \frac{1}{\vartheta}$$

$$t^{\alpha} = \frac{1}{\vartheta} \qquad \vartheta > 0$$

$$\Updownarrow$$

$$t = \frac{1}{\vartheta^{1/\alpha}}$$

$0 \leq t \leq 1$ $\longrightarrow$ $0 \leq \dfrac{1}{\vartheta^{1/\alpha}} \leq 1$

$$\Updownarrow$$

$$1 \leq \vartheta^{\frac{1}{\alpha}}$$

$$\Updownarrow$$

$$1 \leq \vartheta$$

La curva appare così:

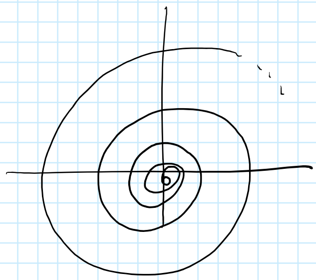
$$\gamma(t) = \left(\vartheta^{-\frac{2}{\alpha}} \cos\vartheta \,,\ \vartheta^{-\frac{2}{\alpha}} \sin\vartheta\right)$$

$$\vartheta \geq 1$$

Equazione polare della curva

$$r(\vartheta) = \vartheta^{-\frac{2}{\alpha}}$$

Spirale



$$\vartheta \mapsto e^{i\vartheta} = (\cos\vartheta + i \sin\vartheta)$$

$$\equiv (\cos\vartheta, \sin\vartheta)$$

Devo vedere se la curva ha lunghezza finita. Uso la formula della lunghezza in coordinate polari.

$$L(\gamma) = \int_1^{\infty} \sqrt{r(\vartheta)^2 + r'(\vartheta)^2}\, d\vartheta$$

dove

$$r(\vartheta) = \vartheta^{-\frac{2}{\alpha}}$$

$$r'(\vartheta) = -\frac{2}{\alpha}\, \vartheta^{-\frac{2}{\alpha}-1}$$

Quindi

$$L(\gamma) = \int_1^{\infty} \sqrt{\vartheta^{-\frac{4}{\alpha}} + \boxed{\frac{4}{\alpha^2}\, \vartheta^{-\frac{4}{\alpha}-2}}}$$

Devo discutere la convergenza dell'integr. improprio $d\vartheta$

Idea! FARE confr. Asint.

$$L(\gamma) = \int_1^{\infty} \sqrt{\vartheta^{-\frac{4}{\alpha}} \left(1 + \frac{4}{\alpha^2}\frac{1}{\vartheta^2}\right)}\, d\vartheta$$

$$= \int_1^{\infty} \vartheta^{-\frac{2}{\alpha}} \sqrt{1 + \frac{4}{\alpha^2}\frac{1}{\vartheta^2}}\, d\vartheta$$

Per il CCA

$$L(\gamma) < \infty \iff \int_1^{\infty} \vartheta^{-\frac{2}{\alpha}}\, d\theta < \infty$$

$$\int_1^{\infty} \frac{1}{\vartheta^{\frac{2}{\alpha}}}\, d\theta < \infty$$

$$\iff \frac{2}{\alpha} > 1$$

$$\iff \boxed{\alpha < 2}$$

$\square$

ES 4 Calcolare la lunghezza del tratto di cicloide:

$$\gamma : [0, 2\pi] \to \mathbb{R}^2$$

$$\gamma(t) = (t - \sin t \,,\, 1 - \cos t)$$

Soluzione.

$$L(\gamma) = \int_0^{2\pi} |\gamma'(t)| \, dt$$

Conti:

$$\gamma'(t) = \left(1 - \cos t, \ \sin t\right)$$

$$|\gamma'(t)| = \sqrt{(1-\cos t)^2 + (\sin t)^2}$$

$$= \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1}}$$

$$= \sqrt{2 - 2\cos t}$$

$$= \sqrt{2} \sqrt{1 - \cos t}$$

Devo calcolare

$$L(\gamma) = \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} \, dt$$

$$1-\sin^2 x$$

$$\boxed{\cos(2x) = \cos^2 x - \sin^2 x}$$

$$= 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\sin x = \sqrt{\frac{1-\cos 2x}{2}} \qquad 2x = t$$

$$\sin\left(\frac{t}{2}\right) = \frac{\sqrt{1-\cos t}}{\sqrt{2}}$$

Quindi:

$$L(\gamma) = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\, dt$$

$$= 2\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt$$

Ok fine.

ES. 5 Sia $\gamma : [0, 1/2] \to \mathbb{R}^2$ la curva data dall'eq. polare

$$r(\theta) = \begin{cases} -\dfrac{1}{\log \theta} & \theta \in (0, \frac{1}{2}] \\ 0 & \theta = 0 \end{cases}$$

Disegnare il supporto di γ e stabilire se γ è rettificabile.

Soluzione.

$$r(\vartheta) \geq 0$$

$r(\vartheta)$ funz. monotona

$$\lim_{\vartheta \to 0^+} -\frac{1}{\log(\vartheta)} = 0$$

Supporto: Non è una spirale.

Lunghezza

$$L(\gamma) = \int_0^{1/2} \sqrt{r^2 + r'^2}\, d\vartheta$$

dove

$$r(\vartheta) = -\frac{1}{\log \vartheta}$$

$$r'(\vartheta) = -\frac{\frac{1}{\vartheta}}{(\log(\vartheta))}$$

$$= \frac{1}{\vartheta (\log \vartheta)^2}$$

F.d.L.

$$L(\gamma) = \int_0^{1/2} \sqrt{\frac{1}{\log^2 \vartheta} + \frac{1}{\vartheta^2 (\log \vartheta)^4}}\, d\vartheta$$

Osservo che

$$\lim_{\vartheta \to 0^+} \vartheta^2 (\log \vartheta)^4 = 0$$

$$\int^{1/2} \frac{1}{} \sqrt{(1) + \vartheta^2 \log^2 \vartheta}\, d\vartheta$$

$$L(r) = \int_0^{1/2} \frac{1}{\vartheta(\log\vartheta)^2}\sqrt{\underbrace{1 + \vartheta^2\log^2\vartheta}_{\substack{\text{per } \vartheta\to 0 \\ 1}}}\, d\vartheta$$

Per il CCA

$L(\gamma) < \infty \iff = \int_0^{1/2} \frac{1}{\vartheta(\log\vartheta)^2}\, d\vartheta < \infty$

$$= \int_0^{1/2} \frac{1}{\vartheta}(\log\vartheta)^{-2}\, d\theta$$

$$= \Big[-(\log\vartheta)^{-1} \Big]_{\theta=0}^{\theta=1/2}$$

$$= -\left(\left(\log\tfrac{1}{2}\right)^{-1} - 0 \right)$$

Converge!

Quindi r è rettificabile. $\square$

$r = r(\vartheta)$

$\gamma(\vartheta) = (r(\vartheta)\cos\vartheta, r(\vartheta)\sin\vartheta)$

$= r(\theta) \; (\cos\theta, \sin\theta)$

La curva γ non è una spirale. $\theta \in [0, \frac{1}{2}]$

Il suo supporto è: