REAL ANALYSIS EXAMS
A.A 2013–14

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Analisi Reale per Matematica – Precompitino – 8 novembre 2013

Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) For $f : X \to [0, \infty]$ measurable we can define $\nu_f : \mathcal{M} \to [0, \infty]$ by $\nu_f(E) = \int_E f \, d\mu$. Prove that ν_f is countably additive.

(ii) If $f \in L^1_\mu(X, \mathbb{R})$ we can still define the set function $\nu_f : \mathcal{M} \to \mathbb{R}$ by the same formula $\nu_f(E) = \int_E f \, d\mu$. Prove that ν_f is countably additive, that is, for every disjoint sequence $A_n \in \mathcal{M}$, if $A = \bigcup_{n=0}^\infty A_n$ we have $\nu_f(A) = \sum_{n=0}^\infty \nu_f(A_n)$. Is this series absolutely convergent?

(iii) If $f, g \in L^1_\mu(X, \mathbb{R})$ and $\nu_f(X) = \nu_g(X)$, then the set

$$\mathcal{C} = \{A \in \mathcal{M} : \nu_f(A) = \nu_g(A)\}$$

is a Dynkin class of subsets of X.

From now on we assume that $\mathcal{E}$ is a semialgebra that generates $\mathcal{M}$, i.e. $\mathcal{M} = \mathcal{M}(\mathcal{E})$.

(iv) Prove that if $f, g \in L^1_\mu(X, \mathbb{R})$ and $\nu_f(E) = \nu_g(E)$ for every $E \in \mathcal{E}$ then $\nu_f(A) = \nu_g(A)$ for every $A \in \mathcal{M}$, and deduce from this that $f(x) = g(x)$ for a.e. $x \in X$.

(v) Can we reach the same conclusion in (v) assuming $f, g \in L^1_\mu(X, \mathbb{C})$ (that is, f and g not necessarily real valued?)

Solution. (i) This is the termwise integration of series of positive functions, an immediate consequence of the monotone convergence theorem: if $(A_n)_n$ is a disjoint sequence of sets in $\mathcal{M}$, with union $A = \bigcup_{n=0}^\infty A_n$, and χ_n, χ are the characteristic functions of A_n, A then $\chi(x) = \sum_{n=0}^\infty \chi_n(x)$, for every $x \in X$; then $f\,\chi = \sum_{n=0}^\infty f\,\chi_n$, and by the aforementioned theorem on termwise integration of series of positive functions we have

$$\int_X f\,\chi = \sum_{n=0}^\infty \int_X f\,\chi_n \quad \text{that is} \quad \nu_f(A) = \sum_{n=0}^\infty \nu_f(A_n).$$

(ii) As in (i) $(A_n)_n$ is a disjoint sequence of sets in $\mathcal{M}$, with union $A = \bigcup_{n=0}^\infty A_n$, and χ_n, χ are the characteristic functions of A_n, A. The result now follows from the theorem on normally convergent series: we have, by(i):

$$\nu_{|f|}(A) = \sum_{n=0}^\infty \nu_{|f|}(A_n) \quad \text{equivalently} \quad \|f\,\chi\|_1 = \sum_{n=0}^\infty \|f\,\chi_n\|_1,$$

and by the theorem on normal convergence the series $\sum_{n=0}^\infty f\,\chi_n$ converges in $L^1(\mu)$ to its pointwise sum $f\,\chi$, by dominated convergence, and of course:

$$\int_X f\,\chi = \sum_{n=0}^\infty \int_X f\,\chi_n \quad \text{that is} \quad \nu_f(A) = \sum_{n=0}^\infty \nu_f(A_n).$$

Since

$$|\nu_f(A_n)| = \left|\int_{A_n} f\right| \le \int_{A_n} |f| = \|f\,\chi_n\|_1, \quad \text{and} \quad \sum_{n=0}^\infty \|f_n\|_1 = \nu_{|f|}(A) < \infty,$$

the convergence is absolute.

(iii) By hypothesis we have $X \in \mathcal{C}$. If $A \in \mathcal{C}$ we have:

$$\nu_f(X \setminus A) = \nu_f(x) - \nu_f(A) = \nu_g(X) - \nu_g(A) = \nu_g(X \setminus A).$$

And since both functions ν_f and ν_g are countably additive, as proved in (ii), the class $\mathcal{C}$ is closed under countable disjoint union.

(iv) The set $\mathcal{C} = \{A \in \mathcal{M} : \nu_f(A) = \nu_g(A)\}$ contains $\mathcal{E}$, which is by hypotehsis intersection closed; and $X \in \mathcal{C}$ (since $\mathcal{E}$ is a semialgebra, $\emptyset$ belongs to $\mathcal{E}$, hence X is a finite disjoint union of elements of $\mathcal{E}$, so that $X \in \mathcal{C}$). Then $\mathcal{C}$ is a Dynkin class, by (iii), and by Dynkin theorem then $\mathcal{C}$ is $\sigma-$algebra, which coincides with the $\sigma-$algebra $\mathcal{M}(\mathcal{E})$ generated by $\mathcal{E}$. We have then

$$\int_E f = \int_E g \quad \text{for every } E \in \mathcal{M}(= \mathcal{M}(\mathcal{E})),$$

and this implies $f = g$ q.o. In fact, let $A = \{f < g\}$; then $\int_A f \le \int_A g$, with equality iff $\int_E (g - f) = 0$, which happens iff $\mu(A) = 0$ (recall that $g(x) > f(x)$ for *every* $x \in A$!); since our hypothesis implies $\int_A f = \int_A g$, we have in fact $\mu(A) = 0$, In the same way we get that $B = \{f > g\}$ has measure 0, so that $\{f \ne g\} = \{f < g\} \cup \{f > g\}$ has measure 0.

(v) Of course yes: the hypothesis $\int_E f = \int_E g$ is equivalent to $\int_E \operatorname{Re} f = \int_E \operatorname{Re} g$ and $\int_E \operatorname{Im} f = \int_E \operatorname{Im} g$, for every $E \in \mathcal{M}$. □

Exercise 2. (i) Using the theorem on differentiability of parameter depending integrals prove that the formula:

(*)
$$\varphi(x) = \int_1^\infty \frac{e^{-xt}}{t}\, dt$$

defines a function $\varphi \in C^1(]0, \infty[, \mathbb{R})$.

(ii) Find for φ' an expression not containing integrals.

(iii) What are the limits

$$\lim_{x \to 0^+} \varphi(x); \quad \lim_{x \to \infty} \varphi(x)?$$

Solution. (i) Clearly $t \mapsto e^{-xt}/t$ belongs to $L^1([1, \infty[)$ for every $x > 0$, so φ is defined for $x > 0$. We have $\partial_x (e^{-xt}/t) = -e^{-xt}$. Given $x > 0$, let $U = [x/2, \infty[$; the function $\gamma(t) = e^{-(x/2)t}$ is in $L^1([1, \infty[)$ and $e^{-yt}(= |-e^{-yt}|) \le \gamma(t)$ for every $y \in U$ and $t \ge 1$. By the theorem on differentiability we get

$$\varphi'(x) = \int_1^\infty \partial_x(e^{-xt}/t)\, dt = \int_1^\infty (-e^{-xt})\, dt = \left[\frac{e^{-xt}}{x}\right]_{t=1}^{t=\infty} = -\frac{e^{-x}}{x}.$$

We have also solved (ii).

(iii) Notice that for fixed $t \ge 1$ the function $x \mapsto e^{-xt}/t$ is decreasing on $]0, \infty[$ (trivially). If $x_j \downarrow 0$ we then have that the sequence $f_j(t) = e^{-x_j t}/t$ is increasing and converges to $t \mapsto 1/t$. By the monotone convergence theorem we then have

$$\int_1^\infty f_j(t)\, dt \uparrow \int_1^\infty \frac{dt}{t} = \infty \quad \text{in other words} \quad \lim_{x \to 0^+} \varphi(x) = \infty.$$

And if $x_j \uparrow \infty$ then $f_j(t) = e^{-x_j t}/t$ is dominated by $f_0 \in L^1([1, \infty[)$ and converges pointwise to 0 so that, by dominated convergence:

$$\lim_{j \to \infty} \int_1^\infty f_j(t)\, dt = 0 \quad \text{in other words} \quad \lim_{x \to \infty} \varphi(x) = 0.$$

Remark. Of course in the second case we can also argue like that: $e^{-xt}/t \le e^{-xt}$ for $t \ge 1$, so that

$$0 < \varphi(x) \le \int_1^\infty e^{-xt}\, dt = \frac{e^{-x}}{x},$$

and $e^{-x}/x \to 0$ as $x \to +\infty$. Moreover, a change of variable $xt = s$ in the integral defining φ shows that

$$\varphi(x) = \int_x^\infty \frac{e^{-s}}{s}\, ds,$$

and we can use easier results on integral functions, e.g. Torricelli's theorem and the like, to prove differentiability of φ.

□

Exercise 3. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence f_n of measurable functions *converges to* 0 *in measure* if for every $t > 0$ we have $\lim_{n \to \infty} \mu(\{|f_n| > t\}) = 0$.

(i) Using Čebičeff inequality prove that if $\|f_n\|_1 \to 0$, then f_n converges to 0 in measure.

(ii) With $X = [0, 1]$ and μ Lebesgue measure, let $f_n = n^2\, \chi_{]0,1/n]}$. Is it true that f_n converges to 0 in measure? and in $L^1(\mu)$ also?

(iii) Assume now that f_n is a uniformly bounded sequence of measurable functions on X (that is, there is a constant $M > 0$ such that $\|f_n\|_\infty \le M$ for every $n \in \mathbb{N}$), and that $\mu(X) < \infty$. Prove that if f_n converges to 0 in measure then it converges to 0 in $L^1(\mu)$ (hint: given $\varepsilon > 0$ write

$$\int_X |f_n| = \int_{\{|f_n|>\varepsilon\}} |f_n| + \int_{\{|f_n|\le\varepsilon\}} |f_n|$$

and estimate separately the two terms).

(iv) A sequence f_n of real–valued measurable functions converges to 0 in measure if and only if the sequence $f_n/(1 + |f_n|)$ converges to 0 in measure.

(v) On a finite measure space a sequence f_n of real–valued measurable functions converges to 0 in measure if and only if the sequence $f_n/(1 + |f_n|)$ converges to 0 in $L^1(\mu)$.

Solution. (i) For every $t > 0$ and every $n \in \mathbb{N}$ we have $\mu(\{f_n > t\}) \le (1/t) \int_X |f_n| = (1/t)\, \|f_n\|_1$; letting $n \to \infty$ in this inequality we get $\lim_{n\to\infty} \mu(\{f_n > t\}) = 0$.

(ii) Given $t > 0$ we have that $\{|f_n| > t\} = \{f_n > t\} =]0, 1/n]$ for $n^2 > t$ (and $\{f_n > t\} = \emptyset$ for $n^2 \le t$) so that $\mu(\{|f_n| > t\} = 1/n$ tends to 0 as $n \to \infty$, and f_n converges to 0 in measure. On the other hand $\|f_n\|_1 = \int_{[0,1]} f_n\, dm = n^2\, m(]0, 1/n]) = n$ for every n, so that $\|f_n\|_1$ tends to infinity, and f_n does not converge in $L^1_m([0, 1])$ (to 0, or to any other function).

(iii) Accepting the hint we write

$$(*)\qquad \int_X |f_n| = \int_{|f_n|>\varepsilon} |f_n| + \int_{\{|f_n|\le\varepsilon\}} |f_n| \le \int_{|f_n|>\varepsilon} M + \int_{\{|f_n|\le\varepsilon\}} \varepsilon \le$$
$$\le M\, \mu(\{|f_n| > \varepsilon\}) + \varepsilon\, \mu(\{|f_n| \le \varepsilon\}) \le M\, \mu(\{|f_n| > \varepsilon\}) + \varepsilon\, \mu(X);$$

by hypothesis $\lim_{n\to\infty} \mu(\{|f_n| > \varepsilon\}) = 0$, so that we may pick $n_\varepsilon \in \mathbb{N}$ such that if $n \ge n_\varepsilon$ then $\mu(\{|f_n| > \varepsilon\}) \le \varepsilon/M$; then

$$\|f_n\|_1 = \int_X |f_n| \le (1 + \mu(X))\, \varepsilon \quad \text{for} \quad n \ge n_\varepsilon;$$

the proof of (iii) is completed.

(iv) First observe that in the definition of convergence to 0 in measure it is not restrictive to assume $0 < t < 1$, that is, a sequence f_n of measurable functions converges to 0 in measure if for every t with $0 < t < 1$ we have $\lim_{n\to\infty} \mu(\{|f_n| > t\}) = 0$ (clear: if $t < s$ we have $\{|f_n| > t\} \supseteq \{|f_n| > s\}$, so that $\lim_{n\to\infty} \mu(\{|f_n| > t\}) = 0$ implies $\lim_{n\to\infty} \mu(\{|f_n| > s\}) = 0$). We now see that for every function $f : X \to \mathbb{R}$ and $0 < t < 1$ and $x \in X$ we have

$$\frac{|f(x)|}{1 + |f(x)|} > t \iff (1 - t)\, |f(x)| > t \iff |f(x)| > \frac{t}{1 - t};$$

after this the answer is immediate: If f_n converges to 0 in measure then $\mu(\{|f_n|/(1 + |f_n|) > t\}) = \mu(\{|f_n| > (t/(1 - t)\}$ tends to 0 as $n \to \infty$, and if $f_n/(1 + |f_n|)$ tends to 0 in measure then $\mu(\{|f_n| > t\}) = \mu(\{|f_n|/(1 + |f_n|) > t/(1 + t)\})$ tends to 0 as $n \to \infty$.

(v) If $f_n/(1 + |f_n|)$ converges to 0 in $L^1(\mu)$, then by (i) it converges to 0 in measure, and by (iv) then also f_n converges to 0 in measure (no need for $\mu(X) < \infty$). And if f_n converges to 0 in measure then by (iv) also $f_n/(1 + |f_n|)$ converges to 0 in measure; since all $f_n/(1 + |f_n|)$ are uniformly bounded by 1, and $\mu(X) < \infty$, (iii) shows that this sequence converges to 0 in $L^1(\mu)$.

□

Analisi Reale per Matematica – Primo compitino – 16 novembre 2013

Exercise 4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Let $f_n \in L^+(X)$ be a sequence of positive measurable functions converging a.e. to $f : X \to [0, \infty]$. Assume that there is a real number $M > 0$ such that $\int_X f_n \le M$ for every $n \in \mathbb{N}$. Prove that then $\int_X f \le M$ (this result is sometimes by some people called Fatou's lemma . . .).

(ii) Let f_n be a sequence in $L^1_\mu(X, \mathbb{R})$ converging a.e. to $f : X \to \mathbb{R}$. Assume that there is a real $M > 0$ such that $\|f_n\|_1 \le M$ for every $n \in \mathbb{N}$. Prove that then $f \in L^1_\mu(X, \mathbb{R})$.

(iii) Show by an example in $L^1_m([0, 1])$ that in the hypotheses of (i) (or (ii)) f_n does not necessarily also converge to f in $L^1_m([0, 1])$.

Solution. (i) Fatou's lemma (our version!):

$$\int_X \liminf_{n\to\infty} f_n = \int_X \lim_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_X f_n \le M,$$

the last inequality is obvious: if all terms of a real sequence are smaller than some real number, then the liminf (and also the limsup, of course) of the sequence is smaller than this number.

(ii) A triviality: apply (i) to the sequence $|f_n| \in L^+(X)$, which converges a.e. to $|f|$ (by continuity of the absolute value function); then $\int_X |f| \le M < \infty$, and f is measurable as limit a.e. of a sequence of measurable functions, so that $f \in L^1(\mu)$.

(iii) The usual counterexample works: $f_n = n\,\chi_{]0,1/n]}$ converges pointwise to 0 (we have $f_n(0) = 0$ for every n, and if $x > 0$ we have $f_n(x) = 0$ as soon as $1/n < x$), and all the integrals are 1, in particular uniformly bounded; but f_n does not converge in L^1_m, as is well known (if it did converge to g, some subsequence ought to converge pointwise a.e. to the limit, but every subsequence converges pointwise to 0, so $g = 0$; and the L^1 norms are constantly 1 ...)

$\square$

EXERCISE 5. (i) Complete the following statement:

- *If $\mu, \nu : \mathcal{M} \to [0, \infty]$ are measures on the $\sigma-$algebra $\mathcal{M}$ of parts of X, $\mathcal{E}$ is a subset of $\mathcal{M}$ closed under intersection, and μ, ν are finite and coincide on $\mathcal{E}$ then μ and ν coincide on all subsets of the $\sigma-$ algebra $\mathcal{M}(\mathcal{E})$ generated by $\mathcal{E}$ that can be covered by ....*

From now on $F : \mathbb{R} \to \mathbb{R}$ is an increasing function and $\mu = dF$ is the associated Radon–Stjelties measure; $\mathcal{B}$ is the $\sigma-$algebra of Borel subsets of $\mathbb{R}$.

(ii) Assume that $F_{]-\infty,0[}$ and $F_{|]0,\infty[}$ are C^1 functions; we can define a measure $\nu : \mathcal{B} \to [0, \infty]$ by $\nu(A) = \int_A F'(x)\,dx$. Prove that $\nu(A) = \mu(A)$ for every Borel set $A \in \mathcal{B}$ with $0 \notin A$. Which condition on F ensures equality for all Borel sets?

Now assume that F is bounded.

(iii) Using the theorem on continuity and differentiability of parameter depending integrals prove that the formula:

$$\varphi(x) = \int_{\mathbb{R}} \frac{\cos(xt)}{1 + |t|}\, d\mu(t) \tag{*}$$

defines a function $\varphi \in C^1(\mathbb{R}, \mathbb{R})$.

(iv) What condition can be added to μ to ensure that $\varphi \in C^2(\mathbb{R})$?

Solution. (i) by a countable subset $\{E_n\}$ of $\mathcal{E}$.

(ii) It is clear that μ and ν coincide on all compact intervals not containing 0: by the fundamental theorem of calculus we have, if $[a, b]$ is an interval on which F is of class C^1:

$$\nu([a, b]) := \int_{[a,b]} F'(x)\,dx = F(b) - F(a) = \mu([a, b]).$$

The family $\mathcal{E}$ of all compact intervals not containing 0 is plainly closed under intersection; and $\mathcal{M}(\mathcal{E})$ is the $\sigma-$algebra $\mathcal{B}$ of all Borel subsets of $\mathbb{R}$ (it is enough to consider intervals with extremes in a dense subset of $\mathbb{R}$). All of $\mathbb{R} \setminus \{0\}$ can be covered by a countable union of compact intervals contained in $]-\infty, 0[\cup]0, \infty[$, e.g. $\{[-n, -1/n],\, [1/n, n] \,:\, n \ge 1\}$. By (i) we conclude that $\nu(A) = \mu(A)$ for every Borel set $A \in \mathcal{B}$ with $0 \notin A$. Clearly $\nu(\{0\}) = 0$, whereas $\mu(\{0\}) = F(0^+) - F(0^-)$, the jump of F at 0. That this jump is 0, that is, continuity of F at 0 is then a necessary condition for $\mu = \nu$ on all of $\mathcal{B}$.

Plainly it is also sufficient: for every Borel set $A \subseteq \mathbb{R}$ we can write $\nu(A) = \nu(A \setminus \{0\}) + \nu(\{0\}) = \nu(A \setminus \{0\})$ and $\mu(A) = \mu(A \setminus \{0\}) + \mu(\{0\})$, so that $\mu(A) = \nu(A)$ iff $\mu(\{0\}) = 0$.

(iii) The function $f(x, t) = \cos(xt)/(1 + |t|)$ is continuous in both variables, and it has a continuous partial derivative with respect to x, $\partial_x f(x, t) = -t \sin(xt)/(1 + |t|)$. Both f and its derivative are dominated by the constant 1, which is in $L^1(\mu)$ since $\mu(\mathbb{R}) = F(\infty) - F(-\infty)$ is finite, by boundedness of F. Then $\varphi \in C^1(\mathbb{R})$.

(iv) We have:

$$\partial_x^2 f(x, t) = \frac{-t^2}{1 + |t|} \cos(xt) \quad \text{so that} \quad |\partial_x^2 f(x, t)| \le \frac{t^2}{1 + |t|};$$

if this function is in $L^1(\mu)$ then $\varphi \in C^2(\mathbb{R})$. This is then a sufficient condition, and it is quite easy to see that $t \mapsto t^2/(1 + |t|)$ is in $L^1(\mu)$ if and only if $t \mapsto t \in L^1(\mu)$. $\square$

Remark. A common misconception is to think that Borel subsets of $\mathbb{R}$, if they are not intervals, are at least countable unions of intervals! Borel sets are much more complicated than that: think for instance to the set K in the lecture notes, 2.6.4, a compact set of positive measure with empty interior, containing no non trivial interval. It is not possible to state without proof that $\nu(A) = \mu(A)$ for every Borel set not containing 0: this requires a proof, first for bounded intervals, then for arbitrary Borel sets, according to the proposition quoted in (i).

Exercise 6. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Prove that if $f \in L^1(\mu)$ then

$$\lim_{t\to\infty} \int_{\{|f|>t\}} |f|\, d\mu = 0.$$

A sequence $f_n \in L^1(\mu)$ is said to be *uniformly integrable* if the limit above is uniform with respect to n, in other words given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$\int_{\{|f_n|>t\}} |f_n|\, d\mu \le \varepsilon \quad \text{for every } n \in \mathbb{N} \text{ and every } t \ge t_\varepsilon.$$

(ii) Prove that if f_n is a sequence of measurable functions dominated by some $g \in L^1(\mu)$ ($|f_n| \le g$ for every $n \in \mathbb{N}$) then f_n is uniformly integrable.

(iii) Prove that if f_n is uniformly integrable then for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$\int_E |f_n| \le \varepsilon \quad \text{for every } n \in \mathbb{N} \text{ and every } E \in \mathcal{M} \text{ with } \mu(E) \le \delta_\varepsilon.$$

(iv) Prove that if $f_n \in L^1(\mu)$ and f_n converges to $0 \in L^1(\mu)$ (that is $\lim_{n\to\infty} \|f_n\|_1 = 0$) then f_n is uniformly integrable (hint: by (i) a finite sequence is always uniformly integrable ...).

(v) With $X = [0, 1]$, $\mathcal{M}$=Borel subsets of $[0, 1]$, and $\mu = m$, Lebesgue measure, let $f_n = n\,(\chi_{]0,1/(2n)]} - \chi_{]1/2n,1/n]})$, $n \ge 1$. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$; for every $t > 0$ find

$$\lim_{n\to\infty} \int_{\{|f_n|>t\}} |f_n|;$$

is f_n uniformly integrable?

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(vi) (Extra) Assume that $f_n \in L^1(\mu)$ is uniformly integrable, that $f_n \to 0$ pointwise a.e. and that $\mu(X) < \infty$. Prove that then $\|f_n\|_1 \to 0$, that is, f_n converges to 0 in $L^1(\mu)$ (hint: write:

$$\int_X |f_n| = \int_{\{|f_n|>t\}} |f_n| + \int_X |f_n|\,\chi_{\{|f_n|\le t\}}, \dots).$$

Solution. (i) Let $t_n > 0$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Then $|f|\,\chi_{\{|f|>t_n\}}$ converges pointwise to 0 (given $x \in X$, as soon as $t_n > f(x)$ we have $\chi_{\{|f|>t_n\}}(x) = 0$), and is dominated by $|f|$; by dominated convergence we have

$$\lim_{n\to\infty} \int_X |f|\,\chi_{\{|f|>t_n\}} = 0,$$

which is what desired.

(ii) Given $\varepsilon > 0$, by (i) we find $t_\varepsilon > 0$ such that $\int_{\{g>t\}} g \le \varepsilon$ for every $t \ge t_\varepsilon$. Since $|f_n| \le g$, we have $\{|f_n| > t\} \subseteq \{g > t\}$ (if $|f_n(x)| > t$, then $g(x) \ge |f_n(x)| > t$), so that

$$|f_n|\,\chi_{\{|f_n|>t\}} \le g|\,\chi_{\{|f_n|>t\}} \le g\,\chi_{\{g>t\}},$$

hence

$$\int_X |f_n|\,\chi_{\{|f_n|>t\}} \le \int_X g\,\chi_{\{g>t\}} \le \varepsilon, \quad \text{for every } t \ge t_\varepsilon.$$

(iii) Given $\varepsilon > 0$ pick $t > 0$ such that $\int_{\{|f_n|>t\}} |f_n| \le \varepsilon/2$ for every $n \in \mathbb{N}$; then for every $E \in \mathcal{M}$ of finite measure we can write:

$$\int_E |f_n| = \int_{E\cap\{|f_n|>t\}} |f_n| + \int_{E\smallsetminus\{|f_n|>t\}} |f_n| \le \int_{\{|f_n|>t\}} |f_n| + \int_{E\cap\{|f_n|\le t\}} |f_n| \le \frac{\varepsilon}{2} + t\,\mu(E);$$

Given t, pick now $\delta = \varepsilon/(2t)$; if $\mu(E) \le \delta$ we have, for every $n \in \mathbb{N}$:

$$\int_E |f_n| \le \varepsilon, \quad \text{as desired.}$$

(iv) Given $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\|f_n\|_1 = \int_X |f_n| \le \varepsilon$ for every $n \ge \varepsilon$. There is now $t_\varepsilon > 0$ such that $\int_{\{|f_n|>t\}} |f_n| \le \varepsilon$ for every $t \ge t_\varepsilon$ and $n = 0, \dots, n_\varepsilon - 1$: simply pick $t(n, \varepsilon)$ such that this holds for $n \in \{0, 1, \dots, n_\varepsilon - 1\}$, and let $t_\varepsilon = \max\{t(0, \varepsilon), \dots, t(n_\varepsilon - 1, \varepsilon)\}$. Then $\int_{\{|f_n|>t\}} |f_n| \le \varepsilon$ for every $n \in \mathbb{N}$, if $t \ge t_\varepsilon$; it is true by construction for $n < n_\varepsilon$, and if $n \ge n_\varepsilon$ we have

$$\int_{\{|f_n|>t\}} |f_n| \le \int_X |f_n| \le \varepsilon.$$

(v) Observe that $|f_n| = n\,\chi_{]0,1/n]}$, so that (as already observed in exercise 1, iii)) the pointwise limit of f_n is the constant 0, and $\int_{[0,1]} |f_n| = 1$, for every n. Given $t > 0$, as soon as $n > t$ we have $\{|f_n| > t\} =]0, 1/n]$, so that, if $n > t$:

$$\int_{\{|f_n|>t\}} |f_n| = \int_{]0,1/n]} n = 1 \quad \text{hence} \quad \lim_{n\to\infty} \int_{\{|f_n|>t\}} |f_n| = 1,$$

which clearly excludes uniform integrability of the sequence.

(vi) Given $\varepsilon > 0$, pick t_ε as in the definition of uniform integrability, Given any $t \ge t_\varepsilon$, accepting the hint we write

$$\|f_n\|_1 = \int_{\{|f_n|>t\}} |f_n| + \int_X |f_n|\,\chi_{\{|f_n|\le t\}} \le \varepsilon + \int_X |f_n|\,\chi_{\{|f_n|\le t\}};$$

Since X has finite measure, the constant function $t = t\,\chi_X$ is in $L^1(\mu)$; moreover the sequence

$$|f_n|\,\chi_{\{|f_n|\le t\}} \le |f_n|\,\chi_X$$

converges a.e. to 0, being dominated by a sequence that does so; the dominated convergence theorem implies then

$$\lim_{n\to\infty} \int_X |f_n|\,\chi_{\{|f_n|\le t\}} = 0,$$

so that there is $n_\varepsilon \in \mathbb{N}$ such that this integral is smaller than ε for $n \ge n_\varepsilon$; then

$$\|f_n\|_1 \le 2\varepsilon \quad \text{for} \quad n \ge n_\varepsilon.$$

Remark. A number of people insist in trying to prove (i) by Cebiceff inequality, which has only a marginal role, if at all, in the proof.

Another common incomprehensible mistake is the following: some people believe that

$$\text{NO!} \quad \int_X |f|\,d\mu \le \|f\|_1\,\mu(X) \quad \text{NO!}.$$

Unless $f = 0$ a.e., in which case both members are 0, this inequality is trivially true iff $\mu(X) \ge 1$.

□

Analisi Reale per Matematica – Secondo precompitino – 16 gennaio 2014

Exercise 7. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = -e^{-|x|} + H(x)$, with $H = \chi_{[0,\infty[}$ the Heaviside step; let μ denote the measure dF associated to F.

(i) Plot F
(ii) Plot $T(x) = VF(]-\infty, x])$, $A = (T + F)/2$, $B = (T - F)/2$, giving formulas for these functions.
(iii) Write a Hahn decomposition for μ, and split μ into singular and absolutely continuous parts (with respect to Lebesgue measure m).
(iv) Compute $\mu^+ \otimes \mu^-(T)$ and $\mu^+ \otimes \mu^-(S)$, where $T = \{(x, y) \in \mathbb{R}^2 : x \le y\}$ and $S = \mathbb{R}^2 \setminus T$.

Solution. (i) Notice that F is right continuous. The plot is very easy:

(ii) Since F is right–continuous T is also right–continuous. We have $T(x) = e^x$ for $x < 0$, so $T(0^-) = 1$; $T(0) = T(0^+) = 2$; $T(x) = 3 - e^{-x}$ for $x \ge 0$. The plot is the following:

We finally get $A(x) = 0$ for $x < 0$, $A(x) = 2 - e^{-x}$ for $x \ge 0$, and $B(x) = e^x$ for $x > 0$, $B(x) = 1$ for $x \ge 0$.

(iii) A positive set for μ is $P = [0, \infty[$, with $Q = \mathbb{R} \setminus P$ negative. The singular part of μ is δ, unit mass at 0, the regular part is $F'(x)\,dx$, with $F'(x) = -\operatorname{sgn} x\, e^{-|x|}$ (for $x \in \mathbb{R} \setminus \{0\}$).

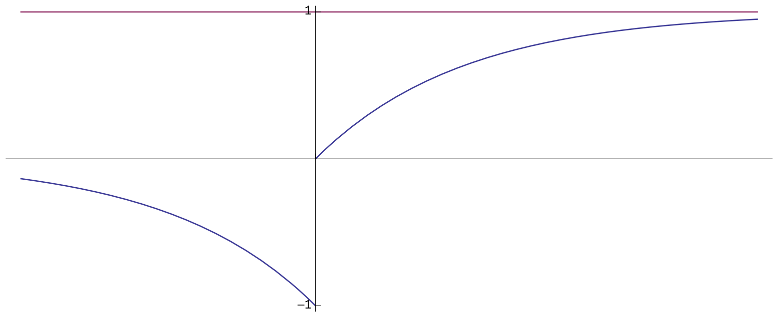

FIGURE 1. Plot of F.

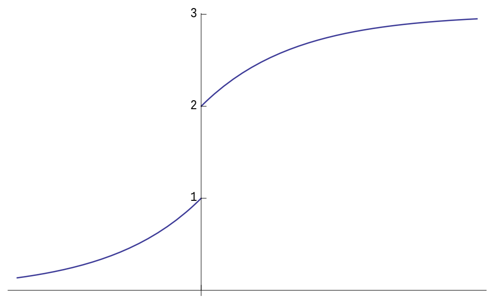

FIGURE 2. Plot of T.

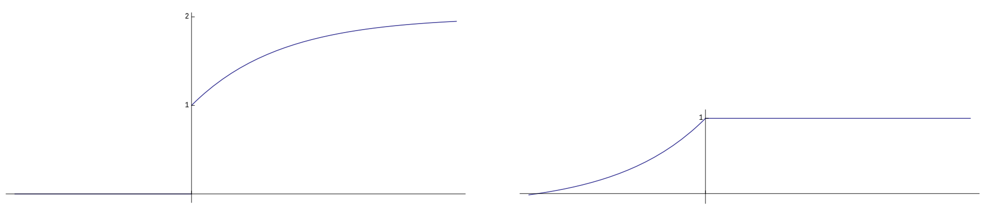

FIGURE 3. Plot of A (left) and of B (right).

(iv) We have $\mu^+ dA(x) = \delta + e^{-x} H(x)\, dx$ and $\mu^- = dB(x) = e^x\, H(-x)\, dx$. To compute $\mu^+ \otimes \mu^-(T)$ we integrate the μ^+-measure of the $y-$sections of T with respect do $d\mu^-(y)$, noting that $T(y) =]-\infty, y]$ for every $y \in \mathbb{R}$, so that $\mu^+(]-\infty, y]) = 0$ for every $y < 0$, and $\mu^+(]-\infty, y]) = A(y) - A(-\infty) = 2 - e^{-y}$ for $y \geq 0$, so that

$$\mu^+ \otimes \mu^-(T) = \int_{\mathbb{R}} \mu^+(T(y))\, d\mu^-(y) = \int_{[0,\infty} (2 - e^{-y})\, d\mu^-(y) = 0,$$

since $\mu^-([0, \infty[) = 0$.

We have $\mu^+ \otimes \mu^-(\mathbb{R}^2) = \mu^+(\mathbb{R})\, \mu^-(\mathbb{R}) = 2 \cdot 1 = 2$; then

$$\mu^+ \otimes \mu^-(S) = \mu^+ \otimes \mu^-(\mathbb{R}^2) - \mu^+ \otimes \mu^-(T) = 2 - 0 = 2.$$

□

EXERCISE 8. (12) For every $f : \mathbb{R} \to \mathbb{C}$ and every $a \in \mathbb{R}$ the *translate of f by a* is the function $\mathrm{tr}_a\, f : \mathbb{R} \to \mathbb{C}$ given by $\mathrm{tr}_a\, f(x) = f(x - a)$. Let $p \geq 1$ be a real number.

(i) [2] Prove that $\lim_{a \to 0} \mathrm{tr}_a\, f(x) = f(x)$ if and only if f is continuous at x.

(ii) [4] Let f be the characteristic function of a compact interval. Prove that if $a(j)$ is sequence in $\mathbb{R}$ with $\lim_{j \to \infty} a(j) = 0$ then

$$\lim_{j \to \infty} \| \mathrm{tr}_{a(j)}\, f - f \|_p = 0.$$

(iii) [2] Prove that (ii) holds for every step function with compact support (a finite linear combination of characteristic functions of compact intervals).

(iv) [4] Prove that (ii) holds for every $f \in L^p(\mathbb{R})$.

Solution. (i) Trivial: this is almost the definition of continuity of f at x (with the increment of the independent variable called $-a$ instead of h!) (ii) Since $\lim_{j\to\infty} a(j) = 0$, the sequence $a(j)$is bounded, say $|a(j)| \leq h$ for every $j \in \mathbb{N}$. Assume that $f = \chi_{[a,b]}$. Then $|\operatorname{tr}_{a(j)} f|^p| \operatorname{tr}_{a(j)} f| =\leq \chi_{[a-h,b+h]} \left(\in L^1(\mathbb{R})\right)$ for every $j \in \mathbb{N}$; and $\lim_{j\to\infty} \operatorname{tr}_{a(j)} f(x) = f(x)$ for every $x \in \mathbb{R} \smallsetminus \{a, b\}$, by (i) (of course, the points of discontinuity of f are a, b). By dominated convergence $\lim_{j\to\infty} \|\operatorname{tr}_{a(j)} f - f\|_p = 0$.

(iii) Let $f = \sum_{k=1}^m \alpha_k f_k$, where $f_k = \chi_{[a_k,b_k]}$. Then of course for every $a \in \mathbb{R}$ we have $\operatorname{tr}_a f = \sum_{k=1}^m \alpha_k \operatorname{tr}_a f_k$, and the result is simply a consequence of additivity of the limits:

$$\|\operatorname{tr}_{a(j)} f - f\|_p = \left\|\sum_{k=1}^m \alpha_k \operatorname{tr}_{a(j)} f_k - \sum_{k=1}^m \alpha_k f_k\right\|_p = \left\|\sum_{k=1}^m \alpha_k (\operatorname{tr}_{a(j)} f_k - f_k)\right\|_p \leq$$
$$\leq \sum_{k=1}^m |\alpha_k| \, \|\operatorname{tr}_{a(j)} f_k - f_k\|_p$$

and the last sum is a finite linear combination of things tending to zero.

(iv) Step functions with compact support are dense in $L^p(\mathbb{R})$ for $p < \infty$ (Lecture Notes, 5.2.2). Given $\varepsilon > 0$ pick g step function with compact support such that $\|f - g\|_p \leq \varepsilon$; we have

$$\|\operatorname{tr}_{a(j)} f - f\|_p = \|\operatorname{tr}_{a(j)} f - \operatorname{tr}_{a(j)} g + \operatorname{tr}_{a(j)} g - g + g - f\|_p \leq \|\operatorname{tr}_{a(j)} f - \operatorname{tr}_{a(j)} g\|_p + \|\operatorname{tr}_{a(j)} g - g\|_p + \|g - f\|_p;$$

we now have

$$|\operatorname{tr}_{a(j)} f - \operatorname{tr}_{a(j)} g\|_p = \|\operatorname{tr}_{a(j)}(f - g)\|_p = \|f - g\|_p \leq \varepsilon$$

(Lebesgue integral is translation invariant, implying $\|\operatorname{tr}_a h\|_p^p = \int_{\mathbb{R}} |h(x-a)|^p \, dx \int_{\mathbb{R}} |h(t)|^p \, dt = \|h\|_p^p$ for every $a \in \mathbb{R}$)

$$\|\operatorname{tr}_{a(j)} f - f\|_p \leq 2\varepsilon + \|\operatorname{tr}_{a(j)} g - g\|_p;$$

taking lim sup, since by (iii) $\lim_{j\to\infty} \|\operatorname{tr}_{a(j)} g - g\|_p = 0$

$$\limsup_{j\to\infty} \|\operatorname{tr}_{a(j)} f - f\|_p \leq 2\varepsilon;$$

for every $\varepsilon > 0$, so that $\limsup_{j\to\infty} \|\operatorname{tr}_{a(j)} f - f\|_p = 0$, as desired. □

EXERCISE 9. Let $(X, \mathcal{M}, \mu)$ be a measure space. We consider for simplicity only real valued functions on X. Let us accept for a moment the following version of the dominated convergence theorem:

. *If $\varphi_n, \gamma_n, \gamma$ are measurable functions on X, $|\varphi_n| \leq |\gamma_n|$ for every n, $\varphi_n \to 0$, $\gamma_n \to \gamma$ a.e. on X, γ_n, $\gamma \in L^1(\mu)$ and $\|\gamma_n\|_1 \to \|\gamma\|_1$, then $\|\varphi_n\|_1 \to 0$.*

Let now p be a real number, $1 \leq p < \infty$.

(i) If f_n, f, g_n, $g : X \to \mathbb{R}$ are measurable, $|f_n| \leq |g_n|$ for every n, $f_n \to f$, $g_n \to g$ a.e., g_n, $g \in L^p(\mu)$ and finally $\|g_n\|_p \to \|g\|_p$, then we have
$$\lim_{n\to\infty} \|f - f_n\|_p = 0.$$
(ii) Prove that if $f_n, f \in L^p(\mu)$, $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ then f_n converges to f in $L^p(\mu)$.
(iii) Prove the version of the dominated convergence theorem stated at the beginning.

The exercise is essentially 5.3.5.5 of the Lecture Notes

Solution. (i) We have $|f - f_n|^p \leq 2^{p-1}(|f|^p + |f_n|^p) \leq 2^{p-1}(|g|^p + |g_n|^p)$. We can then apply (i) with $\varphi_n = |f - f_n|^p$, $\gamma_n = |\gamma_n| = 2^{p-1}(|g|^p + |g_n|^p)$ and $\gamma = 2^p\,|g|$: we have in fact that γ_n converges a.e. to γ and

$$\|\gamma_n\|_1 = \int_X 2^{p-1}(|g|^p + |g_n|^p) = 2^{p-1}\left(\int_X |g|^p + \int_X |g_n|^p\right) = 2^{p-1}(\|g\|_p + \|g_n\|_p),$$

and by hypothesis:

$$\lim_{n\to\infty} \|g_n\|_p = \|g\|_p \quad \text{so that } \lim_{n\to\infty} \|g_n\|_p^p = \|g\|_p^p, \ \lim_{n\to\infty} 2^{p-1}(\|g\|_p^p + \|g_n\|_p^p) = 2^p \|g\|_p^p,$$

by (i) we then have

$$\lim_{n\to\infty} \|\varphi_n\|_1 = 0 \quad \text{equivalently} \quad \lim_{n\to\infty} \int_X |f - f_n|^p = 0,$$

in other words $\lim_{n\to\infty} \|f - f_n\|_p^p = 0$

(ii) Simply take $g_n = |f_n|$.

(iii)Take $g_n = |\gamma_n|$ and $f_n = |\varphi_n|$. Then we are in the hypotheses of the generalized dominated convergence theorem; we repeat the proof in this case, with $0 \le f_n \le g_n$, $f_n \to 0$, $g_n \to g$ .e. and $\|g_n\|_1 := \int_X g_n \to \int_X g = \|g\|_1$: applying Fatou's lemma to the sequence $g_n + f_n \ge 0$ we get:

$$\int_X g \le \liminf_{n\to\infty} \int_X (g_n + f_n) = \liminf_{n\to\infty} \left(\int_X g_n + \int_X f_n \right) = \int_X g + \liminf_{n\to\infty} \int_X f_n,$$

equivalently

$$\int_X g \le \int_X g + \liminf_{n\to\infty} \int_X f_n \iff 0 \le \liminf_{n\to\infty} \int_X f_n.$$

And applying Fatou's lemma to the sequence $g_n - f_n \ge 0$ we get

$$\int_X g \le \liminf_{n\to\infty} \int_X (g_n - f_n) = \liminf_{n\to\infty} \left(\int_X g_n + \int_X (-f_n) \right) = \int_X g + \liminf_{n\to\infty} \int_X (-f_n),$$

equivalently

$$\int_X g = \int_X g - \limsup_{n\to\infty} \int_X f_n \iff \limsup_{n\to\infty} \int_X f_n \le 0.$$

We have proved that $\lim_{n\to\infty} \int_X f_n = 0$; since $f_n \ge 0$, this is exactly the assertion $\lim_{n\to\infty} \|f_n\|_1 = 0$. $\square$

Addenda

We expand somewhat the previous exercises.

0.1. We might have been asked to compute $\mu^+ \otimes \mu^-(S)$ by integrating both ways. The $y-$section of S is $S(y) =]y, \infty[$ with $\mu^+(]y,\infty[) = A(\infty) - A(y^+) = 2 - A(y) = e^{-y}$ if $y \ge 0$, and 2 if $y < 0$, so that

$$\mu^+ \otimes \mu^-(S) = \int_{\mathbb{R}} \mu^+(S(y))\, d\mu^-(y) = \int_{[-\infty,0[} 2\, d\mu^-(y) + \int_{[0,\infty[} e^{-y}\, dB(y) = \int_{[-\infty,0[} 2\, d\mu^-(y),$$

given that $\mu^-([0,\infty[) = 0$; since $dB(y) = e^y\, dy$ we get

$$\mu^+ \otimes \mu^-(S) = \int_{-\infty}^0 2e^y\, dy = 2.$$

The $x-$section of S is $S(x) =]-\infty, x[$ so that $\mu^-(S(x)) = B(x^-) - B(-\infty) = B(x) = e^x \wedge 1$ for every $x \in \mathbb{R}$; on the other hand we have $\mu^+(]-\infty,0[) = 0$, so that

$$\mu^+ \otimes \mu^-(S) = \int_{\mathbb{R}} \mu^-(S(x))\, d\mu^+(x) = \int_{[0,\infty[} e^x \wedge 1\, (d\delta(x) + e^{-x}\, dx) = 1 + \int_0^\infty e^{-x}\, dx = 2.$$

Another question: Find all $p > 0$ such that $f(x) = x$ is in $L^p(|\mu|)$, where $|\mu| = \mu^+ + \mu^-$ is the total variation of μ, and compute $\|f\|_p$ in $L^p(|\mu|)$.

$$d|\mu| = d\mu^+ + d\mu^- = dA + dB = d\delta + e^{-|x|}\, dx.$$

Since $f(0) = 0$, the singular part $d\delta$ is irrelevant; we have

$$\|f\|_p^p = \int_{\mathbb{R}} |x|^p\, e^{-|x|}\, dx = 2\int_0^\infty x^p\, e^{-x}\, dx = 2\,\Gamma(p+1),$$

so that $f \in L^p(\mu)$ for every $p > 0$ (but $p < \infty$) and

$$\|f\|_p = 2^{1/p}(\Gamma(p+1))^{1/p}.$$

0.2. The problem of measurability of the translate has been ignored. We observe that $\mathrm{tr}_a\, f$ may be considered as the composition $f \circ \mathrm{tr}_{-a}$, where now $\mathrm{tr}_{-a} : \mathbb{R} \to \mathbb{R}$ is the map "translation by $-a$", i.e. $x \mapsto x - a$. This map is a self–homeomorphism of $\mathbb{R}$, hence if $f : \mathbb{R} \to \mathbb{K}$ is Borel measurable $\mathrm{tr}_a\, f$ is also Borel measurable; the problem is to show that f Lebesgue measurable implies $\mathrm{tr}_a\, f$ Lebesgue measurable. Given a Borel subset A of $\mathbb{K}$ the set $f^{\leftarrow}(A)$ is a Lebesgue measurable subset of $\mathbb{R}$, hence $f^{\leftarrow}(A) = B \cup N$, with B Borel and N of Lebesgue measure zero. Then $\mathrm{tr}_{-a}^{\leftarrow}(B \cup N) = (B + a) \cup (N + a)$ is Lebesgue measurable, since $B + a$ is Borel and $N + a$ is a Lebesgue null set, as a translate of a Lebesgue null set.

To prove (iv) we have had to observe that $f \mapsto \mathrm{tr}_a\, f$ is an isometry of $L^p(\mathbb{R})$ onto itself, another way of expressing translation invariance of the Lebesgue integral

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Exercise 10. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = -\operatorname{sgn} x \arctan(1/x^2)$, $F(0) = 0$; let μ denote the measure dF associated to F.

(i) Plot F
(ii) Plot $T(x) = VF(]-\infty, x])$, $A = (T+F)/2$, $B = (T-F)/2$, giving formulas for these functions.
(iii) Write a Hahn decomposition for μ, and split μ into singular and absolutely continuous parts (with respect to Lebesgue measure m).
(iv) Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le |x|\}$. Compute $\mu^+ \otimes m(E)$, integrating both the $y-$sections and the $x-$sections.

Solution. (i) Easy (that $F'(0^\pm) = 0$ will be clear from the sequel, when we compute $F'(x)$),

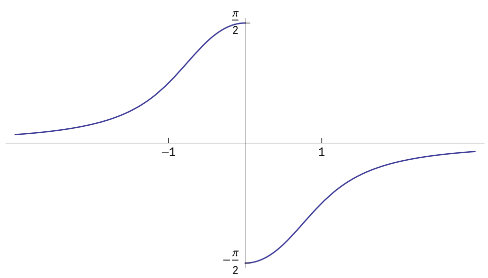

Figure 4. Plot of F.

(ii) Clearly $T(x) = F(x) = \arctan(1/x^2)$ per $x < 0$, since F is incresing and $F(-\infty) = 0$. And $T(0) = T(0^-) + |F(0) - F(0^-)| = \pi/2 + \pi/2 = \pi$, $T(0^+) = T(0) + |F(0^+) - F(0)| = \pi + \pi/2 = 3\pi/2$, finally $T(x) = T(0^+) + F(x) - F(0^+) = 3\pi/2 + \pi/2 - \arctan(1/x^2) = 2\pi - \arctan(1/x^2)$ if $x > 0$.

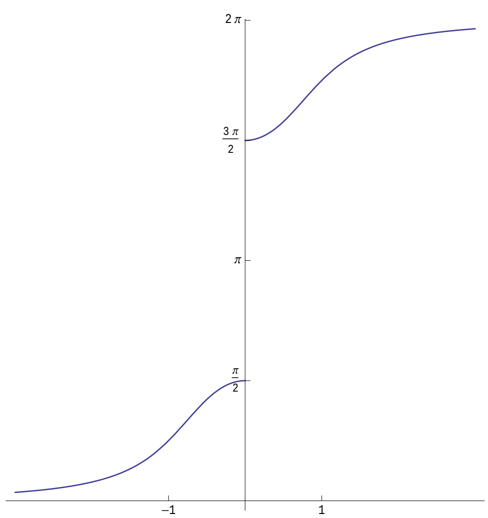

Figure 5. Plot of T.

Easily we have

$$A(x) = \begin{cases} \arctan(1/x^2) & \text{per} \quad x < 0 \\ \pi/2 & \text{per} \quad x = 0\,; \\ \pi - \arctan(1/x^2) & \text{per} \quad x > 0 \end{cases} \qquad B(x) = \begin{cases} 0 & \text{per} \quad x < 0 \\ \pi/2 & \text{per} \quad x = 0\,. \\ \pi & \text{per} \quad x > 0 \end{cases}$$

(iii) A positive set is $P = \mathbb{R} \setminus \{0\}$ with complement $Q = \{0\}$ negative (of measure $\mu(\{0\}) = -\pi$) The absolutely continuous part is exactly the positive part μ^+ of the measure: if $x \ne 0$ we have

$$F'(x) = A'(x) = \operatorname{sgn} x \frac{2/x^3}{1 + (1/x^2)^2} = \operatorname{sgn} x \frac{2x}{1 + x^4} = \frac{2|x|}{1 + x^4},$$

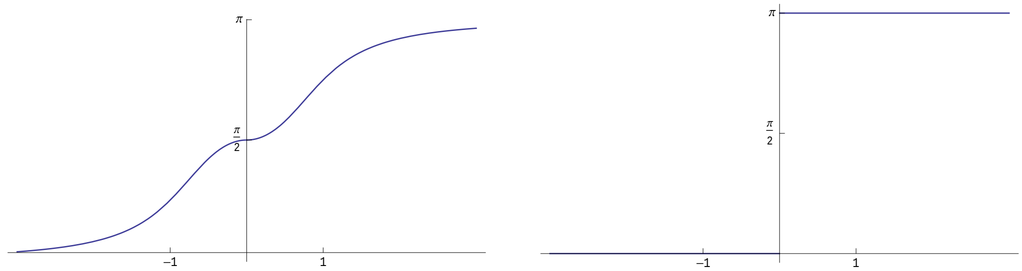

FIGURE 6. Plot of A (left) and B.

a continuous function on $\mathbb{R}$ (in particular, $A \in C^1(\mathbb{R})$). The singular part is $-\pi\,\delta = -\mu^-$,

(iv) The $y-$section $E(y)$ of E is empty for $y < 0$ and is $E(y) =]-\infty, -y] \cup [y, \infty[$ for $y \geq 0$, with measure

$$\mu^+(E(y)) = A(-y) - A(-\infty) + A(\infty) - A(y) = 2\arctan(1/y^2);$$

then

$$\mu^+ \otimes m(E) = \int_{[0,\infty[} 2\arctan(1/y^2)\,dm(y) = \left[y\,2\arctan(1/y^2)\right]_0^\infty - 2\int_0^\infty y\frac{-2y}{1+y^4}\,dy =$$
$$0 + 4\int_0^\infty \frac{y^2}{1+y^4}\,dy = (x^4 = t)4\int_0^\infty \frac{t^{1/2}}{1+t}\,\frac{t^{-3/4}}{4}\,dt = \int_0^\infty \frac{t^{-1/4}}{1+t}\,dt = \int_0^\infty \frac{t^{3/4-1}}{1+t}\,dt$$
$$= B(3/4, 1/4) = \Gamma(3/4)\,\Gamma(1/4) = \frac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}.$$

The $x-$section is $E(x) = [0, |x|]$ for every $x \in \mathbb{R}$ and we have

$$\mu^+ \otimes m(E) = \int_{\mathbb{R}} m([0,|x|])\,d\mu^+(x) = \int_{\mathbb{R}} |x|\,dA(x) = \int_{\mathbb{R}} |x|\,\frac{2|x|}{1+x^4}\,dx = 4\int_0^\infty \frac{x^2}{1+x^4}\,dx,$$

the integral already computed.

REMARK. A surprisingly high number of people got the derivative of F wrong!

$\square$

EXERCISE 11. Let $(X, \mathcal{M})$ be a measurable space. On the set $M(\mathcal{M}, \mathbb{R}) = M$ of all signed measures defined on the $\sigma-$algebra $\mathcal{M}$ (all measures in this problem are defined on $\mathcal{M}$) we introduce the natural partial ordering:

$$\mu \leq \nu \quad \text{meaning that} \quad \mu(E) \leq \nu(E) \quad \text{for every} \quad E \in \mathcal{M}.$$

(i) We say that a positive measure $\lambda : \mathcal{M} \to [0, \infty]$ *dominates* the signed measure $\mu \in M$ if we have $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathcal{M}$. Prove that the set of positive measures that dominate a given signed measure μ has a smallest element $|\mu|$ with respect to the partial order previously introduced. Prove also that if $\mu \in M$, and μ is a finite measure, then the function $E \mapsto |\mu(E)|$ is a measure iff μ has constant sign.

(ii) Assume that $\mu, \nu \in M$ are finite signed measures. Prove that there is a finite positive measure ρ and functions f, $g \in L^1(\rho)$ such that $d\mu = f\,d\rho$ and $d\nu = g\,d\rho$. Prove that the set

$$\{\lambda \in M : \mu \leq \lambda,\ \nu \leq \lambda\}$$

contains a smallest member, denoted $\mu \vee \nu$ and that

$$\mu \vee \nu(E) = \int_E f \vee g\,d\rho \qquad \text{for every } E \in \mathcal{M}.$$

(iii) Define mutual singularity $\mu \perp \nu$ of two measures $\mu, \nu \in M$, and prove that the following are equivalent for two finite measures μ, $\nu \in M$:
  (a) $\mu \perp \nu$.
  (b) $|\mu| \perp |\nu|$.
  (c) $|\mu + \nu| = |\mu| \vee |\nu|$.

(iv) (Extra) Given two finite measures $\mu, \nu \in M$, prove directly that the formula

$$\tau(E) = \sup\{\mu(A) + \nu(E \setminus A) : A \subseteq E,\ A \in \mathcal{M}\} \qquad (E \in \mathcal{M}),$$

defines a measure $\tau \in M$, and that $\tau = \mu \vee \nu$.

Solution. (i) If P, Q is a Hahn decomposition for μ we set, as usual, $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = -\mu(E \cap Q)$, and $|\mu|(E) = \mu^+(E) + \mu^-(E)$; it is well–known that $|\mu|$ is a positive measure. Then:

$$|\mu(E)| = |\mu^+(E) - \mu^-(E)| \le \mu^+(E) + \mu^-(E) = |\mu|(E),$$

so that $|\mu|$ dominates μ. And if λ dominates μ we have, for every $E \in \mathcal{M}$:

$$|\mu(E \cap P)| \le \lambda(E \cap P), \quad |\mu(E \cap Q)| \le \lambda(E \cap Q), \quad \text{equivalently} \quad \mu^+(E) \le \lambda(E \cap P), \quad \mu^-(E) \le \lambda(E \cap Q),$$

so that

$$|\mu|(E) := \mu^+(E) + \mu^-(E) \le \lambda(E \cap P) + \lambda(E \cap Q) = \lambda(E),$$

in other words $|\mu|(E) \le \lambda(E)$. Thus $|\mu|$ is the smallest positive measure that dominates μ. If the function $E \mapsto |\mu(E)|$ is a positive measure, then of course it is the smallest measure that dominates μ, and it must coincide with $|\mu|$: in other words $E \mapsto |\mu(E)$ is a measure iff

$$|\mu(E)| = |\mu^+(E) - \mu^-(E)| = |\mu|(E) = \mu^+(E) + \mu^-(E) \quad \text{for every } E \in \mathcal{M}.$$

Then $\mu(P)$ and $\mu(Q)$ cannot be both nonzero: in this case in fact we have $|\mu(X)| = |\mu(P) + \mu(Q)| < \mu(P) \vee (-\mu(Q))$, while $|\mu|(X) = \mu(P) - \mu(Q) > \mu(P) \vee (-\mu(Q))$.

(ii) The total variations $|\mu|$ and $|\nu|$ are finite positive measures, so that $\rho = |\mu| + |\nu|$ is a finite positive measure. We have $\rho(E) = 0$ iff $|\mu|(E) = |\nu|(E) = 0$, and this clearly happens iff $\mu^\pm(E) = \nu^\pm(E) = 0$, so that $\mu \ll \rho$ and $\nu \ll \rho$; since all measures are finite, the Radon–Nikodym theorem may be applied to show that there exist $f, g \in L^1(\rho)$ such that $d\mu = f\, d\rho$ and $d\nu = g\, d\rho$. Since $f \vee g \in L^1(\rho)$ the formula

$$\lambda(E) = \int_E f \vee g\, d\rho \qquad (E \in \mathcal{M})$$

defines a finite signed measure on $\mathcal{M}$, and since f, $g \le f \vee g$ we clearly have $\mu(E) \le \lambda(E)$ and $\mu(E) \le \lambda(E)$. We have to prove that λ is the smallest measure larger than μ and ν, that is, if τ is a signed measure such that $\mu(E) \vee \nu(E) \le \tau(E)$ for every $E \in \mathcal{M}$, then $\lambda(E) \le \tau(E)$. Set $A = \{f > g\}$, $B = \{f \le g\}$; then $X = A \cup B$, disjoint union and for every $E \in \mathcal{M}$ we get:

$$\begin{aligned}\lambda(E) = \lambda(E \cap A) + \lambda(E \cap B) &= \int_{E \cap A} f\, d\rho + \int_{E \cap B} g\, d\rho = \\ &= \mu(E \cap A) + \nu(E \cap B) \le \tau(E \cap A) + \tau(E \cap B) = \tau(E),\end{aligned}$$

so that $\lambda \le \tau$. Notice that $d|\mu| = f \vee (-f)\, d\rho = |f|\, d\rho$, same for ν and g

(iii) The measures μ and ν are said to be mutually singular if there is a partition A, B of X with B null for μ and A null for ν. For a signed measure a set is null iff the total variation is zero on this set, and for a positive measure a measurable set is null iff it has zero measure, so that (a) and (a) are clearly equivalent. Taking now $\rho = |\mu| + |\nu|$, f, g as in (ii), with $|\mu|(B) = |\nu|(A) = 0$, it is clear that $\mu \perp \nu$ implies $f(x) = 0$ ρ a.e. on B and $g(x) = 0$ ρ a.e. on A so that (assuming $f(x) = 0$ everywhere on B and $g(x) = 0$ everywhere on A) we have $|f + g| = |f| \vee |g|$, equivalently $|\mu + \nu| = |\mu| \vee |\nu|$; and $|f + g| = |f| \vee |g|$ ρ–a.e. on X clearly holds iff $\rho(\{f \ne 0\} \cap \{g \ne 0\}) = 0$: if a, $b \in \mathbb{R}$ are both non zero we have $|a + b| > |a| \vee |b|$ if $\operatorname{sgn} a = \operatorname{sgn} b$ and $|a + b| < |a| \vee |b|$ if $\operatorname{sgn} a \ne \operatorname{sgn} b$.

(iv) (*The proof is a little delicate*) Clearly $\tau(\emptyset) = 0$. Given a disjoint sequence $E_n \in \mathcal{M}$, with union E, we have to prove that

$$\tau(E) = \sum_{n=0}^{\infty} \tau(E_n),$$

where of course we have also to prove that the series $\sum_{n=0}^{\infty} \tau(E_n)$ is convergent, and that $\tau(E) \in \mathbb{R}$ for every $E \in \mathcal{M}$.

Given $A \subseteq E$, $A \in \mathcal{M}$, we set $B = E \setminus A$ and $A_n = E_n \cap A$, $B_n = E_n \cap B = E_n \setminus A_n$. Then A is the disjoint union of the sequence A_n, B is the disjoint union of the sequence B_n so that by countable additivity of μ and ν we have

$$\mu(A) = \sum_{n=0}^{\infty} \mu(A_n), \; \nu(B) = \sum_{n=0}^{\infty} \nu(B_n),$$

so that

$$\mu(A) + \nu(B) = \sum_{n=0}^{\infty} (\mu(A_n) + \nu(B_n)); \tag{*}$$

of course for every n we have $\mu(A_n)+\nu(B_n)\le\tau(E_n)$, and it is tempting to conclude that

$$\sum_{n=0}^{\infty}(\mu(A_n)+\nu(B_n))\le\sum_{n=0}^{\infty}\tau(E_n),$$

but first we have to prove that the last series is convergent. For this we use the fact that there is a finite positive measure ρ that dominates both μ and ν, e.g $\rho=|\mu|+|\nu|$, as above. Then

$$|\mu(A_n)+\nu(B_n)|\le|\mu(A_n)|+|\nu(B_n)|\le|\mu|(A_n)+|\nu|(B_n)\le\rho(A_n)+\rho(B_n)=\rho(E_n);$$

from this it easily follows that

$$|\tau(E_n)|\le\rho(E_n),$$

and since $\rho(E)=\sum_{n=0}^{\infty}\rho(E_n)<\infty$ the series $\sum_0^\infty\tau(E_n)$ is absolutely convergent, in particular convergent, and from (*) we may deduce that

$$\mu(A)+\nu(B)=\sum_{n=0}^{\infty}(\mu(A_n)+\nu(B_n))\le\sum_{n=0}^{\infty}\tau(E_n);$$

we have obtained

$$\mu(A)+\nu(B)\le\sum_{n=0}^{\infty}\tau(E_n)$$

for every partition A, $B=E\setminus A$ of E into measurable sets, so that

$$\tau(E)\le\sum_{n=0}^{\infty}\tau(E_n),$$

But we also know that

$$\sum_{n=0}^{\infty}(\mu(A_n)+\nu(B_n))\le\sum_{n=0}^{\infty}\tau(E_n)$$

for an arbitrary sequence of partitions of E_n into sets A_n, $B_n=E_n\setminus A_n\in\mathcal{M}$, so that we also have the reverse inequality $\tau(E)\ge\sum_{n=0}^{\infty}\tau(E_n)$ (e.g, given $\varepsilon>0$ pick partitions A_n, B_n such that $\mu(A_n)+\nu(B_n)\ge\tau(E_n)-\varepsilon/2^{n+1}$, obtaining

$$\tau(E)\ge\mu(A)+\nu(B)=\sum_{n=0}^{\infty}(\mu(A_n)+\nu(B_n))\ge\sum_{n=0}^{\infty}(\tau(E_n)-\varepsilon/2^{n+1})=\sum_{n=0}^{\infty}\tau(E_n)-\varepsilon).$$

Once we know that τ is a measure, it is clear that it is the smallest measure greater than both μ and ν: given a measure $\lambda\ge\mu,\nu$ we have $\lambda(E)=\lambda(A)+\lambda(E\setminus A)\ge\mu(A)+\nu(E\setminus A)$ for every measurable $A\subseteq E$, whence $\tau(E)\le\lambda(E)$.

Remark. If we do not assume μ finite, we can prove that $E\mapsto|\mu(E)|$ is a measure if and only if there do not exist pairs of sets of finite nonzero measure of different sign. Of course we have also $|\mu|=\mu\vee(-\mu)$, and in general $|\mu|(E)>|\mu(E)|=\mu(E)\vee(-\mu(E))$. Some people have misinterpreted $\mu\vee\nu(E)$ as equal to $\mu(E)\vee\nu(E)$ for every $E\in\mathcal{M}$, whereas we have $\mu(E)\vee\nu(E)\le(\mu\vee\nu)(E)$, with the inequality in general proper: $(\mu\vee\nu)(E)$ is the value at E of the smallest signed measure larger than μ and ν, and it may be strictly larger than both $\mu(E)$ and $\nu(E)$.

□

Exercise 12. Let $(X,\mathcal{M},\mu)$ be a measure space. For every $p>0$ we write L^p for $L^p(\mu)$; let $r>1$ be given. For every $a>0$ let $S(a)=\{f\in L^1\cap L^r:\ \|f\|_r\le a\}$.

(i) Prove that $S(a)$ is closed in L^1 (hint: Fatou's lemma).

Recall that if $1\le p<r\le\infty$ and $\alpha\in]0,1]$ is such that $1/p=\alpha+(1-\alpha)/r$ then for every measurable f

$$\|f\|_p\le\|f\|_1^{\alpha}\,\|f\|_r^{1-\alpha}.$$

(ii) Prove that if $f_n\in S(a)$ converges to f in L^1, then f_n converges to f in L^p, for every $p\in[1,r[$.

(iii) Prove that if $r<\infty$ then every sequence $f_n\in S(a)$ is uniformly integrable, that is, given $\varepsilon>0$ there is $t_\varepsilon>0$ such that

$$\int_{\{|f_n|>t\}}|f_n|\le\varepsilon\qquad\text{for } t\ge t_\varepsilon,\text{ and every } n\in\mathbb{N}.$$

(hint: if $r = 1 + s$ we have

$$a^r \geq \int_X |f|^r \geq \int_{\{|f|>t\}} |f|^{1+s} \geq t^s \int_{\{|f|>t\}} |f|$$

for every $f \in S(a)$ and every $t > 0 \dots$)

From now on we assume $\mu(X) < \infty$.

(iv) Deduce from the above that if $f_n \in S(a)$ converges a.e. to 0 then $\|f_n\|_1 \to 0$ (write

$$\int_X |f_n| = \int_{\{|f_n|>t\}} |f_n| + \int_X |f_n|\, \chi_{\{|f_n|\leq t\}} \dots .$$

and estimate separately the two terms).

(v) Prove that if $f_n \in S(a)$ converges pointwise a.e to f, then f_n converges to f in L^p, for every $p \in [1, r[$.

(vi) If we remove the assumption $\mu(X) < \infty$ then (iv) no longer holds: explain how $f_n = \chi_{[n,n+1]}$ falsifies (iv) and (v) in the measure space $\mathbb{R}$ with Lebesgue measure.

