PLATEAU'S PROBLEM FOR SETS WITH FINITE PERIMETER

1. Brief review on functions with bounded variation

Let $A \subset \mathbb{R}^n$ be an open set.

DEF We say that $f : A \to \mathbb{R}$ is a function with bounded variation $f \in BV(A)$, if $f \in L^1(A)$ and

$$\|Df\|(A) := \sup \left\{ \int_A f \operatorname{div} \varphi \, dx \; ; \; \varphi \in C^1_c(A; \mathbb{R}^n), \ \|\varphi\|_\infty \leq 1 \right\} < \infty .$$

The number $\|Df\|(A)$ is called the total variation of f in A.

Notation $\operatorname{div} \varphi = \sum_{i=1}^n \frac{\partial \varphi^i}{\partial x_i}$ is the divergence of $\varphi = (\varphi_1, \ldots, \varphi_n)$.

Comment If $f \in W^{1,1}(A)$ then

$$\|Df\|(A) = \int_A |\nabla f(x)| \, dx$$

and thus $W^{1,1}(A) \subset BV(A)$. However, $BV(A) \neq W^{1,1}(A)$.

The following characterization of BV functions can be proved using Riesz's representation theorem for linear functionals on continuous functions:

THEOREM Let $f \in BV(A)$ with $A \subset \mathbb{R}^n$ open set. Then there exists a finite Borel measure μ in A and a Borel function $\sigma : A \to \mathbb{R}^n$ such that:

i) $|\sigma(x)| = 1$ for μ-almost every $x \in A$;

ii) the following integration by-parts formula holds:

$$\int_A f \operatorname{div} \varphi \, dx = -\int_A \langle \varphi, \sigma \rangle \, d\mu \qquad \forall \varphi \in C^1_c(A; \mathbb{R}^n).$$

We do not prove this theorem. It tells us that BV functions are precisely L^1-functions whose distributional derivative is a measure.

More important is for us the lower semicontinuity property in $BV(A)$.

THEOREM Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of $BV(A)$ functions such that $f_k \xrightarrow[k \to \infty]{L^1(A)} f$.

Then $f \in BV(A)$ and

$$\|Df\|(A) \leq \liminf_{k \to \infty} \|Df_k\|(A).$$

Proof. By dominated convergence, for any $\varphi \in C_c^1(A;\mathbb{R}^n)$ we have

$$\lim_{k\to\infty} \int_A f_k \operatorname{div}\varphi \, dx = \int_A f \operatorname{div}\varphi \, dx .$$

It follows that, for $\|\varphi\|_\infty \le 1$,

$$\int_A f \operatorname{div}\varphi \, dx \le \liminf_{k\to\infty} \|Df_k\|(A).$$

Taking the sup for $\varphi \in C_c^1(A;\mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$ we get the claim.

$\square$

The second tool that we need is the compactness theorem for BV functions.

<u>THEOREM</u> Let $A \subset \mathbb{R}^n$ be a <u>bounded</u> open set with boundary of class C^1 (or with Lipschitz boundary). Let $f_k \in BV(A)$, $k \in \mathbb{N}$, be a sequence of BV functions such that

$$\sup_{k\in\mathbb{N}} \left\{ \|f_k\|_{L^1(A)} + \|Df_k\|(A) \right\} < \infty .$$

Then there exist $f \in BV(A)$ and a sub-sequence $(f_{k_j})_{j\in\mathbb{N}}$ such that

$$f_{k_j} \xrightarrow[j\to\infty]{L^1(A)} f .$$

A very sketchy idea of the proof is this:

Let $(\chi_\varepsilon)_{\varepsilon>0}$ be a regularizing kernel ("mollifications").