Solution. (i) Assume that $f_n \in S(a)$ converges to $f \in L^1$; then a subsequence converges also a.e. to f, and we may as well assume that the original sequence does so. Then $|f_n|^r$ converges pointwise to $|f|^r$ and by Fatou's lemma

$$\int_X |f|^r \leq \liminf_{n\to\infty} \int_X |f_n|^r \leq a^r,$$

so that $f \in S(a)$, hence $S(a)$ is closed in L^1.

(ii) We have

$$(*) \qquad \|f - f_n\|_p \leq \|f - f_n\|_1^\alpha\, \|f - f_n\|_r^{1-\alpha},$$

and since $f \in S(a)$ we get

$$\|f - f_n\|_r \leq \|f\|_r + \|f_n\|_r \leq 2a \qquad \text{for every } n \in \mathbb{N},$$

so that (*) gives

$$\|f - f_n\|_p \leq \|f - f_n\|_1^\alpha\, (2a)^{1-\alpha},$$

and we conclude (since $\alpha > 0$ we have $\lim_{n\to\infty} \|f - f_n\|_1^\alpha = 0$). □

(iii) Following the hint we have, for $t > 0$ and $f \in S(a)$:

$$\int_{\{|f|>t\}} |f| \leq \frac{a^r}{t^s}$$

and since $s > 0$ we have $\lim_{t\to\infty} a^r/t^s = 0$, so that given $\varepsilon > 0$ we may pick $t_\varepsilon > 0$ such that $a^r/t^s \leq \varepsilon$ for $t \geq t_\varepsilon$.

(iv) Given $\varepsilon > 0$ we pick $\bar{t} > 0$ such that the first term on the right hand–side is smaller than ε for every $n \in \mathbb{N}$, so that

$$(*) \qquad \int_X |f_n| \leq \varepsilon + \int_X |f_n|\, \chi_{\{|f_n|\leq\bar{t}\}}.$$

The sequence $f_n\, \chi_{|f_n|>\bar{t}\}}$ converges to 0 a.e. in X, being dominated by $|f_n|$, which does so, and it is dominated also by the constant $\bar{t} = \bar{t}\,\chi_X$, which is in $L^1(\mu)$ since $\mu(X) < \infty$. By dominated convergence we have

$$\lim_{n\to\infty} \int_X |f_n|\, \chi_{\{|f_n|\leq t\}} = 0.$$

Taking lim sup in (*) we then get

$$\limsup_{n\to\infty} \|f_n\|_1 \leq \varepsilon \qquad \text{for every } \varepsilon > 0,$$

the desired conclusion.

(v) We are of course tempted to apply (iv) to $g_n = f - f_n$, and to do that we need to know that $f - f_n \in S(a)$ (or something similar). By repeating the proof of (i) we see that $S(a)$ is pointwise closed: if $f_n \to f$ a.e., and $\|f_n\|_r \leq a$, Fatou's lemma says that $\|f\|_r \leq a$; and $\mu(X) < \infty$, so that $L^r \subseteq L^1$ if $r > 1$, as supposed. Then $f \in S(a)$, hence $g_n = f - f_n \in S(2a)$, so that $g_n \to 0$ in L^1, and by (ii) we have $\|g_n\|_p \to 0$ for every p with $1 \leq p < r$.

(vi) Clearly we have $\|f_n\|_r = 1$ for every $n \in \mathbb{N}$ and $r > 0$, so that $f_n \in S(1)$ for every n and $r > 0$. And f_n converges pointwise to the zero function, but since $\|f_n\|_p = 1$ for every $p > 0$ and every $n \in \mathbb{N}$ in no L^p space we have convergence of f_n to any function.

Remark. Fatou's lemma concerns the liminf of measurable functions *with respect to pointwise convergence a.e*: it is *true* that if f_n converges to f in L^1 then we have, for every $r > 0$:

$$\int_X |f|^r \le \liminf_{n\to\infty} \int_X |f_n|^r,$$

but the route for proving this fact is through pointwise converging subsequences, to which Fatou's lemma may be applied. Try to prove it!

Analisi Reale per Matematica – Primo appello invernale – 3 febbraio 2014

Exercise 13. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = H(-x)\arctan(1/x^3) + H(x)\left(\frac{\pi}{2} - \arctan(1/x^3)\right), \quad F(0) = 0;$$

$H = \chi_{[0,\infty[}$ is the Heaviside step; let μ denote the measure dF associated to F.

(i) Plot F.
(ii) Plot $T(x) = VF(]-\infty, x])$, $A = (T + F)/2$, $B = (T - F)/2$, giving formulas for these functions.
(iii) Write a Hahn decomposition for μ, and split μ into singular and absolutely continuous parts (with respect to Lebesgue measure m); compute also the derivative F'.
(iv) Let $E = \{(x, y) \in \mathbb{R}^2 : -|x| \le |y| \le |x|\}$. Compute $\mu^+ \otimes m(E)$.

Solution. (i) Plot of F is easy:

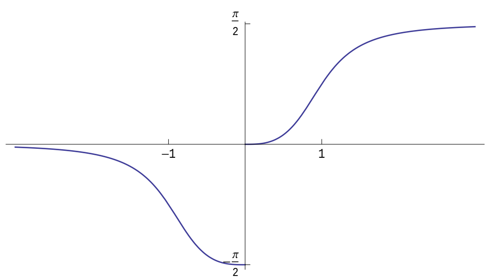

Figure 7. Plot of F.

(ii) We have (notice that F is right–continuous, so that T, A, B are also right–continuous):

$$T(x) = \begin{cases} -\arctan(1/x^3) & x < 0 \\ \pi/2 & x = 0 \\ 3\pi/2 - \arctan(1/x^3) & x > 0 \end{cases}.$$

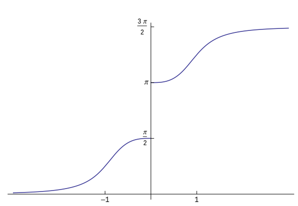

Figure 8. Plot of T.

and

$$A(x) = \begin{cases} 0 & x < 0 \\ \pi/2 & x = 0 \\ \pi - \arctan(1/x^3) & x > 0 \end{cases}; \quad B(x) = \begin{cases} -\arctan(1/x^3) & x < 0 \\ \pi/2 & x = 0 \\ \pi/2 & x > 0 \end{cases}.$$

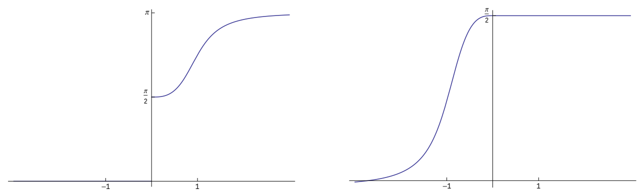

FIGURE 9. Plots of A, B.

(iii) A positive set for μ with negative complement is $P = [0, \infty[$ with $Q =]-\infty, 0[$ negative. We have

$$d\mu = F'(x)\,dx + \frac{\pi}{2}\,\delta \qquad F'(x) = -\operatorname{sgn} x\,\frac{-3/x^4}{1+(1/x)^6} = 3\operatorname{sgn} x\,\frac{x^2}{1+x^6} \quad (x \neq 0);$$

The negative part μ^- is absolutely continuous, $d\mu^- = ((3x^2)/(1+x^6))H(-x)\,dx$; the positive part has $\pi\,\delta/2$ as singular part, and $((3x^2)/(1+x^6))H(x)\,dx$ as absolutely continuous part.

(iv) We compute the measure by integrating in $d\mu^+$ the $m-$measure of the $x-$sections of E; since $\mu^+(]-\infty, 0[) = 0$, we need to compute the $x-$section $E(x)$ of E only for $x \geq 0$. We have $E(x) = [-x, x]$ so that $m(E(x)) = 2x$, and (

$$\mu^+ \otimes m(E) = \int_{[0,\infty[} 2x\,d\mu^+(x) = 6\int_0^\infty \frac{x^3}{1+x^6}\,dx;$$

(since $|x| = 0$ for $x = 0$ the singular part does not contribute to the integral). The integral is easily reduced to Euler's functions by the change of variable $x^6 = t$:

$$\int_0^\infty \frac{t^{1/2}}{1+t}\,\frac{t^{-5/6}}{6}\,dt = \frac{1}{6}\int_0^\infty \frac{t^{2/3-1}}{1+t}\,dt =$$
$$= \frac{1}{6}\,B(1/3, 2/3) = \frac{1}{6}\Gamma(1/3)\,\Gamma(2/3) = \frac{\pi}{6\sin(\pi/3)},$$

so that

$$\mu^+ \otimes m(E) = \frac{2\pi}{3}\,\sqrt{3}.$$

□

EXERCISE 14. Consider the function

$$f(x) = \int_{\mathbb{R}} e^{-xt}\,d\mu^+(t),$$

where μ^+ is the positive part of the measure μ defined in Exercise 1).

(i) Find the domain of existence D of f (=the set of all $x \in \mathbb{R}$ such that the integral defining $f(x)$ is finite) and discuss continuity of f on D.

(ii) Find the largest $k \in \mathbb{N}$ such that $f \in C^k(D)$.

(iii) Prove that $f \in C^\infty(\operatorname{int}(D))$, where $\operatorname{int}(D)$ is the interior of D

Solution. (i) Since $\mu^+(]-\infty, 0[) = 0$ the function is ($\pi/2$ is the integral with respect to the singular part $\pi\,\delta/2$)

$$f(x) = \frac{\pi}{2} + 3\int_0^\infty e^{-xt}\,\frac{t^2}{1+t^6}\,dt;$$

Clearly the integral is finite if and only if $x \geq 0$ (if $x < 0$ the integrand has limit $+\infty$ as $t \to \infty$, and the integral cannot converge); in other words $D = [0, \infty[$. And since for $x \in D$:

$$0 \leq e^{-xt}\,\frac{t^2}{1+t^6} \leq \frac{t^2}{1+t^6}, \quad \text{and} \quad t \mapsto \frac{t^2}{1+t^6} \in L^1([0, \infty[),$$

by the theorem on continuity of parameter depending integrals f is continuous on $D = [0, \infty[$.

(ii) Differentiating under the integral sign we get (we forget the factor 3)

$$\left|(-t)e^{-xt}\,\frac{t^2}{1+t^6}\right| \leq \frac{t^3}{1+t^6}, \quad \text{and} \quad t \mapsto \frac{t^3}{1+t^6} \in L^1([0, \infty[),$$

so that $f \in C^1(D)$; another differentiation brings

$$0 \le t^2 e^{-xt}\,\frac{t^2}{1+t^6} \le \frac{t^4}{1+t^6}, \quad \text{and} \quad t \mapsto \frac{t^4}{1+t^6} \in L^1([0,\infty[),$$

so that $f \in C^2(D)$ and

$$f''(x) = \int_0^\infty e^{-xt}\,\frac{t^4}{1+t^6}\,dt.$$

another differentiation brings $t^5/(1+t^6)$, which is not in $L^1([0,\infty[)$, so it' s doubtful that $f \in C^3(D)$. We provisionally say that $f \in C^2(D)$, but $f \notin C^3(D)$ (having not yet completely proved it).

(iii) Of course $\mathrm{int}(D) =]0,\infty[$. We observe that if $x > 0$ then $t \mapsto e^{-xt}\,t^n/(1+t^6)$ belongs to $L^1([0,\infty[)$ for every $n \in \mathbb{N}$. Then $f \in C^\infty(]0,\infty[)$: given $x > 0$, if $y \ge x/2$ we have

$$\left|(-t)^k\,e^{-yt}\,\frac{t^2}{1+t^6}\right| \le e^{-yt}\,\frac{t^{2+k}}{1+t^6} \le e^{-xt/2}\,\frac{t^{2+k}}{1+t^6}(= \gamma_x(t) \in L^([0,\infty[)),$$

and the theorem on differentiation of parameter depending integrals says that $f \in C^k(]0,\infty[)$ for every $k \in \mathbb{N}$.

We can now prove that $f \notin C^3(D)$; in fact $f \in C^3(\mathrm{int}(D))$ and for $x > 0$ we have

$$f'''(x) = -3\int_0^\infty e^{-xt}\,\frac{t^5}{1+t^6}\,dt;$$

for $x_j \downarrow 0$ sequence $f_j(t) = 3e^{-x_j t}t^5/(1+t^6)$ converges increasingly to $3t^5/(1+t^6)$; by monotone convergence:

$$\lim_{j\to\infty}\int_0^\infty f_j(t)\,dt = 3\int_0^\infty \frac{t^5}{1+t^6}\,dt = \infty,$$

which implies $\lim_{x\to 0^+} f'''(x) = -\infty$.

Remark. Some people interpreted $f(x)$ as

$$f(x) = \frac{\pi}{2} + \int_0^\infty e^{-xt}\,\frac{3x^2}{1+x^6}\,dx,$$

or something similar; a fatal error which rendered everything incomprehensible.

□

Exercise 15. Let $(X, \mathcal{M})$ be a measurable space, and let $\nu : \mathcal{M} \to \tilde{\mathbb{R}}$ be a signed measure; as usual we denote $\nu^\pm$ the positive and negative parts of ν, and with $|\nu| = \nu^+ + \nu^-$ the total variation. As with positive measures, a signed measure is said to be $\sigma-$finite if X can be covered by a sequence $E_n \in \mathcal{M}$ of sets of finite $\nu-$measure.

(i) Prove that for an $E \in \mathcal{M}$ we have $\nu(E) \in \mathbb{R}$ iff $\nu^\pm(E) < \infty$, and iff $|\nu|(E) < \infty$. Deduce that a signed measure is $\sigma-$finite iff its total variation is $\sigma-$finite.

(ii) Prove that if $\mu, \nu : \mathcal{M} \to [0,\infty]$ are positive measures then $\mu + \nu$ is $\sigma-$finite iff μ and ν are both $\sigma-$finite.

(iii) Prove that if $\mu,\ \nu : \mathcal{M} \to \tilde{\mathbb{R}}$ are $\sigma-$finite signed measures there is a $\sigma-$finite positive measure $\rho : \mathcal{M} \to [0,\infty]$ and measurable functions f, g integrable in the extended sense such that $d\mu = f\,d\rho$ and $d\nu = g\,d\rho$.

(iv) Prove that if $\mu, \nu : \mathcal{M} \to \mathbb{C}$ are complex measures there are: a finite positive measure $\rho : \mathcal{M} \to [0,\infty[$ and functions $f, g \in L^1(\rho)$ such that

$$\mu(E) = \int_E f\,d\rho, \quad \nu(E) = \int_E g\,d\rho \quad \text{for every } E \in \mathcal{M}.$$

Solution. (i) Recall that if $\nu(E)$ is finite then for every $F \in \mathcal{M}$ with $F \subseteq E$ we also have $\nu(F) \in \mathbb{R}$: for $\nu(E) = \nu(F) + \nu(E \setminus F)$, and a meaningful sum of extended reals is finite iff both summands are finite. Then $\nu(E) \in \mathbb{R}$ implies $\nu^+(E) = \nu(E \cap P) \in \mathbb{R}$, and $\nu^-(E) = -\nu(E \cap Q) \in \mathbb{R}$; in turn this implies $|\nu|(E) = \nu^+(E) + \nu^-(E) < \infty$, and if $|\nu|(E)$ is finite then $|\nu(E)| \le |\nu|(E)$ implies $\nu(E) \in \mathbb{R}$. The remaining assertion on $\sigma-$finiteness is then trivial.

(ii) Of course $E \in \mathcal{M}$ has finite $\mu + \nu)-$measure iff both $\mu(E)$ and $\nu(E)$ are finite. If $(E_m)_{m\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ are countable coverings of X by sets of finite $\mu-$ and $\nu-$measure, respectively, then $(E_m \cap F_n)_{(m.n)\in\mathbb{N}\times\mathbb{N}}$ is countable covering of X by sets of finite $(\mu + \nu)-$measure, thus proving that if μ and

ν are both $\sigma-$finite then $\mu+\nu$ is also $s-$finite. The converse is trivial: a cover by a sequence $E_n \in \mathcal{M}$ with $(\mu+\nu)(E_n) < \infty$ is also a cover by sets of finite μ and ν measure.

(iii) If μ and ν are $\sigma-$finite then by (i) $|\mu|$ and $|\nu|$ are $s$finite, and by (ii) $\rho = |\mu| + |\nu|$ is $\sigma-$finite. Moreover $|\mu(E)| \le |\mu|(E) \le \rho(E)$ and $|\nu(E)| \le |\nu|(E) \le \rho(E)$ clearly imply $\mu \ll \rho$ and $\nu \ll \rho$. The Radon–Nikodym theorem then applies to conclude the question (to be more precise: if $\mu = \mu^+ - \mu^-$ then one at least of these measures, say μ^+, is finite, and the Radon–Nikodym theorem implies that $d\mu^+ = u\,d\rho$ for some positive $u \in L^1(\rho)$; for μ^- we can only assert that there is $v \ge 0$ measurable such that $\mu^-(E) = \int_E v\,d\rho$ for every $E \in \mathcal{M}$; then $f = u - v$ is such that $d\mu = f\,d\rho$; same for ν).

(iii) A complex measure has necessarily finite real and imaginary parts; we can the take

$$\rho = |\mu_r| + |\mu_\iota| + |\nu_r| + |\nu_\iota|.$$

Remark. (i) It is not correct, when ν is a signed measure, to write $\nu(E) < \infty$ to signify that $\nu(E)$ is a real number; one ought to write $-\infty < \nu(E) < \infty$ to indicate that $\nu(E) \in \mathbb{R}$.

(ii) If μ and ν are positive measures they are not necessarily positive and negative parts of a signed measure; this certainly does not happen if they are both infinite, and even when $\mu-\nu$ is meaningful (which is the case iff one of the two measures is finite) we shall in general have $(\mu-\nu)^+(E) \le \mu(E)$ and $(\mu-\nu)^-(E) \le \nu(E)$, with equality if and only if $\mu \perp \nu$ (LN, Exercise 6.2.5.1).

□

Exercise 16. Let $(X\mathcal{M}, \mu)$ be a measure space. We denote by $L(\mu)$ the set of all (equivalence classes modulo equality $\mu-$a.e. of) measurable complex valued functions. We say that a subset S of $L(\mu)$ is $\sigma-$closed in $L(\mu)$ if it contains the pointwise limits of its a.e. converging sequences: in other words, S $\sigma-$closed means that whenever a sequence $f_n \in S$ converges a.e. to $f \in L(\mu)$, then $f \in S$.

(i) Prove that if $S \subseteq L(\mu)$ is $\sigma-$closed, then $S \cap L^p(\mu)$ is closed in $L^p(\mu)$, for every $p \ge 1$.

(ii) Let $F \subseteq \mathbb{C}$ be closed in $\mathbb{C}$, and denote by $L_\mu(X, F)$ the set of all $f \in L(\mu)$ such that $\mu(f^{\leftarrow}(\mathbb{C} \setminus F)) = 0$. Prove that $L_\mu(X, F)$ is $\sigma-$closed in $L(\mu)$.

(iii) Let $\varphi : [0, \infty[\to [0, \infty[$ be continuous, and let $a > 0$. Prove that the set

$$S = \left\{ f \in L(\mu) : \int_X \varphi(|f(x)|)\, d\mu(x) \le a \right\}$$

is $\sigma-$closed in $L(\mu)$.

Recall that a sequence $f_n \in L(\mu)$ is said to *converge in measure* to $f \in L(\mu)$ if for every $t > 0$ we have $\lim_{n\to\infty} \mu(\{|f - f_n| > t\}) = 0$. We have seen (accept it) that

. (*) *If f_n converges to f in measure then some subsequence of f_n converges to f almost uniformly.*

(iv) Prove that if $S \subseteq L(\mu)$ is $\sigma-$closed then it is also closed under convergence in measure of sequences (first prove that almost uniform convergence implies a.e. convergence).

(v) Assuming $\mu(X) < \infty$, prove that a subset S of $L(\mu)$ is $\sigma-$closed iff it is closed under convergence in measure of sequences.

(vi) Let $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}(\mathbb{R})$ and $\mu = m$, Lebesgue measure; let

$$S = \{\chi_{[n.n+1[} : n \in \mathbb{N}\}.$$

Prove that S is closed under convergence in measure, but it is not $\sigma-$closed.

Solution. (i) Recall that a subset of the normed space $L^p(\mu)$ is said to be closed in $L^p(\mu)$ when it contains the limits of its sequences converging in the space $L^p(\mu)$. Thus we have to prove that if a sequence $f_n \in L^p \cap S$ converges to $f \in L^p(\mu)$ in the $p-$norm we actually have $f \in S \cap L^p(\mu)$. Not necessarily f_n converges pointwise a.e. to f, but we know that some subsequence $f_{n(k)}$ converges to f also a.e.; and since S is $\sigma-$closed we get $f \in S$, as a.e. limit of the sequence $f_{n(k)} \in S$.

(ii) Assume that $f_n \in S$ converges a.e to f. Let $N_n = f_n^{\leftarrow}(\mathbb{C} \setminus F)$, and let $N = \{x \in X : f_n(x) \text{ does not converge to } f(x)\}$. Then $M = N \cup (\bigcup_{n=0}^{\infty} N_n)$ has measure 0; and if $x \in X \setminus M$ we have $f_n(x) \in F$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$. Then $f(x) \in F$, because F is closed in $\mathbb{C}$. It follows that $f(X \setminus M) \subseteq F$, so that $f^{\leftarrow}(\mathbb{C} \setminus F) \subseteq M$ has measure 0.

(iii) Assume that $f_n \in S$ converges a.e. to the measurable function f. Then $|f_n|$ converges a.e to $|f|$, and by continuity of φ on $[0, \infty[$) we have that $\varphi \circ |f_n|$ is a sequence of measurable functions that converges a.e to $\varphi \circ |f|$. Then $\lim_{n\to\infty} \varphi \circ |f_n| = \liminf_{n\to\infty} \varphi \circ |f_n| = \varphi \circ |f|$, and by Fatou's lemma:

$$\int_X \varphi \circ |f| \le \liminf_{n\to\infty} \int_X \varphi \circ |f_n| \le a.$$

(iv) By the above recalled fact, convergence in measure implies almost uniform convergence of a subsequence. Next we observe that almost uniform convergence implies pointwise convergence a.e.: the set of points at which an almost uniformy convergent sequence does not converge has measure smaller that any number $\delta > 0$ (given $\delta > 0$, if $E(\delta) \in \mathcal{M}$ is such that $\mu(E(\delta)) \le \delta$ and the sequence converges uniformly in $X \setminus E(\delta)$, then the set of points of non convergence is contained in $E(\delta)$) and has then measure zero. Then a $\sigma-$closed set is also closed under convergence in measure: if $f_n \in S$ converges to f in measure then some subsequence $f_{n(k)}$ converges to f almost uniformly, in particular also a.e; if S is $\sigma-$closed then $f \in S$, as the a.e. limit of $f_{n(k)} \in S$ (same argument as in (i)).