Assume $f_k \in BV(\mathbb{R}^n)$ and define

$$f_k^\varepsilon(x) = f_k * \chi_\varepsilon(x) = \int_{\mathbb{R}^n} f_k(x-y)\,\chi_\varepsilon(y)\,dy.$$

Step 1: For fixed $\varepsilon>0$, $(f_k^\varepsilon)_{k\in\mathbb{N}}$ is equibounded and equicontinuous and thus compact for the uniform convergence (Ascoli-Arzelà theorem).

Then $(f_k^\varepsilon)_{k\in\mathbb{N}}$ is totally bounded in L^1

Step 2: As $\varepsilon \to 0^+$, the "family" $(f_k^\varepsilon)_{k\in\mathbb{N}}$ converges in L^1 (uniformly in k) to the "family" $(f_k)_{k\in\mathbb{N}}$.

It follows that also $(f_k)_{k\in\mathbb{N}}$ is totally bounded. $\square$

2. Sets with finite perimeter in $\mathbb{R}^n$

Let $E\subset\mathbb{R}^n$ be an $\mathcal{L}^n$-measurable set and let $A\subset\mathbb{R}^n$ be open.

DEF We say that E is a set with finite perimeter in A if $\chi_E \in BV(A)$, where $\chi_E(x) = 1$ if $x\in E$ and is 0 otherwise. In this case, we call

$$P(E;A) = \|D\chi_E\|(A)$$

the perimeter of E in A. When $A = \mathbb{R}^n$ we let

$$P(E) = P(E;\mathbb{R}^n).$$

<u>EXAMPLE</u> Let $E = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the unit ball and consider $A = \mathbb{R}^n$.

By the divergence theorem, for any $\varphi \in C^1_c(A;\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \chi_E(x) \operatorname{div}\varphi(x)\,dx = \int_E \operatorname{div}\varphi(x)\,dx =$$

$$= \int_{\{|x|=1\}} \langle \varphi(x), \underset{\substack{\text{Exterior}\\ \text{unit normal}\\ \text{to the sphere}}}{\underset{\uparrow}{x}} \rangle \, dH^{n-1} \overset{\text{Cauchy-Schwarz}}{\leq} \int_{\{|x|=1\}} |\varphi(x)|\,dH^{n-1} \leq$$

$$\underset{\text{if } \|\varphi\|_\infty \leq 1}{\leq} H^{n-1}(\{|x|=1\}) = \text{Area of the sphere}.$$

Taking the sup in φ we deduce that

$$P(E) \leq H^{n-1}(\partial E).$$

If we choose $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $\varphi(x) = x$ where $|x| = 1$ (and we can!), then in the Cauchy-Schwarz inequality we have

$$\langle \varphi(x), x \rangle = \langle x, x \rangle = 1.$$

The previous computation gives

$$\int_{\mathbb{R}^n} \chi_E \operatorname{div} \varphi \, dx = H^{n-1}(\partial E)$$

and we conclude that

$$P(E) = H^{n-1}(\partial E).$$

The same computation works for any set $E \subset \mathbb{R}^n$ with smooth boundary.

General conclusion: the perimeter P is a weak version of "area of the boundary" of a set.

3. <u>Solution of a weak form of the Plateau's problem</u>

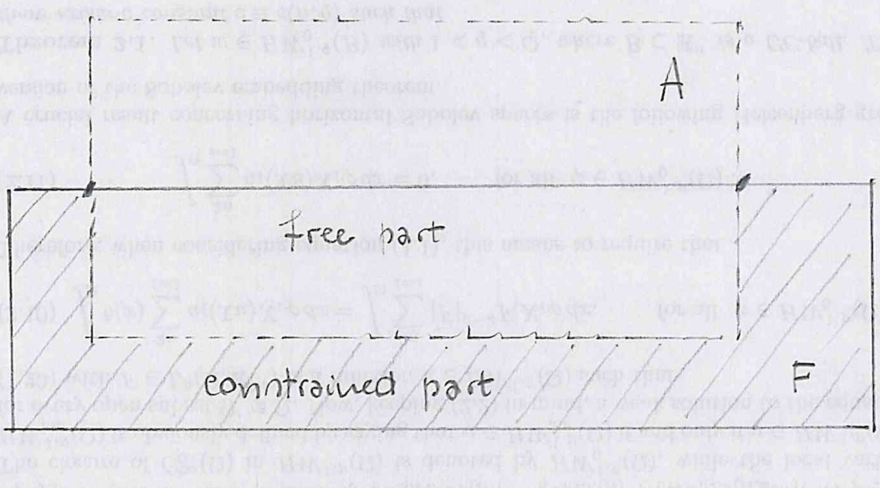
Let $A \subset \mathbb{R}^n$ be a bounded open set.