(v) As seen in (iv), on every measure space $\sigma-$closure implies closure under convergence in measure. On a finite measure space pointwise convergence a.e. implies almost uniform convergence (Severini–Egoroff theorem). Now it is immediate that almost uniform convergence implies convergence in measure: given $t, \varepsilon > 0$ pick $E \in \mathcal{M}$ with $\mu(E) \le \varepsilon$ such that in $X \setminus E$ the sequence f_n converges uniformly to f; we then get $\bar{n} \in \mathbb{N}$ such that for $n \ge \bar{n}$ we have $|f(x) - f_n(x)| \le t$ for every $x \in X \setminus E$; then $\{|f - f_n| > t\} \subseteq E$ for every $n \ge \bar{n}$ so that $\mu(\{|f - f_n| > t\}) \le \mu(E) \le \varepsilon$ for every $n \ge \bar{n}$. We have proved that in a finite measure space cconvergence a.e. implies convergence in measure, so that any subset of $L(\mu)$ closed under convergence in measure is also $\sigma-$closed, if $\mu(X) < \infty$.

(vi) The whole sequence $n \mapsto \chi_{[n,n+1[}$ converges pointwise to the 0 function, which is not in S, so that S is not $\sigma-$closed. Any sequence $f_n \in S$ with infinite range shares a subsequence with $n \mapsto \chi_{[n,n+1[}$, and if it converges in measure to some measurable function, this function has then to be 0. But for every n we have $\mu(\{\chi_{[n,n+1[} > 1/2\}) = \mu([n, n+1[) = 1$; no subsequence of $n \mapsto \chi_{[n,n+1[}$ can converge in measure to the zero function. Then a sequence of S converging in measure has to be eventually constant, and so its limit is in S.

Remark. (i) Some people thought that to answer they had to prove that if $f_n \in S \cap L^p(\mu)$ converges pointwise a.e. to some f, then $f \in L^p(\mu)$ (or even more, that in this case $\lim_{n\to\infty} \|f - f_n\|_p = 0$). This is of course not true, even in finite measure spaces. Take e.g $[0, 1]$ with Lebesgue measure μ; if $S = L(\mu)$ we have $S \cap L^p(\mu) = L^p(\mu)$ (by definition $L^p(\mu)$ consists of measurable functions); the function $x \mapsto 1/x$, which is in no $L^p(\mu)$ for $p \ge 1$ is the limit of an increasing sequence of step–functions, which are in every $L^p(\mu)$!

(ii) The set $L_\mu(X, F) \cap L^1(\mu)$ was used in Lecture Notes, exercise 3.2.19.4.

(iii) There is a dangerous misunderstanding: dominated convergence is *not* implied by the fact that the integrals of functions in the sequence are bounded! it requires much more, that *all functions in the sequence* are dominated by a single function with finite integral!

(iv) Some people believe that convergence a.e. of f_n to f means that $\lim_{n\to\infty} \mu(\{|f - f_n| > 0\}) = 0$. This is not true: $f_n = \chi_{[n,n+1[}$ converges pointwise to 0, yet $\mu(\{|f - f_n| > 0\}) = 1$ for every n.

□

Analisi Reale per Matematica – Secondo appello invernale – 24 febbraio 2014

Exercise 17. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 1 - e^{x+1} & \text{if} \quad x < -1 \\ \arcsin x & \text{if} \quad -1 \le x < 1 \\ 1 - e^{-(x-1)} & \text{if} \quad x \ge 1 \end{cases}.$$

Let μ denote the signed measure dF associated to F.

(i) Plot F.

(ii) Compute the derivative $F'(x)$ where it exists; find the points of $\mathbb{R}$ with non–zero $\mu-$measure, and write the decomposition of μ into absolutely continuous and singular part (with respect to Lebesgue measure $m = dx$, of course).

(iii) Write μ^+ and μ^- with their absolutely continuous and singular part. Find a Hahn decomposition for μ.

(iv) Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le |x|\}$. Compute $|\mu| \otimes m(E)$.

(you can answer (ii) and (iii) without explicitly computing the total variation function of F)

Solution. (i) The plot is very easy:

(ii) The function F is continuous everywhere but at the points $-1, 1$, where it has a jump of $-\pi/2$, so that $\mu(\{-1\}) = \mu(\{1\}) = -\pi/2$. It is of class C^1 in its set of continuity, $]-\infty, -1[\cup]-1, 1[\cup]1, \infty[$,

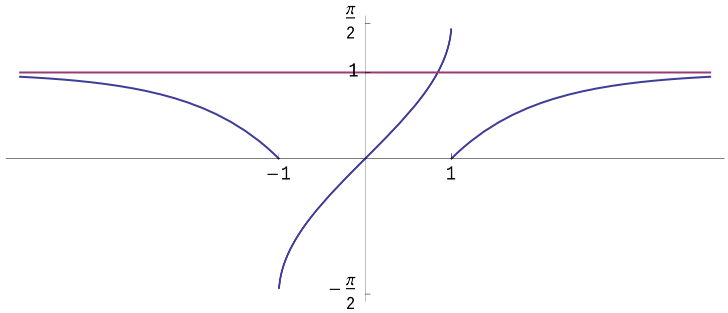

FIGURE 10. Plot of F.

where the derivative is

$$F'(x) = -e^{x+1}\,\chi_{]-\infty,-1[}(x) + \frac{1}{\sqrt{1-x^2}}\,\chi_{]-1,1[}(x) + e^{-(x-1)}\,\chi_{]1,\infty[}(x);$$

thus the measure is locally absolutely continuous on each of the three intervals $]-\infty,-1[$, $]-1,1[$, $]1,\infty[$, and we have $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$ with $\mu_{\mathrm{a}} = F'(x)\,dx$ and $\mu_{\mathrm{s}} = -\pi/2(\delta_{-1}+\delta_1)$.

(iii) The positive part of μ is

$$\mu^+ = \mu_{\mathrm{a}}^+ + \mu^+\mathrm{s} = (F'(x))^+\,dx + 0 = \frac{1}{\sqrt{1-x^2}}\,\chi_{]-1,1[}(x) + e^{-(x-1)}\,\chi_{]1,\infty[}(x),$$

absolutely continuous, and the negative part is

$$\mu^- = \mu_{\mathrm{a}}^- + \mu^-\mathrm{s} = (F'(x))^-\,dx + \frac{\pi}{2}(\delta_{-1}+\delta_1) = e^{x+1}\,\chi_{]-\infty,-1[}(x) + \frac{\pi}{2}(\delta_{-1}+\delta_1);$$

a positive set for μ is $]-1,1[\cup]1,\infty[$. with the complement $\infty,-1]\cup\{1\}$ negative.

For future use we notice that

$$|\mu| = \mu^+ + \mu^-1 = |F'(x)|\,dx + \frac{\pi}{2}(\delta_{-1}+\delta_1) =$$
$$= e^{x+1}\,\chi_{]-\infty,-1[}(x) + \frac{1}{\sqrt{1-x^2}}\,\chi_{]-1,1[}(x) + e^{-(x-1)}\,\chi_{]1,\infty[}(x) + \frac{\pi}{2}(\delta_{-1}+\delta_1).$$

(iv) We have, observing that the $x-$section of E is $[0,|x|]$, so that E is the trapezoid of the function $x \mapsto |x|$:

$$|\mu|\otimes m(E) = \int_{\mathbb{R}} m([0,|x|])\,d|\mu|(x) = \int_{\mathbb{R}} |x|\,d|\mu|(x) =$$
$$= \int_{]-\infty,-1[} |x|\,e^{x+1}\,dx + \int_{]-1,1[} \frac{|x|}{\sqrt{1-x^2}}\,dx + \int_{[1,\infty[} |x|\,e^{-(x-1)}\,dx + \frac{\pi}{2}\,|-1| + \frac{\pi}{2}\,|1| =$$

(in the first integral we set $-x$ in place of x)

$$= \pi + \int_1^\infty x\,e^{-x+1}\,dx + \int_1^\infty x\,e^{-x+1}\,dx + 2\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx =$$
$$= \pi + 2e\left(\left[-xe^{-x}\right]_1^\infty + \int_1^\infty e^{-x}\,dx\right) + 2\left[-\sqrt{1-x^2}\right]_0^1 =$$
$$= \pi + 2e(e^{-1}+e^{-1}) + 2 = \pi + 6.$$

□

EXERCISE 18. Consider the function

$$f(x) = \int_{\mathbb{R}} \sin(xt)\,d|\mu|(t),$$

where $|\mu|$ is the total variation of the measure μ defined in Exercise 1).

(i) Find the domain of existence D of f (=the set of all $x\in\mathbb{R}$ such that the integral defining $f(x)$ is finite) and discuss continuity of f on D.

(ii) Is $f\in C^\infty(D)$? Otherwise, find the largest $k\in\mathbb{N}$ such that $f\in C^k(D)$.

Solution. (i) Is immediate: we only have to notice that $|\mu|$ is a finite measure ($|\mu|(\mathbb{R}) = 2(1+\pi)$), so that the constant 1 is in $L^1(|\mu|)$, and moreover $|\sin(xt)| \le 1$, so that f is defined for every $x \in \mathbb{R}$; since $x \mapsto \sin(xt)$ is continuous the theorem on continuity of parameter depending integrals (with a unique global dominating function $\gamma(t) = 1$) implies continuity of f on $D = \mathbb{R}$.

(ii) Setting $g(x,t) = \sin(xt)$. we have that $(\partial_x g)^k(x,t) = t^k h(x,t)$, with $h(x,t) = \pm\sin(xt)$ or $h(x,t) = \pm\cos(xt)$; in any event we have

$$|(\partial_x g)^k(x,t)| \le |t|^k \quad \text{for every } x, t \in \mathbb{R} \text{ and every } k \in \mathbb{N}.$$

If $\gamma(t) = |t|^k$ is in $L^1(|\mu|)$ for every k, the differentiabilty part of the teorem allows us to say that $f \in C^\infty(\mathbb{R})$. It is clear that $|t|^k \in L^1(|\mu|)$; the function is bounded on compact sets, hence summable there; and clearly $t \mapsto |t|^k\, e^{t+1}$, $t \mapsto |t|^k\, e^{-(t-1)}$ belong to $L^1_m(]-\infty,-1[)$ and $L^1_m(]1,\infty[)$, respectively.

Remark. One can also observe that

$$f(x) = \int_{-\infty}^{-1} \sin(xt)\, e^{t+1}\, dt + \int_{-1}^{1} \frac{\sin(xt)}{\sqrt{1-t^2}}\, dt + \int_1^\infty \sin(xt)\, e^{-(t-1)}\, dt + \frac{\pi}{2}\int_{\mathbb{R}} \sin(xt)\, d(\delta_{-1}(t) + \delta_1(t)) =$$

(put $t = -s$ in the first integral, and integrate with respect to the singular part)

$$= \int_\infty^1 \sin(-xs)\, e^{-s+1}\,(-ds) + \int_{-1}^{1} \frac{\sin(xt)}{\sqrt{1-t^2}}\, dt + \int_1^\infty \sin(xt)\, e^{-(t-1)}\, dt + \frac{\pi}{2}(\sin(-x) + \sin x) =$$

$$= \int_{-1}^{1} \frac{\sin(xt)}{\sqrt{1-t^2}}\, dt,$$

in other words we have

$$f(x) = \int_{-1}^{1} \frac{\sin(xt)}{\sqrt{1-t^2}}\, dt,$$

and the fact that $f \in C^\infty(\mathbb{R})$ becomes immediate, since t^k is bounded on $[-1,1]$, and $t \mapsto 1/\sqrt{1-t^2}$ is in $L^1_m([-1,1])$

Many people still have not understood how to use the theorem to prove continuity of the parameter depending integral: proving that $t \mapsto \sin(xt)$ is in $L^1(|\mu|)$ for every $x \in \mathbb{R}$ proves only that f is defined on all of $\mathbb{R}$, for continuity one has to find a dominating function in $L^1(|\mu|)$ which does not depend on x, at least locally (here it is easy to find a globally dominating function, the constant 1).

□

Exercise 19. Given a set X and a positive function $w : X \to [0,\infty]$, for every subset A of X we define the *sum of w over A* as

$$\sum_A w = \sum_{x\in A} w(x) := \sup\left\{\sum_{x\in F} w(x),\ F \subseteq A,\ F \text{ finite}\right\} \qquad (\text{finite or } \infty).$$

It is easy to prove (accept it) the *unrestricted associativity of infinite sums*, that is, for every disjoint family $(A_\lambda)_{\lambda\in\Lambda}$ of subsets of X we have, if $A = \bigcup_{\lambda\in\Lambda} A_\lambda$

$$\sum_{x\in A} w(x) = \sum_{\lambda\in\Lambda}\sum_{x\in A_\lambda} w(x).$$

(i) Prove that the formula $\mu_w(A) = \sum_{x\in A} w(x)$ defines a measure $\mu_w : \mathcal{P}(X) \to [0,\infty]$.

From now on we assume that X is an infinite set, that w is finite–valued, and that $\mu_w(X) = \infty$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{P}(X)$ consisting of finite and cofinite subsets; define the set function $\mu : \mathcal{A} \to [0,\infty]$ by the formula $\mu(A) = \mu_w(A)$ if A is finite, and $\mu(A) = \infty$ if A is cofinite.

(ii) Prove $\mu(A) = \mu_w(A)$ for every $A \in \mathcal{A}$, and that μ is a premeasure.

(iii) Prove that if μ^* is the outer measure associated to μ by the usual procedure, then $\mu_w(E) \le \mu^*(E)$ for every $E \subseteq X$, and that $\mu^*(E) = \infty$ for every uncountable $E \subseteq X$. Prove that $\mu_w(E) = \mu^*(E)$ for every countable $E \subseteq X$.

(iv) Prove that every subset A of X is μ^*−measurable (recall that measurability of A may be checked with sets E of finite μ^*−measure ...).

We now specialize by taking $X = \mathbb{R}$ and $w : \mathbb{R} \to [0,\infty[$ defined by $w(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Z}$, and $w(x) = 2^{-x} \wedge 1$ for $x \in \mathbb{Z}$.

(v) Exhibit a subset E of $\mathbb{R}$ for which $\mu_w(E) < \mu^*(E)$.

(vi) Given $0 < p < q$ find $f \in L^p(\mu_w) \setminus L^q(\mu_w)$ and $g \in L^q(\mu_w) \setminus L^p(\mu_w)$ (you may take $f(x) = g(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$, you need to define f and g only on $\mathbb{Z}$ ...).

Solution. (i) Clearly the empty sum is 0, $\sum_\emptyset w = 0$ (by definition), so that $\mu_w(\emptyset) = 0$. And unrestricted associativity with Λ countable is exactly countable additivity of μ_w.

(ii) Notice that since w is finite–valued we have $\mu_w(E) < \infty$ fo every finite subset E of X. If A is cofinite For every $A \subseteq X$ we have $\mu_w(X) = \mu_w(A) + \mu_w(X \setminus A)$; if A is cofinite then $\mu_w(X \setminus A) < \infty$. and by the hypothesis $\mu_w(X) = \infty$ we get $\mu_w(A) = \infty$, so that $\mu_w(A) = \mu(A)$ for cofinite sets. Countable additivity of μ on $\mathcal{A}$ is now obvious, since $\mu(A) = \mu_w(A)$ for every $A \in \mathcal{A}$ and μ_w is countably additive. Then μ is a premeasure.

(iii) Given $E \subseteq X$, assume that $(A_n)_{n\in\mathbb{N}}$ is a countable cover of E by elements of $\mathcal{A}$. Then, first by monotonicity and then by countable subadditivity of μ_w:

$$\mu_w(E) \le \mu_w\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \mu_w(A_n) = \sum_{n\in\mathbb{N}} \mu(A_n),$$

the last equality due to the fact that $\mu_w(A_n) = \mu(A_n)$ for every $n \in \mathbb{N}$ since $A_n \in \mathcal{A}$. We have proved that

$$\mu_w(E) \le \sum_{n\in\mathbb{N}} \mu(A_n) \quad \text{for every countable cover of } E \text{ by elements of } \mathcal{A},$$

which implies $\mu_w(E) \le \mu^*(E)$. If E is countable then $\{x\} : x \in E\}$ is a countable cover of E by elements of A and $\mu_w(E) = \sum_{x\in E} w(x) = \sum_{x\in E} \mu(\{x\})$, so that $\mu^*(E) \le \mu_w(E)$, and equality $\mu^*(E) = \mu_w(E)$ then holds. Finally, if E is uncountable then every countable cover $(A_n)_{n\in\mathbb{N}}$ by elements of $\mathcal{A}$ must contain a cofinite set, since a countable union of finite sets is at most countable and cannot contain the uncountable set E. Then $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$, because one element of the sum is infinite. We have seen that for every countable cover $(A_n)_{n\in\mathbb{N}}$ of E by elements of $\mathcal{A}$ we have $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$: this is equivalent to say that $\mu^*(E) = \infty$.

(iv) We know that a set $A \subseteq X$ is μ^*-measurable if and only if $\mu^*(E) = \mu^*(A \cap E) + \mu^*(E \setminus A)$ for every $E \subseteq X$ with $\mu^*(E) < \infty$. This means that E is countable; then also $E \cap A$ and $E \setminus A$ are countable, being subsets of E. But then $\mu_w(E) = \mu_w(E \cap A) + \mu_w(E \setminus A)$ by additivity of μ_w, and on each of these sets μ_w and μ^* coincide, as seen in (iii); in other words the preceding equality is exactly $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$.

(v) The set $E = \mathbb{R} \setminus \mathbb{Z}$ is uncountable, so that $\mu^*(E) = \infty$, by (iii). But $\mu_w(E) = \sum_{x\in E} w(x) = 0$ since w is identically zero on E.

(vi) We try to define $f = f_\alpha \in L^p(\mu_w) \setminus L^q(\mu_w)$ in the following way: $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{N}$, $f(n) = 2^{n\alpha}$ per $n \in \mathbb{N}$, with $\alpha > 0$ to be determined. We have

$$\int_{\mathbb{R}} (f(x))^p \, d\mu_w(x) = \sum_{x\in\mathbb{R}} (f(x))^p \, w(x) = \sum_{n=0}^{\infty} 2^{n\alpha p} \, 2^{-n} = \sum_{n=0}^{\infty} 2^{n(\alpha\, p-1)} = \sum_{n=0}^{\infty} (2^{\alpha\, p-1})^n,$$

a geometric series that converges iff $\alpha\, p - 1 < 0 \iff \alpha < 1/p$. Similarly

$$\int_{\mathbb{R}} (f(x))^q \, d\mu_w(x) = \sum_{n=0}^{\infty} (2^{\alpha\, q-1})^n,$$

converging iff $\alpha < 1/q$ Since $0 < p < q$ we have $1/q < 1/p$; for any $\alpha \in [1/q, 1/p[$ we have $f_\alpha \in L^p(\mu_w) \setminus L^q(\mu_w)$. To get $g \in L^q(\mu_w) \setminus L^p(\mu_w)$ we take instead $g = g_\alpha(x) = 0$ for $x \notin -\mathbb{N}$, and $g(-n) = 1/(n+1)^\alpha$ for every $n \in \mathbb{N}$, We have

$$\int_{\mathbb{R}} (g(x))^p \, d\mu_w(x) = \sum_{x\in\mathbb{R}} (g(x))^p \, w(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha\, p}},$$

and the last series converges iff $\alpha\, p > 1 \iff \alpha > 1/p$, For $\alpha \in]1/q, 1/p]$ we get $g_\alpha \in L^q(\mu_w) \setminus L^p(\mu_w)$. □

Exercise 20. Let $F_n : \mathbb{R} \to \mathbb{R}$ be a sequence of increasing functions such that $F_n(-\infty) = 0$, $F_n(\infty) = 1$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} F_n(x)$ exists for every $x \in \tilde{\mathbb{R}}$; calling this limit $F(x)$, it is clear that the formula $x \mapsto F(x)$ defines an increasing function $F : \mathbb{R} \to \mathbb{R}$, with $F(-\infty) = 0$ and $F(\infty) = 1$. Let $\mu_n = dF_n$ and $\mu = dF$ be the measures associated to the functions F_n and F.

(i) Prove that there is a countable subset D of $\mathbb{R}$ such that on $C = \mathbb{R} \setminus D$ the functions F, F_n are all continuous.

Let $\mathcal{A} = \mathcal{A}_C$ denote the subalgebra of $\mathcal{P}(\mathbb{R})$ consisting of finite disjoint unions of intervals with extremes in C.

(ii) Prove that for every $\mathcal{A}-$simple function g we have

$$\int_{\mathbb{R}} g\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} g\,d\mu_n$$

(recall that $\mathcal{A}-$simple functions are step functions, linear combinations of characteristic functions of intervals with extremes in C ...).

(iii) Here, and here only, assume $F_n(x) = (1 + \tanh(nx))/2$. Plot some F_n and F, and describe all intervals I such that $\lim_{n\to\infty} \mu_n(I) \neq \mu(I)$.

(iv) (Back to general F_n and F) Assume that $f : \mathbb{R} \to \mathbb{R}$ is in the uniform closure of the space of $\mathcal{A}-$simple functions, that is, there is a sequence g_k of $\mathcal{A}-$simple functions converging uniformly on $\mathbb{R}$ to f. Then

$$(*) \qquad \int_{\mathbb{R}} f\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f\,d\mu_n$$

(v) [extra] Prove that $f : \mathbb{R} \to \mathbb{R}$ is in the uniform closure of the space of $\mathcal{A}-$simple functions if and only if f is continuous on $\mathbb{R} \setminus C$, has finite left and right limits at every $c \in C$, and both limits $\lim_{x\to\pm\infty} f(x)$ exist finite.

Solution. (i) It is well known that an increasing function has a set of points of discontinuity that is at most countable; if D_n is the set of jump points of F_n, and $D(F)$ is the set of discontinuity points of F, setting $D = D(F) \cup \left(\bigcup_{n\in\mathbb{N}} D_n\right)$, the set D is countable, and at every point of $C = \mathbb{R} \setminus D$ the limit function F and all functions F_n are continuous.

(ii) For every interval I with extremes $\inf I, \sup I \in C$ we have $\mu_n(I) = F_n(\sup I) - F_n(\inf I)$ and also $\mu(I) = F(\sup I) - F(\inf I)$, because F_n and F are continuous at $\inf I, \sup I \in C$. Then, since $F(\sup I) = \lim_{n\to\infty} F_n(\sup I)$ and similarly for $\inf I$ we get

$$\lim_{n\to\infty} \mu_n(I) = \lim_{n\to\infty} (F_n(\sup I) - F_n(\inf I)) = \lim_{n\to\infty} F_n(\sup I) - \lim_{n\to\infty} F_n(\inf I) = F(\sup I) - F(\inf I) = \mu(I).$$

If g is $\mathcal{A}-$simple we have $g = \sum_{k=1}^m \alpha_k\, \chi_{I(k)}$, where each $I(k)$ is an interval with extremes in C; then

$$\int_{\mathbb{R}} g\,d\mu_n = \sum_{k=1}^m \alpha_k\, \mu_n(I(k));$$

taking limits as $n \to \infty$ we get

$$\lim_{n\to\infty} \left(\sum_{k=1}^m \alpha_k\, \mu_n(I(k))\right) = \sum_{k=1}^m \alpha_k \lim_{n\to\infty} \mu_n(I(k)) = \sum_{k=1}^m \alpha_k\, \mu(I(k)) = \int_{\mathbb{R}} g\,d\mu;$$

thus proving what required.

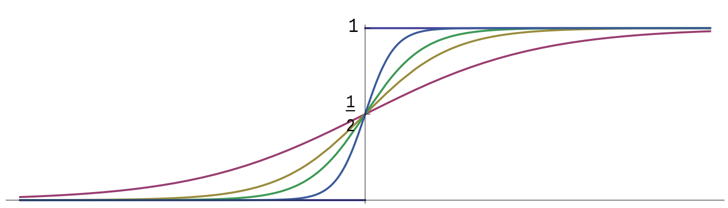

FIGURE 11. Plots of F_n, F.

(iii) The plot is easy (if one remembers the plot of tanh!). If $F(x) = \lim_{n\to\infty} F_n(x)$, we have $F(x) = 1$ for $x > 0$, $F(x) = 0$ for $x < 0$, and $F(0) = 1/2$. All functions F_n are continuous, only the limit function F is discontinuous at 0, with a jump of 1. Then we may take $D = \{0\}$, and $C = \mathbb{R} \setminus \{0\}$. By (ii) the only intervals I which can make $\mu(I) \neq \lim_{n\to\infty} \mu_n(I)$ are those with an extreme in 0. And in fact this happens for $\{0\}$ and for every interval not containing 0 with 0 as an extreme: if $I = \{0\}$ we have $\mu_n(I) = 0$ for every n, but $\mu(\{0\}) = 1$. Assume now that $\inf I = 0$, that $\sup I > 0$ and that $0 \notin I$. Then $\mu_n(I) = F_n(\sup I) - F_n(0) = F_n(\sup I)$ for every n, so that $\lim_{n\to\infty} \mu_n(I) = \lim_{n\to\infty} F_n(\sup I) = 1$; but $\mu(I) = 1 - F(0^+) = 1 - 1 = 0$ if $0 \notin I$, whereas $\mu(I) = 1 - F(0^-) = 1$ if $0 \in I$. Similarly for intervals with $0 = \sup I$.

(iv) Write

$$\left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} f\,d\mu_n\right| = \left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} g_k\,d\mu + \int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n + \int_{\mathbb{R}} g_k\,d\mu_n - \int_{\mathbb{R}} f\,d\mu_n\right| \le$$
$$\le \left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} g_k\,d\mu\right| + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right| + \left|\int_{\mathbb{R}} g_k\,d\mu_n - \int_{\mathbb{R}} f\,d\mu_n\right| \le$$
$$\int_{\mathbb{R}} |f - g_k|\,d\mu + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right| + \int_{\mathbb{R}} |g_k - f|\,d\mu_n \le$$
$$\le \int_{\mathbb{R}} \|f - g_k\|_\infty\,d\mu + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right| + \int_{\mathbb{R}} \|g_k - f\|_\infty\,d\mu_n;$$

Since $g_k \to f$ uniformly, given $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $\|f - g_k\|_\infty \le \varepsilon$; then we have

$$\left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} f\,d\mu_n\right| \le \int_{\mathbb{R}} \varepsilon\,d\mu + \int_{\mathbb{R}} \varepsilon\,d\mu_n + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right| = \varepsilon\,\mu(\mathbb{R}) + \varepsilon\,\mu_n(\mathbb{R}) + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right|.$$

We have obtained:

$$\left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} f\,d\mu_n\right| \le 2\varepsilon + \left|\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n\right|;$$

taking limsup for $n \to \infty$, since $\int_{\mathbb{R}} g_k\,d\mu - \int_{\mathbb{R}} g_k\,d\mu_n \to 0$ by (ii) we get

$$\limsup_{n\to\infty} \left|\int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} f\,d\mu_n\right| \le 2\varepsilon,$$

and we conclude, since $\varepsilon > 0$ is arbitrary.

(v) Since uniform convergence "preserves limits" (meaning that if g_k converges uniformly to f, and $\lim_{x\to c} g_k(x) = l_k$ for some $c \in \tilde{\mathbb{R}}$ then $\lim_{k\to\infty} l_k = l$ is finite and $l = \lim_{x\to c} f(x)$), the condition is clearly necessary: all $\mathcal{A}-$simple functions are continuous on $\mathbb{R} \setminus C$, have finite left and right limits at every $c \in C$, and have both limits as $x \to \pm\infty$ finite. We have to prove that any $f : \mathbb{R} \to \mathbb{C}$ that verifies these conditions is the uniform limit of a sequence of $\mathcal{A}-$simple functions. It is clearly enough to prove it for real valued functions. We prove that if $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the stated limit conditions, then for every $\varepsilon > 0$ there are $\mathcal{A}-$simple functions u, v such that $f - \varepsilon \le u \le f \le v \le f + \varepsilon$. Let $\alpha = \inf f$ and $\beta = \sup f$. Given $c \in \tilde{\mathbb{R}}$ we find $\mathcal{A}-$simple functions u_c, v_c such that $u_c \le f \le v_c$, and $f(x) - \varepsilon \le u_c(x) \le v_c(x) \le f(x) + \varepsilon$ for all x in a nbhd U_c of c in $\tilde{\mathbb{R}}$. For $c \in C$ we pick $a_c, b_c \in C$ such that $a_c < c < b_c$, $f(c^-) - \varepsilon \le f(x) \le f(c^-) + \varepsilon$ for $x \in]a_c, c[$, $f(c^+) - \varepsilon \le f(x) \le f(c^+) + \varepsilon$ for $c < x < b_c$, and we define u_c, $v_c : \mathbb{R} \to \mathbb{R}$ by $u_c(c) = f(c) = v_c(c)$ and:

$$u_c(x) = \alpha \quad \text{for } x \le a_c \text{ and } x \ge b_c;\ u_c(x) = f(c^-) - \varepsilon \text{ for } a_c < x < c;\ u_c(x) = f(c^+) - \varepsilon \text{ for } c < x < b_c;$$
$$v_c(x) = \beta \quad \text{for } x \le a_c \text{ and } x \ge b_c;\ v_c(x) = f(c^-) + \varepsilon \text{ for } a_c < x < c;\ v_c(x) = f(c^+) + \varepsilon \text{ for } c < x < b_c;$$

Moreover we set $U(c) =]a_c, b_c[$. For $c \in \mathbb{R} \setminus C$ the functions u_c and v_c are defined in the same way, except that now $f(c^-) = f(c^+) = f(c)$. For $c = \infty$ pick $a \in C$ such that $f(-\infty) - \varepsilon \le f(x) \le f(-\infty) + \varepsilon$ for $x \in]-\infty, a[$, and define $u_{-\infty}(x) = f(-\infty) - \varepsilon$ for $x < a$, $u_{-\infty}(x) = \alpha$ for $x \ge a$, whereas $v_{-\infty} = f(-\infty) + \varepsilon$ for $x < a$, and $v_{-\infty}(x) = \beta$ for $x \ge a$; $U(-\infty) = [-\infty, a[$. An analogous construction defines u_∞ and v_∞. Since $\tilde{\mathbb{R}}$ is compact, there exist $c(1), \dots, c(m) \in \tilde{\mathbb{R}}$ such that $\tilde{\mathbb{R}} = \bigcup_{k=1}^m U(c(k))$. We set $u = u_{c(1)} \vee \cdots \vee u_{c(m)}$ and $v = v_{c(1)} \wedge \cdots \wedge v_{c(m)}$, and we claim that for every $x \in \mathbb{R}$ we have $f(x) - \varepsilon \le u(x) \le f(x) \le v(x) \le f(x) + \varepsilon$. The inequality $u(x) \le f(x)$ is due to the fact that $u_c(x) \le f(x)$ for every x and every c, same for $f(x) \le v(x)$. And given $x \in \mathbb{R}$ we have $x \in U_{c(k)}$ for some $k \in \{1, \dots, m\}$, so that

$$f(x) - \varepsilon \le u_{c(k)}(x) \le u(x) \le f(x); \qquad f(x) + \varepsilon \ge v_{c(k)}(x) \ge v_c(x) \ge f(x).$$

□

Analisi Reale per Matematica – Recupero luglio 14/7/14

Exercise 21. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} (1 + (|x| - 1)^2)^{-1} & \text{if } \ |x| > 1 \\ \sin(\pi x/2) & \text{if } \ -1 \le x \le 1 \end{cases}.$$

Let μ denote the signed measure dF associated to F.

(i) Plot F.

(ii) Compute the derivative $F'(x)$ where it exists; find the points of $\mathbb{R}$ with non–zero $\mu-$measure, and write the decomposition of μ into absolutely continuous and singular part (with respect to Lebesgue measure $m = dx$, of course).

(iii) Find a Hahn decomposition for μ. Write μ^+ and μ^- with their absolutely continuous and singular part.

(iv) Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le |x|\}$. Compute $|\mu| \otimes m(E)$.

(you can answer (ii) and (iii) without explicitly computing the total variation function of F)

Solution. (i) For $x > 1$ the function F is $t \mapsto 1/(1 + t^2)$ translated forward by 1, and for $x < -1$ it is the symmetric of that. Easy the plot:

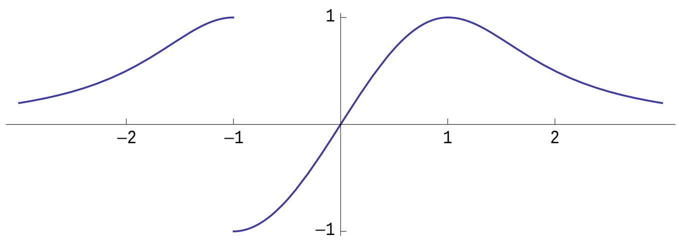

FIGURE 12. Plot of F.

(ii) Plainly F is everywhere continuous but at $x = -1$, where $F(-1^-) = 1$ and $F(-1^+) = -1$. Moreover $F \in C^1(\mathbb{R} \setminus \{-1\})$, with derivative:

$$F'(x) = \frac{2(|x|-1)}{(1+(|x|-1)^2)^2}\,\chi_{]-\infty,-1[}(x) + \frac{\pi}{2}\cos(\pi x/2)\,\chi_{]-1,1[}(x) + \frac{2(1-|x|)}{(1+(|x|-1)^2)^2}\,\chi_{[1,\infty[}(x) \quad (x \neq -1).$$

Thus -1 is the only point of $\mathbb{R}$ with nonzero $\mu = \mu_F-$measure, and $\mu(\{-1\}) = F(1^+) - F(-1^-) = (-1) - 1 = -2$; the absolutely continuous part is $F'(x)\,dx$, the singular part is $-2\,\delta_{-1}$.

(iii) A Hahn decomposition for μ is $P =]-\infty, -1[\cup]-1, 1[$, positive set, with complement $\{-1\} \cup]1, \infty[$ negative set. The positive part is absolutely continuous and is $\mu^+ = F'(x)\,\chi_{]-\infty,1]}\,dx$; the negative part is $\mu^- = 2\delta_{-1} - F'(x)\,\chi_{]1,\infty[}\,dx$; this is also the decomposition of μ^- into singular and absolutely continuous part.

(iv) We observe that the set E is exactly the trapezoid of the function $x \mapsto |x|$; the measures being $\sigma-$finite we then have:

$$|\mu| \otimes m(E) = \int_{\mathbb{R}} |x|\,d|\mu|(x) = \int_{\mathbb{R}} |F'(x)|\,dx + \int_{\mathbb{R}} |x|\,2d\delta_{-1}(x) = 2 + \int_{|x|\ge 1} |F'(x)|\,dx + \int_{|x|<1} |F'(x)|\,dx =$$

(notice that $|F'(x)|$ is an even function)

$$\begin{aligned}
=&2 + 2\int_1^\infty x\left(-\frac{d}{dx}\left(\frac{1}{1+(x-1)^2}\right)\right)dx + 2\int_0^1 x\,\frac{d}{dx}(\sin(\pi x/2))\,dx = \\
=&2\left(1 - \left[\frac{x}{1+(x-1)^2}\right]_1^\infty + \int_1^\infty \frac{dx}{1+(x-1)^2} + [x\,\sin(\pi x/2)]_0^1 - \int_0^1 \sin(\pi x/2)\,dx\right) = \\
=&2\left(1 + 1 + [\arctan(x-1)]_0^\infty + 1 + \frac{2}{\pi}[\cos(\pi x/2)]_0^1\right) = 6 + \pi - \frac{4}{\pi}.
\end{aligned}$$

□

EXERCISE 22. Consider the function

$$f(x) = \int_{\mathbb{R}} \frac{e^{-xt^2}}{1+|t|}\,dt.$$

(i) Find the domain of existence D of f (=the set of all $x \in \mathbb{R}$ such that the integral defining $f(x)$ is finite) and the limits of f as x tends to the boundary points of D in $\tilde{\mathbb{R}}$.

(ii) Prove that $f \in C^0(D)$. Is $f \in C^\infty(D)$? Otherwise, find the largest $k \in \mathbb{N}$ such that $f \in C^k(D)$.

Solution. (i) It is easy to see that $D =]0,\infty[$: we have

$$\frac{e^{-xt^2}}{1+|t|} \le e^{-xt^2}$$

and if $x > 0$ the integral $\int_{\mathbb{R}} e^{-xt^2}\,dt$ is finite; if $x \le 0$ we have

$$\frac{e^{-xt^2}}{1+|t|} \ge \frac{1}{1+|t|},$$

and $\int_{\mathbb{R}} dt/(1+|t|) = \infty$. We have to compute the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$. If $x_j \downarrow 0$ the sequence of functions $f_j(t) = e^{-x_j t^2}/(1+|t|)$ is increasing and converges for every $t \in \mathbb{R}$ to $1/(1+|t|)$. By monotone convergence we have

$$\lim_{j\to\infty} f(x_j) = \int_{\mathbb{R}} \frac{dt}{1+|t|} = \infty,$$

so that $\lim_{x\to 0^+} f(x) = \infty$. And if $x_j > 0$ tends to ∞, and $a = \min\{x_j\}$ we have for every $j \in \mathbb{N}$:

$$0 \le f_j(t) = \frac{e^{-x_j t^2}}{1+|t|} \le e^{-at^2}$$

since $t \mapsto e^{-at^2}$ is in $L^1(\mathbb{R})$, and $f_j(t) \to 0$ for every $t \in \mathbb{R}$, the dominated convergence theorem applies to say that $\lim_{j\to\infty} f(x_j) = 0$, so that $\lim_{x\to\infty} f(x) = 0$.

(ii) We prove that $f \in C^\infty(D)$. The $n-$th derivative with respect to x of the integrand is

$$\partial_x^n \left(\frac{e^{-xt^2}}{1+|t|}\right) = (-1)^n t^{2n}\,\frac{e^{-xt^2}}{1+|t|}.$$

Given $x_0 > 0$ pick any $a > 0$ strictly smaller that x_0, e.g. $a = x_0/2$. If $x \in U = [a,\infty[$ we have, for every $t \in \mathbb{R}$:

$$\left|\partial_x^n \left(\frac{e^{-xt^2}}{1+|t|}\right)\right| \le t^{2n}\,e^{-at^2},$$

and it is clear that since $a > 0$ the function $t \mapsto t^{2n}\,e^{-at^2}$ belongs to $L^1(\mathbb{R})$, for every $n \in \mathbb{N}$. □

Exercise 23. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a measurable function; let $A = \{|f| > 1\}$, $B = \{0 < |f| \le 1\}$.

(i) Is it true that $f = f\,\chi_A + f\,\chi_B$?

(ii) Assume that there exists a real number $q > 0$ such that $f \in L^q(\mu)$. Prove that then (recall that $\mathrm{Coz}(f) = \{|f| > 0\}$):

$$\mu(\mathrm{Coz}(f)) = \lim_{p\to 0^+} \int_X |f|^p\,d\mu.$$

(iii) In the hypotheses of (ii), assuming $\mu(\mathrm{Coz}(f)) \ne 1$ compute

$$\lim_{p\to 0^+} \|f\|_p$$

(distinguish the cases $\mu(\mathrm{Coz}(f)) > 1$ and $\mu(\mathrm{Coz}(f)) < 1$ ...).

We denote by $L^0(\mu)$ the set of all measurable functions $f : X \to \mathbb{C}$ such that $\mu(\mathrm{Coz}(f)) < \infty$.

(iv) Prove that $L^0(\mu)$ is a vector space of functions; moreover if $0 < p \le q \le \infty$ then $L^p(\mu) \subseteq L^0(\mu) + L^q(\mu)$ and $L^0(\mu) \cap L^q(\mu) \subseteq L^p(\mu)$ (hint: $f = f\,\chi_{\mathrm{Coz}(f)}$ ...).

(v) In $X = \mathbb{R}$ with Lebesgue measure find a function f such that $f \in L^p(\mathbb{R})$ for every $p > 0$, but $\mu(\mathrm{Coz}(f)) = \infty$.

Solution. (i) If $|f(x)| > 1$ then $\chi_A(x) = 1$ and $\chi_B(x) = 0$ so that (i) holds; if $0 < |f(x)| \le 1$ then $\chi_A(x) = 0$ and $\chi_B(x) = 1$ so that again (i) holds. The only remaining case is $|f(x)| = 0$; then $f(x) = 0 = \chi_A(x) = \chi_B(x)$ and still (i) holds. The answer is affirmative.

(ii) By (i) we have, for every $p > 0$:

$$|f|^p = |f|^p\,\chi_A + |f|^p\,\chi_B;$$

if $p(j) \downarrow 0$ the sequence $|f|^{p(j)}\,\chi_A$ is decreasing and the sequence $|f|^{p(j)}\,\chi_B$ is increasing. The first sequence converges pointwise to χ_A, being dominated if $p(j) < q$ by the summable function $|f|^q\,\chi_A$; by dominated convergence we have

$$\lim_{j\to\infty}\int_X |f|^{p(j)}\,\chi_A = \int_X \chi_A = \mu(A).$$

The second sequence converges to χ_B and as observed is increasing; by monotone convergence we get

$$\lim_{j\to\infty}\int_X |f|^{p(j)}\,\chi_B = \int_X \chi_B = \mu(B).$$

Then

$$\lim_{p\to 0^+}\int_X |f|^p = \lim_{p\to 0^+}\left(\int_X |f|^p\,\chi_A + \int_X |f|^p\,\chi_B\right) = \mu(A)+\mu(B) = \mu(\mathrm{Coz}(f)),$$

since $\mathrm{Coz}(f)$ is the disjoint union of A and B.

Remark. Many people seem to forget that $a^{1/n}$ *decreases* to 1 if $a > 1$, and *increases* to 1 if $0 < a < 1$! the sets A and B were called into existence to remind of this fact ...

(iii) We have seen that if $f \in L^q(\mu)$ for some finite $q > 0$ then $\lim_{p\to 0^+}\|f\|_p^p = \mu(\mathrm{Coz}(f))$; setting $a(p) = \|f\|_p^p$ we have

$$\|f\|_p = a(p)^{1/p} = \exp(\log a(p)/p);$$

if $\mu(\mathrm{Coz}(f)) < 1$ then $\lim_{p\to 0^+}\log a(p)/p = -\infty$, so that $\lim_{p\to 0^+}\|f\|_p = 0$; if $\mu(\mathrm{Coz}(f)) > 1$ then $\lim_{p\to 0^+}\log a(p)/p = +\infty$, so that $\lim_{p\to 0^+}\|f\|_p = \infty$.

Remark. If $\mu(\mathrm{Coz}(f)) = 1$ then we get

$$\lim_{p\to 0^+}\|f\|_p = \exp\left(\int_{\mathrm{Coz}(f)}\log|f(x)|\,dx\right),$$

the geometric mean of f, as seen on the Weekly, ninth week.