Let $F \subset \mathbb{R}^n$ be a fixed $\mathcal{L}^n$-measurable set such that $P(F) < \infty$

This set will play the role of "boundary condition".

We consider the following family of admissible sets:

$$\mathcal{A}_F(A) = \left\{ E \subset \mathbb{R}^n : \begin{array}{l} E \text{ is } \mathcal{L}^n\text{-measurable with} \\ \chi_E = \chi_F \text{ on } \mathbb{R}^n \setminus A \text{ a.e.} \end{array} \right\}.$$



THEOREM Let $A \subset \mathbb{R}^n$ be a bounded open set and let $F \subset \mathbb{R}^n$ be such that $P(F) < \infty$. The minimum

$$\min \{ P(E) : E \in \mathcal{A}_F(A) \}$$

is attained.

Proof. Without loss of generality we can assume that F is bounded (because A is bounded).

The infimum

$$L = \inf \{ P(E) : E \in \mathcal{A}_F(A) \}$$

is finite because $F \in \mathcal{A}_F(A)$ and $P(F) < \infty$.

Let $E_k \in \mathcal{A}_F(A)$ be a minimizing sequence:

$$\lim_{k\to\infty} P(E_k) = L.$$

Then we have:

$$\sup_{k\in\mathbb{N}} P(E_k) < \infty$$

and

$$\mathcal{L}^n(E_k) = \mathcal{L}^n(E_k \cap A) + \mathcal{L}^n(E_k \cap (\mathbb{R}^n \setminus A))$$

$$\leq \mathcal{L}^n(A) + \mathcal{L}^n(F \cap (\mathbb{R}^n \setminus A))$$

$$\leq \mathcal{L}^n(A) + \mathcal{L}^n(F) < +\infty \quad \forall k \in \mathbb{N}.$$

F is bounded $\Rightarrow$ finite measure.

Since F and A are bounded there exists $R>0$ such that for all $k \in \mathbb{N}$ we have

$$E_k \underset{\text{compactly contained}}{\subset\subset} B_R = \{x \in \mathbb{R}^n : |x| < R\}.$$

By the compactness theorem for BV functions (B_R has smooth boundary) there exist $f \in BV(B_R)$ and a subsequence of $(E_k)_{k\in\mathbb{N}}$ – still called $(E_k)_{k\in\mathbb{N}}$ – such that

$$\chi_{E_k} \xrightarrow[k\to\infty]{L^1(B_R)} f. \quad \text{– better: } L^1(\mathbb{R}^n)$$

Possibly taking a further subsequence, we also have

$$\lim_{k\to\infty} \chi_{E_k}(x) = f(x) \quad \text{for a.e. } x \in B_R .$$

(for a.e. $x \in \mathbb{R}^n$)

Since $\chi_{E_k}(x) \in \{0,1\}$ it follows that $f(x) \in \{0,1\}$ for a.e. x. That is: f is the characteristic function of a set $E \subset \mathbb{R}^n$:

$$f = \chi_E \quad \text{with} \quad E \subset \mathbb{R}^n \quad \mathcal{L}^n\text{-measurable.}$$

By lower semicontinuity with respect to the L^1-convergence

$$P(E) = \|D\chi_E\|(\mathbb{R}^n) \leq \liminf_{k\to\infty} \|D\chi_{E_k}\|(\mathbb{R}^n) = \liminf_{k\to\infty} P(E_k) = L .$$