(iv) Since $\mathrm{Coz}(f+g) \subseteq \mathrm{Coz}(f)\cup\mathrm{Coz}(g)$ we have that $\mu(\mathrm{Coz}(f+g)) \le \mu(\mathrm{Coz}(f)) + \mu(\mathrm{Coz}(g))$; and $\mathrm{Coz}(\alpha f) = \mathrm{Coz}(f)$ for $\alpha \in \mathbb{K}\smallsetminus\{0\}$, so that $L^0(\mu)$ is a linear space of functions. From (i) we have $f = f\,\chi_A + f\,\chi_B$. If $f \in L^p(\mu)$ then $\mu(A) < \infty$ (e.g., by Čebičeff's inequality: $\mu(A)^{1/p} \le \|f\|_p$) so that $f\,\chi_A \in L^0(\mu)$; since $|f|\,\chi_B \le 1$ and $0 < p < q$ we have $|f|^q\,\chi_B \le |f|^p\,\chi_B \le |f|^p$ if q is finite, so that $f\,\chi_B \in L^q(\mu)$; and $f\,\chi_B \in L^\infty(\mu)$ in any case. We have proved that $L^p(\mu) \subseteq L^0(\mu) + L^q(\mu)$. It remains to prove that if $f \in L^0(\mu)\cap L^q(\mu)$ then $f \in L^p(\mu)$. Assume first that $q < \infty$; using the fact that $f = f\,\chi_{\mathrm{Coz}(f)}$ we apply Hölder's inequality to $|f|^p$ and $\chi_{\mathrm{Coz}(f)}$, with conjugate exponents q/p and $1 - p/q = (q-p)/p$, obtaining

$$\|f\|_p \le \|f\|_q\,\mu(\mathrm{Coz}(f))^{1/p-1/q},$$

(as in the variation of L^p in spaces of finite measure) and then that $f \in L^p(\mu)$, so that $L^0(\mu)\cap L^q(\mu) \subseteq L^p(\mu)$, if $0 < p < q < \infty$. If $q = \infty$ we have only to integrate the inequality $|f|^p \le \|f\|_\infty^p\,\chi_{\mathrm{Coz}(f)}$ to get

$$\int_X |f|^p \le \|f\|_\infty^p\,\mu(\mathrm{Coz}(f) \Longrightarrow \|f\|_p \le \|f\|_\infty\,(\mu(\mathrm{Coz}(f))^{1/p},$$

so that $f \in L^p(\mu)$.

(v) Easy: $f(x) = e^{-|x|}$, or $f(x) = e^{-x^2}$. □

Exercise 24. Let X be a set, and let $\phi : \mathcal{P}(X) \to [0,\infty]$ be an outer measure; let $\mathcal{M} = \mathcal{M}(\phi)$ be the $\sigma-$algebra of $\phi-$measurable sets.

(i) What is the definition of a $\phi-$measurable set?
(ii) Let $A \subseteq X$ be $\phi-$measurable; prove that if $E \subseteq A$ and $F \subseteq X\smallsetminus A$ then $\phi(E\cup F) = \phi(E)+\phi(F)$
(iii) Let $(A(n))_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{M}$, and assume that $E(n) \subseteq A(n)$ for every $n \in \mathbb{N}$. Prove that

$$\phi\left(\bigcup_{n=0}^{\infty}E(n)\right) = \sum_{n=0}^{\infty}\phi(E(n)).$$

Given $E \subseteq X$ we say that $\bar E \in \mathcal{M}$ is a *measurable cover* of E if $\bar E \supseteq E$, and for every measurable subset B of $\bar E \smallsetminus E$ we have $\phi(B) = 0$.

(iv) Assume that for a given $E \subseteq X$ there is an $E_0 \in \mathcal{M}$ with $E_0 \supseteq E$ and $\phi(E_0) < \infty$. Consider $\mathcal{M}(E) = \{A \in \mathcal{M} : A \supseteq E\}$; prove that there exists $\bar{E} \in \mathcal{M}(E)$ such that $\phi(\bar{E}) = \min\{\phi(A) : A \in \mathcal{M}(E)\}$, and that this set $\bar{E}$ is a measurable cover of E.

Solution. (i) A set $A \in \mathcal{P}(X)$ is said to be $\phi-$measurable when it splits additively every set $E \in \mathcal{P}(X)$, that is

$$\phi(E) = \phi(E \cap A) + \phi(E \setminus A), \quad \text{for every } E \subseteq X.$$

(ii) Testing measurability of A against $E \cup F$ we get:

$$\phi(E \cup F) = \phi((E \cup F) \cap A) + \phi((E \cup F) \setminus A) = \phi(E) + \phi(F),$$

since $(E \cup F) \cap A = (E \cap A) \cup (F \cap A) = E \cup \emptyset = E$, and $(E \cup F) \setminus A = (E \cup F) \cap (X \setminus A) = (E \cap (X \setminus A)) \cup (F \cap (X \setminus A)) = \emptyset \cup F = F$.

(iii) By Carathèodory's theorem $\mathcal{M}$ is a tribe, so that $\bigcup_{n=0}^{m} A(n)$ is measurable for every m; by (ii) (with $E = \bigcup_{n=0}^{m} E(n)$ and $F = \bigcup_{n=m+1}^{\infty} E(n)$) we get

$$(*) \qquad \phi\Big(\bigcup_{n=0}^{\infty} E(n)\Big) = \phi\Big(\bigcup_{n=0}^{m} E(n)\Big) + \phi\Big(\bigcup_{n=m+1}^{\infty} E(n)\Big) \geq \phi\Big(\bigcup_{n=0}^{m} E(n)\Big);$$

an easy induction on m, using (ii), shows that

$$\phi\Big(\bigcup_{n=0}^{m} E(n)\Big) = \sum_{n=0}^{m} \phi(E(n)),$$

so that (*) gives:

$$(**) \qquad \phi\Big(\bigcup_{n=0}^{\infty} E(n)\Big) \geq \sum_{n=0}^{m} \phi(E(n)),$$

and passing to the limit as $m \to \infty$ we obtain:

$$\phi\Big(\bigcup_{n=0}^{\infty} E(n)\Big) \geq \sum_{n=0}^{\infty} \phi(E(n));$$

countable subadditivity of ϕ gives the reverse inequality.

(iv) Let's prove the existence of the minimum. Take a sequence $A(n) \in \mathcal{M}$, with $A(n) \in \mathcal{M}(E)$, $\mu(A(n)) < \infty$, and $\lim_{n\to\infty} \mu(A(n)) = \inf\{\mu(A) : A \in \mathcal{M}(E)\}(:= \alpha)$. Since $\mathcal{M}$ is closed under countable intersection we have that $\bar{E} = \bigcap_{n\in\mathbb{N}} A(n)$ belongs to $\mathcal{M}$; since also $E \subseteq \bar{E}$, we have $\bar{E} \in \mathcal{M}(E)$, so that in particular $\phi(\bar{E}) \geq \alpha$; but by monotonicity of ϕ we also have $\phi(\bar{E}) \leq \phi(A(n))$ for every n, so that $\phi(\bar{E}) = \alpha$. We have proved that α is a minimum. If $B \subseteq \bar{E} \setminus E$ is measurable, then also $\bar{E} \setminus B$ is measurable and contains E, so that $\bar{E} \setminus B \in \mathcal{M}(E)$; since we have $\phi(B) < \infty$, we also have $\phi(\bar{E} \setminus B) = \phi(\bar{E}) - \phi(B) < \phi(\bar{E})$ if $\phi(B) > 0$; minimaliti of $\phi(\bar{E})$ then foces $\phi)B) = 0$

$\bar{E} = F$ is a measurable cover of E.

Remark. In (iii) we are essentially repeating the proof needed to prove part of Carathèodory's theorem (that $\mathcal{M}(\phi)$ is closed under countable disjoint union and that ϕ is countably additive on it, see the Lecture Notes). It is in general false that the set $\bigcap_{A\in\mathcal{M}(E)} A$ is measurable; this intersection is not an intersection of countably many sets; e.g., if ϕ is the Lebesgue outer measure on $\mathbb{R}$ then this intersection is E for every subset E of $\mathbb{R}$, measurable or not.

□

Analisi Reale per Matematica – II Recupero – 2 settembre 2014

Exercise 25. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 1 - e^{-(|x|-1)} & \text{if } \ |x| > 1 \\ 1 + \cos(\pi x) & \text{if } \ -1 \leq x \leq 1 \end{cases}.$$

Let μ denote the signed measure dF associated to F.

(i) Plot F.

(ii) Compute the derivative $F'(x)$ where it exists, and write the decomposition of μ into absolutely continuous and singular part (with respect to Lebesgue measure $m = dx$, of course).

(iii) Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq F(x)\}$. Compute $\mu^+ \otimes \mu^-(E)$.

(iv) Find all $p > 0$ such that the function $f(x) = x$ belongs to $L^p(|\mu|)$.

Solution. (i)

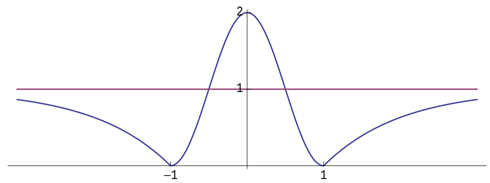

FIGURE 13. Plot of f.

(ii) We have that $\mu \ll dx$ (F is continuous and piecewise C^1) so that

$$d\mu = F'(x)\,dx = \operatorname{sgn} x\, e^{-(|x|-1)}\,\chi_{\mathbb{R}\smallsetminus[-1,1]}(x)\,dx - \pi\,\sin(\pi x)\,\chi_{[-1,1]}(x)\,dx.$$

(iii) A Hahn decomposition for μ is $P = [-1, 0] \cup [1, \infty[$ and $Q = \mathbb{R} \smallsetminus P$. All measures considered are finite so that (notice that for every $x \in \mathbb{R}$ the $x-$section of E is $E(x) = [0, F(x)]$):

$$\mu^+ \otimes \mu^-(E) = \int_{\mathbb{R}} \mu^-(E_x)\,d\mu^+(x) = \int_{\mathbb{R}} \mu^-([0, F(x)])\,(F'(x))^+\,dx =$$

(of course $(F'(x))^+ = 0$ for $x \in \mathbb{R} \smallsetminus P$, so that the integral may be restricted to $P = [-1, 0] \cup [1, \infty[$)

$$= \int_{-1}^{0} \mu^-([0, F(x)])F'(x)\,dx + \int_{1}^{\infty} \mu^-([0, F(x)])F'(x)\,dx.$$

If $0 \le F(x) \le 1$ we have $\mu^-([0, F(x)]) = F(0) - F(F(x)) = 2 - (1 + \cos(\pi F(x)) = 1 - \cos(\pi F(x))$, whereas if $1 \le F(x) \le 2$ we get $\mu^-(F(x)) = 2$, so that, noticing that $F(x) \le 1$ for $x \ge 1$:

$$\int_{1}^{\infty} \mu^-([0, F(x)])F'(x)\,dx = \int_{1}^{\infty}(1 - \cos(\pi F(x))\,F'(x)\,dx = \left[F(x) - \frac{\sin(\pi F(x))}{\pi}\right]_1^{\infty} = 1.$$

For $x \in [-1, 0]$ we get $F(x) \le 1$ for $x \in [-1, -1/2]$, $F(x) \ge 1$ for $x \in [-1/2, 0]$, so that

$$\begin{aligned}\int_{-1}^{0} \mu^-([0, F(x)])F'(x)\,dx &= \int_{-1}^{-1/2}(1 - \cos(\pi F(x)))\,F'(x)\,dx + \int_{-1/2}^{0} 2\,F'(x)\,dx =\\ &= \left[F(x) - \frac{\sin(\pi F(x))}{\pi}\right]_{-1}^{-1/2} + 2(F(0) - F(-1/2)) =\\ &= F(-1/2) - F(-1) + 2 = 3.\end{aligned}$$

We have obtained $\mu^+ \otimes \mu^-(E) = 4$.

(iv) Clearly $d|\mu|(x) = |F'(x)|\,dx$ so that, by evenness of the functions involved:

$$\int_{\mathbb{R}} |x|^p\,d|\mu| = 2\int_0^{\infty} |x|^p\,|F'(x)|\,dx = 2\int_0^1 x^p\,|F'(x)|\,dx + 2e\int_1^{\infty} x^p\,e^{-x}\,dx,$$

an integral plainly finite for every $p > 0$. □

EXERCISE 26. Consider the function

$$f(x) = \int_0^{\infty} \frac{e^{-xt}}{(1 + t)^2}\,dt.$$

(i) Find the domain of existence D of f (=the set of all $x \in \mathbb{R}$ such that the integral defining $f(x)$ is finite) and prove that $f \in C^0(D)$. Observe that $0 \in D$, and compute $f(0)$.

(ii) Let $E = \operatorname{int}(D)$ be the interior of D. Is $f \in C^\infty(E)$? Otherwise, find the largest $k \in \mathbb{N}$ such that $f \in C^k(E)$.

(iii) Compute (justifying the answer!)

$$\lim_{x\to 0^+} \int_0^{\infty} \frac{t\,e^{-xt}}{(1 + t)^2}\,dt.$$

Is it true that $f \notin C^1(D)$?

Solution. (i) If $x \geq 0$ we have (setting $g(x,t) = e^{-xt}/(1+t)^2$):

$$|g(x,t)| = \frac{e^{-xt}}{(1+t)^2} \leq \frac{1}{(1+t)^2},$$

and the last function is summable on $[0,\infty[$, so that $[0,\infty[\subseteq D$; if $x<0$, then clearly $\lim_{t\to\infty} g(x,t) = \infty$, and the integral cannot be finite. Then $D = [0,\infty[$, and since the integrand is continuous in the varaible x, and dominated for every $x \in D$ by the function $t \mapsto 1/(1+t^2)$, which is in $L^1([0,\infty[)$ as observed, f is continuous on D. Finally

$$f(0) = \int_0^\infty \frac{dt}{(1+t)^2} = \left[\frac{-1}{1+t}\right]_0^\infty = 1.$$

(ii) $E =]0,\infty[$. For $x \in E$, we pick a with $0<a<x$; the set $U = [a,\infty[$ is a nbhd of x, and for $y \in U$, $t \in [0,\infty[$ and $k \in \mathbb{N}$ we have

$$\left|\partial_1^k g(y,t)\right| = \left|(-t)^k \frac{e^{-yt}}{(1+t)^2}\right| = t^k\,\frac{e^{-yt}}{(1+t)^2} \leq t^k\,\frac{e^{-at}}{(1+t)^2},$$

and the last function is clearly in $L^1([0,\infty[)$.

(iii) If $x_j \to 0$ we have $t\,e^{-x_j t}/(1+t)^2 \uparrow t/(1+t)^2$ for every $t \geq 0$; by monotone convergence we get

$$\lim_{j\to\infty} \int_0^\infty \frac{t\,e^{-xt}}{(1+t)^2}\,dt = \int_0^\infty \frac{t}{(1+t)^2}\,dt,$$

and this last integral is clearly infinite. This shows that

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \int_0^\infty \frac{-t\,e^{-xt}}{(1+t)^2}\,dt = -\infty,$$

which implies that $f'(0^+) = -\infty$, so that f is not in $C^1([0,\infty[)$

□

Exercise 27. Let $(X,\mathcal{M},\mu)$ be a measure space, and let $\rho : X \to [0,\infty]$ be a positive measurable function. We define $\nu = \nu_\rho : \mathcal{M} \to [0,\infty]$ by

$$\nu(E) = \int_E \rho\,d\mu \qquad \text{for every } E \in \mathcal{M}.$$

(i) It is well–known that ν is a measure; state the theorem that implies this fact.
(ii) Prove that if $\varphi : X \to [0,\infty[$ is a positive $\mathcal{M}-$measurable simple function then

$$\int_X \varphi\,d\nu = \int_X \varphi\,\rho\,d\mu.$$

(iii) Prove that if $f : X \to [0,\infty]$ is positive and $\mathcal{M}-$measurable, then

$$\int_X f\,d\nu = \int_X f\,\rho\,d\mu.$$

Deduce from this fact that an $\mathcal{M}-$measurable f belongs to $L^1(\nu)$ iff $f\,\rho \in L^1(\mu)$.
(iv) Let $I = \{\rho = \infty\}$. Prove that if $\mu(I) > 0$ then I is a $\nu-$atom of infinite measure (that is, $\nu(I) = \infty$ and if $E \in \mathcal{M}$ and $E \subseteq I$, then $\nu(E) = 0$ or $\nu(E) = \infty$).
(v) Given real numbers α, β with $0 < \alpha < \beta$, let $A = A_{\alpha\,\beta} = \{\alpha < \rho \leq \beta\}$. Prove that a measurable subset $E \in \mathcal{M}$ of A has finite $\nu-$measure if and only if it has finite $\mu-$measure.
(vi) Prove that X has $\sigma-$finite $\nu-$measure if and only if $I = \{\rho = \infty\}$ has $\mu-$measure 0, and $\mathrm{Coz}(\rho) = \{\rho > 0\}$ has $\sigma-$finite $\mu-$measure.

Solution. (i) Theorem on termwise integration of series with positive terms (immediate consequence of additivity of integrals and monotone convergence).

(ii) and (iii) are found in the Lecture Notes, exercise 3.2.5.2, with solution in 3.2.7, page 44.

(iv) If $\rho(x) = \infty$ for every $x \in E$, and $\mu(E) > 0$ then of course $\int_E \rho\,d\mu = \infty$, whereas if $\mu(E) = 0$ then $\int_E \rho\,d\mu = 0$. This clearly proves the assert.

(v) If $E \subseteq A$ we have $\alpha < \rho(x) \leq \beta$ for every $x \in E$, so that if $E \in \mathcal{M}$ we get

$$\alpha\,\mu(E) \leq \int_E \rho\,d\mu \leq \beta\,\mu(E), \text{ that is: } \alpha\,\mu(E) \leq \nu(E) \leq \beta\,\mu(E),$$

which clearly implies that $\mu(E) < \infty \iff \nu(E) < \infty$.

(vi) By (iv) the sets $A_n = \{1/n < \rho \le n\}$ have $\sigma-$finite $\nu-$measure if and only if they have $\sigma-$finite $\mu-$measure. Since $\mathrm{Coz}(\rho) \smallsetminus I = \bigcup_{n=1}^{\infty} A_n$, we have that this set has $\sigma-$finite $\nu-$measure iff it has $\sigma-$finite $\mu-$measure. If $\mathrm{Coz}(\rho)$ has $\sigma-$finite $\mu-$measure and $\mu(I) = 0$, then $\nu(I) = 0$, hence $\mathrm{Coz}(\rho) = (\mathrm{Coz}(\rho) \smallsetminus I) \cup I$ has also $\sigma-$finite $\nu-$measure, and since $Z(\rho) = \{\rho = 0\}$ has clearly $\nu-$measure 0, X has $\sigma-$finite $\nu-$measure.

Conversely, if X has $\sigma-$finite $\nu-$measure then every subset of X has $\sigma-$finite measure; this implies $\nu(I) = 0$, since otherwise $\nu(I) = \infty$, and I has no subset of finite $\nu-$measure, as observed in (iv); but $\nu(I) = 0$ iff $\mu(I) = 0$. Then $\mathrm{Coz}(\rho)$ has also $\sigma-$finite $\mu-$measure.

□

Exercise 28. Recall that if X is a set, $\mathcal{A}$ is an algebra of parts of X, and $\mu : \mathcal{A} \to [0, \infty]$ is a premeasure, then the outer measure associated to μ is the set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\mu^*(E) = \inf \left\{ \sum_{n=0}^{\infty} \mu(A_n) :\ A_n \in \mathcal{A},\ E \subseteq \bigcup_{n=0}^{\infty} A_n \right\}.$$

We assume now that $\mathcal{A} = \mathcal{M}$ is a $\sigma-$algebra, and that $\mu : \mathcal{M} \to [0, \infty]$ is a measure.

(i) Prove that in this case for every $E \subseteq X$ we have $\mu^*(E) = \inf\{\mu(A) :\ A \in \mathcal{M},\ A \supseteq E\}$, and there is $A = A_E \in \mathcal{M}$ with $A \supseteq E$ such that $\mu(A) = \mu^*(E)$.

For a non–empty set X, we define $\phi : \mathcal{P}(X) \to [0, 1]$ by $\phi(\emptyset) = 0$, $\phi(X) = 1$, and $\phi(A) = 1/2$ for every proper non–empty subset of X.

(ii) Prove that ϕ is an outer measure, which is not a measure unless the cardinality of X is less than $\dots (|X| \le ?)$. Determine the measurable subsets $\mathcal{M}(\phi)$ of ϕ. What is the outer measure associated to the measure $\phi_{|\mathcal{M}(\phi)}$?

Solution. (i) Set, for every $E \in \mathcal{P}(X)$:

(1) $T(E) = \{\sum_{n=0}^{\infty} \mu(A_n) :\ A_n \in \mathcal{M},\ E \subseteq \bigcup_{n=0}^{\infty} A_n\}$;
(2) $S(E) = \{\mu(A) : A \in \mathcal{M},\ E \subseteq A\}$.

By definition we have $\mu^*(E) = \inf T(E)$, and we want to prove that $\inf S(E) = \inf T(E)$. Clearly $S(E) \subseteq T(E)$ (consider the cover $(A, \emptyset, \emptyset, \dots)$, so that $\inf S(E) \ge \inf T(E)$; but given any countable cover A_n of E by elements of $\mathcal{M}$, we have $A = \bigcup_{n=0}^{\infty} A_n \in \mathcal{M}$, since $\mathcal{M}$ is a $\sigma-$algebra, and $A \supseteq E$ so that $\mu(A) \in S(E)$; and by countable subadditivity of μ we get:

$$\mu(A) \le \sum_{n=0}^{\infty} \mu(A_n) (\in T(E)).$$

We have proved that every element of $T(E)$ is larger than some element of $S(E)$, and this clearly implies that $\inf S(E) \le \inf T(E)$. Now existence of $A \supseteq E$ which actually realizes the minimum of $S(E)$ is clear: pick a sequence A_n, with $E \subseteq A_n \in \mathcal{M}$ and $\lim_{n\to\infty} \mu(A_n) = \inf S(E)$, and put $A = \bigcap_{n\in\mathbb{N}} A_n$.

(ii) Isotony of ϕ is obvious. Given a sequence E_n of subsets of X, not all empty, with union $E = \bigcup_{n=0}^{\infty} E_n$, we get $\phi(E) \ge 1/2$, and $\sum_{n=0}^{\infty} \phi(E_n) \ge 1$ if at least two sets E_n are non–empty, or $E_n = X$ for some $n \in \mathbb{N}$, in which case also $\phi(E) = 1$, so that ϕ is countably subadditive. It is immediate that ϕ is a measure if $|X| \le 2$ (for $|X| = 2$ it is the probability measure of a fair coin tossing!); and if a, b, c are three distinct elements of X, then $\{a, b, c\}$ is the disjoint union of the singletons $\{a\}, \{b\}, \{c\}$, but $\phi(\{a\}) + \phi(\{b\}) + \phi(\{c\}) = 3/2 > 1 \ge \phi(\{a, b, c\})$, so that ϕ is not additive if $|X| \ge 3$. We have proved that ϕ is a measure iff $|X| \le 2$. Assuming $|X| \ge 3$, we have that the only measurable subsets are $\emptyset$ and X; these sets are certainly measurable, for every outer measure. And if $\emptyset \subsetneqq A \subsetneqq X$, pick a two elements set $E = \{a, b\}$ with $a \in A$ and $b \in X \smallsetminus A$; we get $\phi(E \cap A) = \phi(\{a\}) = 1/2$ and $\phi(E \smallsetminus A) = \phi(\{b\}) = 1/2$, and $\phi(E) = 1/2$ since X, by hypothesis, has at least three elements. Then $1/2 = \phi(E) < \phi(A \cap E) + \phi(E \smallsetminus A) = 1/2 + 1/2 = 1$, so that A does not split E additively. Thus $\mathcal{M}(\phi) = \{\emptyset, X\}$; the outer measure associated is then $\phi^* : \mathcal{P}(X) \to [0, \infty]$ defined by $\phi^*(E) = 1$ for $E \in \mathcal{P}(X)$ non–empty, and $\phi^*(\emptyset) = 0$.

□

Analisi Reale per Matematica – III recupero – 23 settembre 2014

Exercise 29. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} \sin(\pi x) & \text{if } \quad -1 \le x < 1 \\ \arctan x & \text{if } \quad x \ge 1 \\ 0 & \text{if } \quad x < -1 \end{cases} .$$

Let μ denote the signed measure dF associated to F.

(i) Plot F, and write F as the difference of two increasing functions, $F = A - B$; plot A and B.

(ii) Compute the derivative $F'(x)$ where it exists, and write the decomposition of μ into absolutely continuous and singular part (with respect to Lebesgue measure $m = dx$, of course); find a Hahn decomposition for μ.

(iii) Let $E = \{(x, y) \in \mathbb{R}^2 : 0 < y \le F(x)\}$. Compute $\mu^- \otimes \mu^+(E)$.

Solution. (i) The plot is easy:

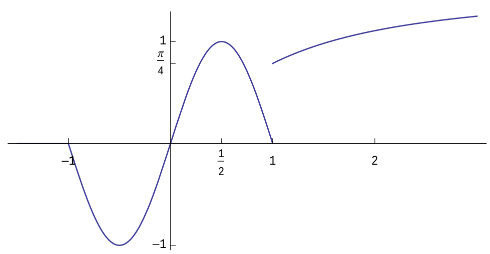

Figure 14. Plot of F.

Next, computing the positive variation only we have

$A(x) = 0$ for $x < -1/2$; $A(x) = 1 + \sin(\pi x)$ for $-1/2 \le x < 1/2$; $A(x) = 2$ for $1/2 \le x < 1$;
$A(x) = 2 + \pi/4 + \arctan x - \pi/4 (= 2 + \arctan x)$ for $x \ge 1$.

Computing the negative variation (with opposite sign) we get:

$B(x) = 0$ for $x < -1$; $B(x) = -\sin(\pi x)$ for $-1 \le x < -1/2$; $B(x) = 1$ for $-1/2 \le x < 1/2$;
$B(x) = 2 - \sin(\pi x)$ for $1/2 \le x < 1$; $B(x) = 2$ for $x \ge 1$.

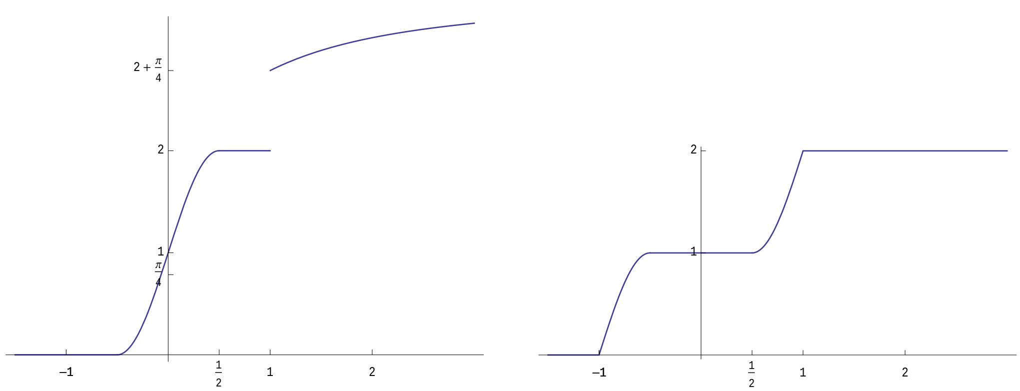

Figure 15. Plot of A (right) and B (left).

(ii) Of course the derivative $F'(x)$ exists but for the points ± 1 and is

$$F'(x) = \pi \cos(\pi x)\, \chi_{]-1,1[} + \frac{1}{1 + x^2}\, \chi_{]1,\infty[} \qquad \text{on} \quad \mathbb{R} \setminus \{\pm 1\}$$

We then have $dF(x) = F'(x)\, dx + (\pi/4)\, \delta_1$, the first $F'(x)\, dx$ is the absolutely continuous part, the second $(\pi/4)\, \delta_1$ the singular part. At the point 1 there is a jump of $\pi/4$, that is $\mu(\{1\}) = \mu^+(\{1\}) = \pi/4$. A Hahn decomposition is $P =]-1/2, 1/2[\cup \{1\}$, $Q = \mathbb{R} \setminus P$.

(iii) All measures are finite, so that Fubini's theorem is certainly applicable; if $E(x) = \{y \in \mathbb{R} : (x, y) \in E\}$ is the $x-$section of E we get $E(x) =]0, F(x)]$, non–empty iff $F(x) > 0$, hence iff $x \in]0, \infty[$

$$\mu^- \otimes \mu^+(E) = \int_{\mathbb{R}} \mu^+(E(x))\, d\mu^-(x) = \int_{]0,\infty[} \mu^+(E(x))\, d\mu^-(x) = \int_{[1/2,1[} \mu^+(E(x))\, d\mu^-(x) =$$

(the first equality because $\mu^+(E(x)) \neq 0$ only if $x \in]0, \infty[$, the second because $\mu^-(]0, 1/2[) = \mu^-(\{1\}) = 0$)

$$= \int_{[1/2,1[} (A(F(x)) - A(0))\, (F'(x))^-\, dx = \int_{1/2}^{1} \sin(\pi x)(-\pi \cos(\pi x))\, dx = \left[\frac{\cos^2(\pi x)}{2}\right]_{1/2}^{1} = \frac{1}{2}.$$

Then $\mu^- \otimes \mu^+(E) = 1/2$. □

Exercise 30. Consider the function

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{-x\,t^2}}{1 + t + t^2}\, dt.$$

(i) Find the domain of existence D of f (=the set of all $x \in \mathbb{R}$ such that the integral defining $f(x)$ is finite) and prove that $f \in C^0(D)$. Compute the limits of f at the boundary points of D in the extended real line.
(ii) Let $E = \operatorname{int}(D)$ be the interior of D. Is $f \in C^\infty(E)$? Otherwise, find the largest $k \in \mathbb{N}$ such that $f \in C^k(E)$.
(iii) Compute (justifying the answer!)

$$f'(0^+) := \lim_{x\to 0^+} \frac{f(x) - f(0)}{x}.$$

Solution. (i) Set $g(x, t) = e^{-x\,t^2}/(1 + t + t^2)$. Clearly $D = [0, \infty[$: if $x \geq 0$ then $|g(x, t)| = g(x, t) \leq 1/(1 + t + t^2)$, and $t \mapsto 1/(1 + t + t^2) \in L^1(\mathbb{R})$, since the denominator is never 0 and the function is $O(1/t^2)$ as $t \to \pm\infty$; this inequality proves also continuity of f on D, since $x \mapsto f(x, t)$ is continuous; and if $x < 0$ then $\lim_{t\to\pm\infty} f(x, t) = \infty$. The boundary points are 0 and ∞. At 0 the limit is $f(0)$ and

$$f(0) = \int_{-\infty}^{\infty} \frac{dt}{1 + t + t^2} = \int_{-\infty}^{\infty} \frac{dt}{1 - 1/4 + 1/4 + t + t^2} = \int_{-\infty}^{\infty} \frac{dt}{3/4 + (t + 1/2)^2} =$$

$$= \frac{2}{\sqrt{3}} \int_{-\infty}^{\infty} \frac{2/\sqrt{3}\, dt}{1 + (2/\sqrt{3}(t + 1/2))^2} = \frac{2}{\sqrt{3}} \left[\arctan((2/\sqrt{3}(t + 1/2))\right]_{-\infty}^{\infty} = \frac{2\pi}{\sqrt{3}}.$$

And if x_j is any sequence of positive numbers with limit ∞ we have $\lim_{j\to\infty} g(x_j, t) = 0$, and $|g(x_j, t)| \leq 1/(1 + t + t^2)$ for every $t \in \mathbb{R}$, so that $\lim_{j\to\infty} f(x_j) = 0$ by dominated convergence.

(ii) We have $\partial_x^k g(x, t) = (-t^2)^k g(x, t)$, for $x, t \in \mathbb{R}$. Given $x > 0$ pick any a with $0 < a < x$ (e.g. $a = x/2$). We get, if $x \geq a$:

$$|\partial_x^k g(x, t)| = \frac{t^{2k}}{1 + t + t^2}\, e^{-xt^2} \leq \frac{t^{2k}}{1 + t + t^2}\, e^{-at^2},$$

and clearly the last function is in $L^1(\mathbb{R})$. Then $f \in C^\infty(]0, \infty[)$.

(iii) Since f is continuous at 0, the difference quotient $(f(x) - f(0))/x$ is quotient of functions with limit 0. The de l'Hôpital's rule may be applied; if $\lim_{x\to 0^+} f'(x)$ exists, then this limit is $f'(0^+)$. And if $x_j \downarrow 0$ we have $t^2\, g(x_j, t) \uparrow t^2/(1 + t + t^2)$; by monotone convergence we get

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} t^2\, g(x_j, t)\, dt = \int_{-\infty}^{\infty} \frac{t^2}{1 + t + t^2} = \infty,$$

that is $\lim_{j\to\infty}(-f'(x_j)) = \infty$. We have proved that $f'(0^+) = -\infty$. □

Exercise 31. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Given a finite signed measure $\nu : \mathcal{M} \to \mathbb{R}$, prove that the following are equivalent:
(a) There is $\rho \in L^1_\mu(X, \mathbb{R})$ such that

$$\nu(E) = \int_E \rho\, d\mu, \qquad \text{for every } E \in \mathcal{M}.$$

(b) ν is supported by a set $M \in \mathcal{M}$ of $\sigma-$finite $\mu-$measure (recall that this means that $X \setminus M$ is a $\nu-$null set), and $\nu \ll \mu$.

From now on assume that $\mu(X) < \infty$.

(ii) Prove that for a finite valued finitely additive set function $\nu : \mathcal{M} \to \mathbb{R}$ the following are equivalent:
   (1) ν is a signed measure (i.e. ν is countably additive) and $\nu \ll \mu$.
   (2) ν is $\varepsilon - \delta$ absolutely continuous with respect to μ (i.e. given $\varepsilon > 0$ there is $\delta > 0$ such that $E \in \mathcal{M}$ and $\mu(E) \le \delta$ imply $|\nu(E)| \le \varepsilon$).

(iii) With $X = [0, 1]$, $\mathcal{M} = \mathcal{B}([0, 1])$ and μ =Lebesgue measure, give an example of a measure $\nu : \mathcal{M} \to [0, \infty]$ such that $\nu \ll \mu$, but ν is not $\varepsilon - \delta$ absolutely continuous with respect to μ.

In what follows $\nu : \mathcal{M} \to [0, \infty]$ is a positive not necessarily finite measure.

(iv) Let $\mathcal{I}(\nu)$ be the $\sigma-$ideal of $\mathcal{M}$ consisting of all $E \in \mathcal{M}$ of $\sigma-$finite $\nu-$measure. Prove that there is $A \in \mathcal{I}(\nu)$ such that $\mu(A) \ge \mu(E)$ for every $E \in \mathcal{I}(\nu)$.

(v) Assume now also $\nu \ll \mu$. Given A like in (iv), prove that if ν is $\sigma-$finite then $\nu(X \setminus A) = 0$, and if ν is not $\sigma-$finite then $X \setminus A$ is for ν an atom of infinite measure. Conclude that there is $\rho \in \mathcal{L}^+_{\mathcal{M}}$ such that

$$\nu(E) = \int_X \rho \, d\mu, \qquad \text{for every } E \in \mathcal{M}.$$

Solution. (i) (a) implies (b): since $\rho \in L^1(\mu)$ the set $M = \mathrm{Coz}(\rho)$ has $\sigma-$finite $\mu-$measure. Clearly $\nu(E) = \int_E \rho \, d\mu = 0$ if $E \cap M = \emptyset$, since ρ is identically zero on E. (b) implies (a): we consider the restrictions μ_M and ν_M of μ and ν to the $\sigma-$algebra $\mathcal{N} = \{E \cap M : E \in \mathcal{N}\}$ traced by $\mathcal{M}$ on M. Since ν_M is a finite signed measure, μ_M is $\sigma-$finite and $\nu_M \ll \mu_M$, the Radon–Nikodym theorem applies to prove that there is a function $g \in L^1_{\mu_M}(M, \mathbb{R})$ such that

$$\nu_M(E) = \int_E g \, d\mu_M$$

for every $E \in \mathcal{N}$. If we define $\rho : X \to \mathbb{R}$ by $\rho(x) = g(x)$ for $x \in M$ and $\rho(x) = 0$ for $x \in X \setminus M$, we clearly have (a).

(ii) (1) implies (2) We know that for a finite signed measure absolute continuity implies $\varepsilon - \delta-$absolute continuity: the proof is in the Lecture Notes, Exercise 6.2.5.3(ii), and is not repeated here. Alternatively we can use (i): it is well–known that the indefinite integral of an L^1 function is $\varepsilon - \delta$ absolutely continuous.

(2) implies (1). Assume that E_n is disjoint sequence in $\mathcal{M}$, with union E. We want to prove that $\nu(E) = \sum_{n=0}^{\infty} \nu(E_n)$. Set $F_m = \bigcup_{n=m+1}^{\infty} E_n$; we have, for every $m \in \mathbb{N}$:

$$\nu(E) = \nu\left(\bigcup_{n=0}^{m} E_n \cup F_m\right) = \nu\left(\bigcup_{n=0}^{m} E_n\right) + \nu(F_m) = \sum_{n=0}^{m} \nu(E_n) + \nu(F_m)$$

by finite additivity. If we know that $\lim_{m\to\infty} \nu(F_m) = 0$, then taking limits as $m \to \infty$ in the previous formula we get the assert. Now the series $\sum_{n=0}^{\infty} \mu(E_n)$ converges (its sum is $\mu(E) < \infty$) so that also

$$\mu(F_m) = \sum_{n=m+1}^{\infty} \mu(E_n) \to 0 \quad \text{for} \quad m \to \infty,$$

and the $(\varepsilon, \delta)-$absolute continuity condition clearly then implies $\lim_{m\to\infty} |\nu(F_m)| = 0$.

(iii) Define $\nu : \mathcal{M} \to [0, \infty]$ by $\nu(E) = \int_E dx/x$. For every $\delta > 0$ we have $\mu(]0, \delta]) = \delta$, but $\nu(]0, \delta]) = \infty$ (see Lecture Notes, Exercise 6.2.5.3(iii))

(iv) Let $\alpha = \sup\{\mu(E) : E \in \mathcal{I}(\nu)\}$, and let $A_n \in \mathcal{I}(\nu)$ be such that $\alpha = \sup\{\mu(A_n) : n \in \mathbb{N}\}$. Then $A = \bigcup_{n=0}^{\infty} A_n \in \mathcal{I}(\nu)$ (a $\sigma-$ideal is closed under countable union) and obviously $\mu(A) \ge \mu(A_n)$ for every $n \in \mathbb{N}$, so that $\mu(A) = \alpha$.

(v) If ν is $\sigma-$finite then $X \in \mathcal{I}(\sigma)$, and clearly $\mu(X) \ge \mu(A)$, so that $\mu(X) = \mu(A)$, equivalently $\mu(X \setminus A) = 0$, so that also $\nu(X \setminus A) = 0$, by absolute continuity. If ν is not $\sigma-$finite, then $X \setminus A$ is not of $\sigma-$finite $\nu-$measure (since A is and X is not); if a measurable subset F of $X \setminus A$ has finite non zero $\nu-$measure then also $\mu(F) > 0$ (by $\nu \ll \mu$) so that $\mu(A \cup F) = \mu(A) + \mu(F) > \mu(A)$, with $A \cup F \in \mathcal{I}(\nu)$, contradicting maximality of $\mu(A)$. Then every measurable subset of $X \setminus A$ has either zero or infinite $\nu-$measure, that is, $X \setminus A$ is $\nu-$atom of infinite measure.

The Radon Nikodym theorem can be applied to the restriction to A of the measures, and gives the existence of measurable positive g such that $\nu(E) = \int_E g \, d\mu$ for every measurable $E \subseteq A$; we now extend g to a measurable $\rho : X \to [0, \infty]$; given $x \in X \setminus A$ we set $\rho(x) = 0$ if ν is $\sigma-$finite, whereas if ν is not $\sigma-$finite we set $\rho(x) = \infty$. It is easy to verify that ρ is a density for ν. $\square$

EXERCISE 32. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $p \ge 1$ be a real number.

(i) Prove the dominated convergence theorem for $L^p(\mu)$: if f_n, f are measurable, $f_n \to f$ a.e., and for every $n \in \mathbb{N}$ we have $|f_n| \le g$ a.e. with $g \in L^p(\mu)$, then f_n converges to f in the L^p-norm.

Accept for the moment the following result:

. *(*) If $f_n \in L^p$ is a sequence of bounded variation ($\sum_{n=1}^{\infty} \|f_n - f_{n-1}\|_p < \infty$) then f_n is a.e. convergent on X, and there is $g \in L^p(\mu)$ such that $|f_n| \le g$ a.e., for every $n \in \mathbb{N}$.*

(ii) Deduce from (*) the following: if for $1 \le p, q < \infty$ we have $L^q(\mu) \subseteq L^p(\mu)$, then the inclusion of $L^q(\mu)$ in $L^p(\mu)$ is a continuous map of normed spaces.

(iii) Prove (*).

Solution. (i) Since $|f(x)| \le g(x)$ for a.e. $x \in X$, we have $|f|^p \le g^p \in L^1(\mu)$, so that $f \in L^p(\mu)$. We have to prove that $\|f - f_n\|_p \to 0$ as $n \to \infty$, equivalently that $\lim_{n\to\infty} \int_X |f - f_n|^p \, d\mu = 0$ and this is immediate, by the dominated convergence in $L^1(\mu)$:

$$|f - f_n|^p \le (|f| + |f_n|)^p \le 2^{p-1}(|f|^p + |f_n|^p) \le 2^p \, g \in L^1(\mu),$$

and clearly $|f(x) - f_n(x)|^p \to 0$ for a.e. $x \in X$.

(ii) We have to prove that if $f_n \to f$ in $L^q(\mu)$ then $f_n \to f$ also in $L^p(\mu)$. If this is not true, then there is a subsequence $f_{n(k)}$ of f_n, no subsequence of which converges to f in $L^p(\mu)$: in fact there is $\alpha > 0$ such that $\|f - f_n\|_p > \alpha$ for infinitely many $n \in \mathbb{N}$. This subsequence still converges to f in $L^q(\mu)$ and has then a subsequence of bounded variation in $L^q(\mu)$; we may as well assume that the original sequence is of bounded variation in $L^q(\mu)$, but $\|f - f_n\|_p > \alpha > 0$ for every $n \in \mathbb{N}$. By (*) the sequence is dominated by a $g \in L^q(\mu)$; and since $g \in L^p(\mu)$, also, the sequence is dominated in $L^p(\mu)$; again by (*) the sequence converges pointwise a.e., necessarily to f; but then by (i) it converges to f in $L^p(\mu)$, a contradiction.

(iii) The argument that proves (*) is actually part of the theorem on normally convergent series on $L^p(\mu)$, used for proving completeness of $L^p(\mu)$ (LN, 5.2). We repeat the argument here: the series $f_0 + \sum_{n=1}^{\infty}(f_n - f_{n-1})$ is normally convergent in $L^p(\mu)$ so that, if $g_m = |f_0| + \sum_{n=0}^{m} |f_n - f_{n-1}|$ we have (remember that $\||h|\|_p = \|h\|_p$ for every measurable h):

$$\|g_m\|_p \le \||f_0|\|_p + \sum_{n=0}^{m} \||f_n - f_{n-1}|\|_p \le \|f_0\|_p + \sum_{n=0}^{\infty} \|f_n - f_{n-1}\|_p = S,$$

so that, for every $m \in \mathbb{N}$:

$$\int_X g_m^p \, d\mu \le S^p.$$

The sequence g_m^p is an increasing sequence of positive functions; if G is its pointwise limit, by dominated convergence we have

$$\int_X G \, d\mu = \lim_{m\to\infty} \int_X g_m^p \, d\mu \le S^p;$$

then G belongs to $L^1(\mu)$, in particular if $N = \{G = \infty\}$ then $\mu(N) = 0$; set $g(x) = (G(x))^{1/p}$ if $G(x) < \infty$, and $g(x) = 0$ if $x \in N$. Then $g \in L^p(\mu)$ and $g_m \uparrow g$ on $X \setminus N$ and this implies that the series

$$f_0(x) + (f_1(x) - f_0(x)) + \cdots + (f_n(x) - f_{n-1}(x)) + \ldots$$

is absolutely convergent, hence also convergent, on $X \setminus N$. Notice that the $m-$th partial sum of this series is $f_m(x)$; clearly we have $|f_m(x)| \le g_m(x) \le g(x)$ on $X \setminus N$. The proof is concluded. $\square$