Moreover:

$$E_k \cap (\mathbb{R}^n \setminus A) = F \cap (\mathbb{R}^n \setminus A)$$

$$\Downarrow$$

$$E \cap (\mathbb{R}^n \setminus A) = F \cap (\mathbb{R}^n \setminus A),$$

This means that $E \in \mathcal{A}_F(A)$

The argument shows that

$$P(E) = L = \underset{\uparrow}{\inf} \{ P(E) : E \in \mathcal{A}_F(A) \} .$$

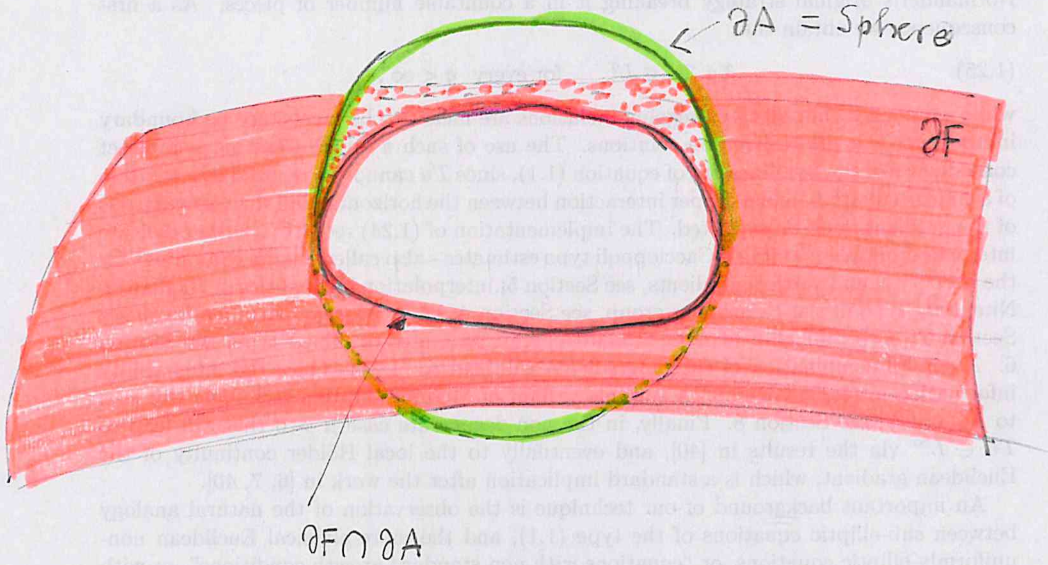
It is a min.

□

COMMENTS

1. If E is the solution of the previous minimum problem then $\partial E \cap A$ is a "minimal hypersurface" that has a certain regularity. When $n=3$, $\partial E \cap A$ is analytic.

2. If $\partial F \cap \partial A$ is a smooth manifold:



We expect that for a solution E of the minimum problem we have coincidence of sets

$$\partial E \cap \partial A = \partial F \cap \partial A.$$

However, this is not clear at all.

How the "boundary value" is attained is not clear.

EXERCISE Let $A \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Fix a real number m such that $0 < m < \mathcal{L}^n(A)$. Consider the family of sets

$$\mathcal{A}_m = \left\{ E \subset \mathbb{R}^n : \begin{array}{l} E \text{ is } \mathcal{L}^n\text{-measurable}, \\ E \subset A \text{ and } \mathcal{L}^n(E) = m \end{array} \right\}.$$

Show that the minimum

$$(*) \qquad \min\{ P(E; A) : E \in \mathcal{A}_m \}$$

Relative perimeter in A

is attained.

OPEN PROBLEM Assume that A is convex. Is it true that a solution E of Problem (*) is convex?

Regularity results and Simon's cone

Definition Let $E \subset \mathbb{R}^n$ be a set with locally finite perimeter in the open set $A \subset \mathbb{R}^n$. We say that E is locally perimeter minimizing in A if for any ball $B \subset\subset A$ and for any $F \subset \mathbb{R}^n$ with $E \Delta F \subset\subset B$ we have

$$P(E; B) \leq P(F; B).$$

The solutions of Plateau Problem are perimeter minimizing in this sense.

Theorem Let $E \subset \mathbb{R}^n$ be locally perimeter minimizing in the open set $A \subset \mathbb{R}^n$. Then

1) If $n \leq 7$, the hypersurface $\partial E \cap A$ is analytic

2) If $n \geq 8$, the boundary $\partial E \cap A$ is of class C^∞ outside a singular set of Hausdorff dimension $\leq n-8$.

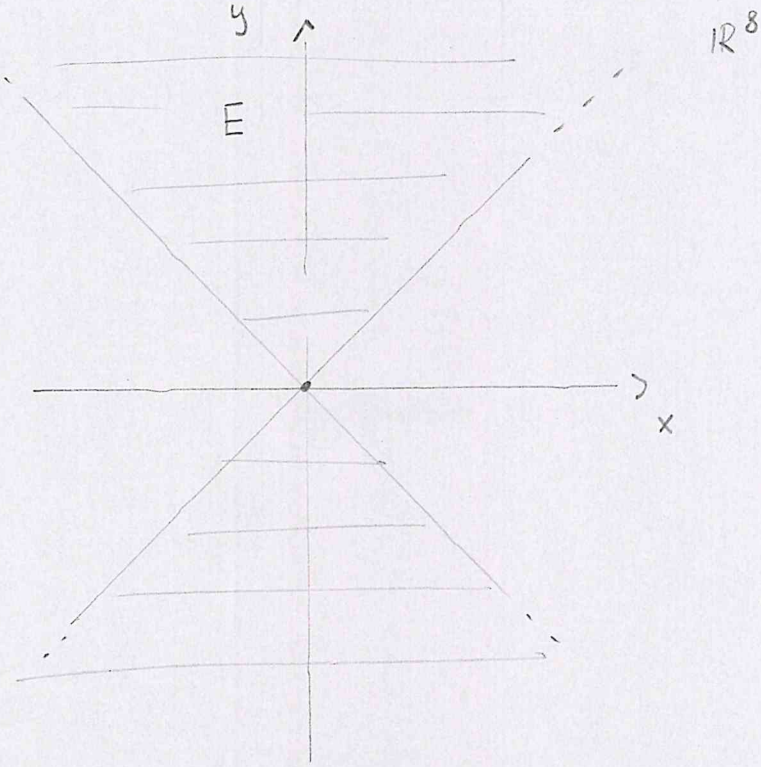
This is De Giorgi - Almgren regularity theorem.

Here we will show that statement 2) is sharp using the example of Simon's cone.

In $\mathbb{R}^8$ consider the coordinates $(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4$ and let

$$E = \{ (x,y) \in \mathbb{R}^8 \,;\, |x| < |y| \}$$

This is Simon's cone:



The boundary $\Sigma = \partial E = \{|x| = |y|\}$ has a singular point at the vertex $0 \in \Sigma$. The dimension of this singular set is zero.

<u>Theorem</u> For any ball $B \subset \mathbb{R}^n$ and for any set $F \subset \mathbb{R}^n$ such that $E \Delta F \subset B$ we have $P(E; B) \leq P(F; B)$.

The proof is based on a calibration technique. We look for a vector-field

$$V \in C^1(B; \mathbb{R}^n)$$

such that:

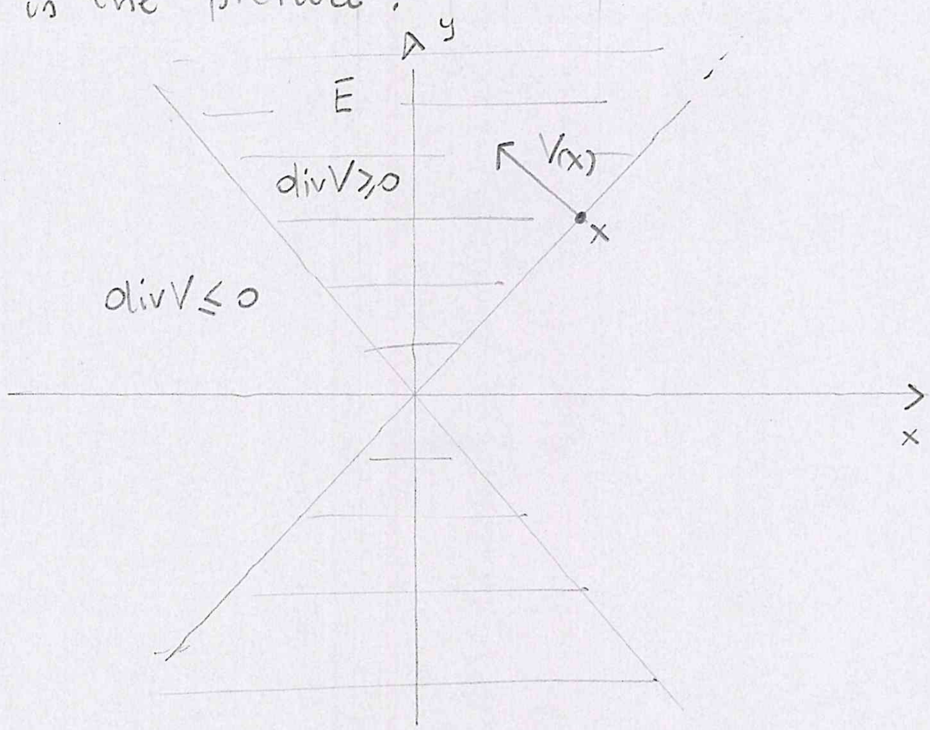
(1) $|V(x)| = 1$ for all $x \in B$;

(2) $V(x) = \nu_E(x)$ interior normal of $\Sigma = \partial E$ at $x \in \Sigma$;

(3) $\operatorname{div} V(x) \geq 0$ for $x \in E \cap B$;

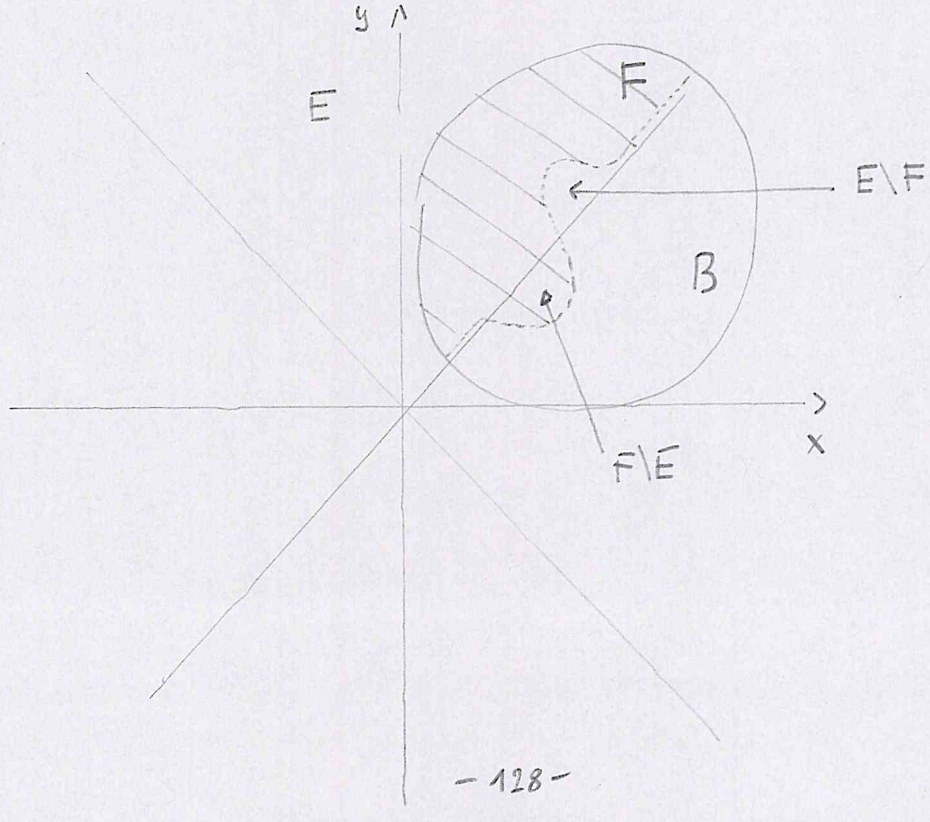
(4) $\operatorname{div} V(x) \leq 0$ for $x \in B \setminus E$

This is the picture:



We can assume that the competing set is regular, e.g., $\partial F \cap B$ is of class C^1.

Assume there exists V satisfying (1)-(4).
The problem with the singular point $0 \in \partial E$ will be discussed later.



By the divergence theorem

$$0 \overset{(3)}{\leq} \int_{E \setminus F} \operatorname{div} V(x)\, dx = \int_{\partial E \setminus F} \overbrace{\langle V, -\nu_E \rangle}^{= -1} dH^{n-1} + \int_{\partial F \cap E} \overbrace{\langle V, \nu_F \rangle}^{\leq 1} dH^{n-1}$$

interior normal to ∂F

$$\overset{(1)+(2)}{\leq} H^{n-1}(\partial F \cap E) - H^{n-1}(\partial E \setminus F)$$

In a similar way,

$$0 \overset{(4)}{\geq} \int_{F \setminus E} \operatorname{div} V(x)\, dx = \int_{\partial F \setminus E} \overbrace{\langle V, -\nu_F \rangle}^{-1 \,\shortparallel} dH^{n-1} + \int_{\partial E \cap F} \overbrace{\langle V, \nu_E \rangle}^{= 1} dH^{n-1}$$

$$\geq H^{n-1}(\partial E \cap F) - H^{n-1}(\partial F \setminus E).$$

Summing the two inequalities we get

$$H^{n-1}(\partial E \cap B) \leq H^{n-1}(\partial F \cap B).$$

This is the minimality of E in B.

We construct the calibration V for the cone E.

Let $f \in C^\infty(\mathbb{R}^{2n})$, where $n \in \mathbb{N}$, be

$$f(x,y) = \frac{1}{4}\left(|x|^4 - |y|^4\right), \qquad x, y \in \mathbb{R}^n.$$

We consider the cone $E = \{(x,y) \in \mathbb{R}^{2n} ; f(x,y) < 0\}$.

The gradient of f is

$$\nabla f(x,y) = \left(|x|^2 x, \; -|y|^2 y\right)$$

and for $(x,y) \neq (0,0)$ we define

$$V(x,y) = -\frac{\nabla f(x,y)}{|\nabla f(x,y)|} = \frac{(-|x|^2 x, \; |y|^2 y)}{\sqrt{|x|^6 + |y|^6}}$$

Clearly, $V \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathbb{R}^{2n})$, and $V|_{\partial E \setminus \{0\}}$ is the interior normal ν_E and $|V| = 1$.

Computations:

$$\frac{\partial}{\partial x_i}\left(-\frac{|x|^2 x_i}{\sqrt{|x|^6 + |y|^6}}\right) = \dots = -\frac{(|x|^2 + 2x_i^2)(|x|^6 + |y|^6) - 3|x|^6 x_i^2}{(|x|^6 + |y|^6)^{3/2}}$$

and summing for $i = 1, \dots, n$

$$\sum_{i=1}^n \dots = -\frac{(n|x|^2 + 2|x|^2)(|x|^6 + |y|^6) - 3|x|^8}{(\dots)^{3/2}}$$

and $$\sum_{i=1}^n \frac{\partial}{\partial y_i}\left(\frac{|y|^2 y_i}{\sqrt{|x|^6 + |y|^6}}\right) = \frac{(n+2)|y|^2(|x|^6 + |y|^6) - 3|y|^8}{(\dots)^{3/2}}$$

So we get the following formula for the divergence

$$\operatorname{div} V(x,y) = \frac{(n+2)(|y|^2-|x|^2)(|x|^6+|y|^6) - 3(|y|^8-|x|^8)}{(\dots)^{3/2}}.$$

where $|x|^6+|y|^6 = (|x|^2+|y|^2)(|x|^4-|x|^2|y|^2+|y|^4)$.

The formula reads:

$$\operatorname{div} V(x,y) = \underbrace{(|y|^4-|x|^4)}_{\substack{\vee \text{ in } E \\ 0}} \frac{(n-1)|x|^4-(n+2)|x|^2|y|^2+(n-1)|y|^4}{(\dots)^{3/2}}$$

The discriminant of the polynomial $p(t) = (n-1)t^2-(n+2)t+(n$ is

$$\Delta = (n+2)^2-4(n-1)^2 \leq 0 \iff n \geq 4$$

The conclusion is that for $n \geq 4$ we have

$$\operatorname{div} V(x,y) \geq 0 \iff (x,y) \in E.$$

The argument is formal except for the fact that V has a singularity at $(0,0) \in \partial E$.

However, there exists a constant $0 < C < \infty$ such that

$$|\operatorname{div} V(x,y)| \leq \frac{C}{\sqrt{|x|^2+|y|^2}}, \quad (x,y) \neq (0,0).$$

Letting $D_\varepsilon = \{(x,y) \in \mathbb{R}^{2n} ; \ |x|^2+|y|^2 < \varepsilon^2\}$, we can use the divergence theorem in $\mathbb{R}^{2n} \setminus D_\varepsilon$, $\varepsilon > 0$.

The errors due to integration on D_ε and ∂D_ε are infinitesimal for $\varepsilon \to 0^+$. In fact, we have

$$\int_{\partial D_\varepsilon} \overbrace{|\langle V, \nu_{D_\varepsilon} \rangle|}^{\leq 1} \, dH^{2n-1} \leq \varepsilon^{2n-1} \, H^{2n-1}(\partial D_1)$$

and

$$\int_{D_\varepsilon} |\operatorname{div} V(x,y)| \, dx\,dy \leq \frac{C}{\varepsilon} \, \varepsilon^{2n} \, \mathcal{L}^{2n}(D_1).$$

$\square$

Area minimality of holomorphic manifolds

Let $F: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a holomorphic function. In the coordinates $(z,w) \in \mathbb{C}^2$ this means that

$$\frac{\partial F}{\partial \bar z} = 0 \quad \text{and} \quad \frac{\partial F}{\partial \bar w} = 0 .$$

Consider the set

$$M = \{ (z,w) \in \mathbb{C}^2 ; \; F(z,w) = 0 \}.$$

The restricted set $M \cap \{ \nabla F \neq 0 \}$ is a holomorphic manifold of real dimension 2 that is embedded in $\mathbb{R}^4 \equiv \mathbb{C}^2$. Where $\nabla F = 0$ M may have singularities

<u>EXAMPLE.</u> With $F(z,w) = z^2 - w^3$, the manifold

$$M = \{ (z,w) \in \mathbb{C}^2 : \; z^2 = w^3 \}$$

has a singular point at $(0,0) \in M$, where two sheets $z = \pm \sqrt{w^3}$ touch.

Where $\nabla F \neq 0$ the tangent space $T_{(z,w)} M$ is a complex sub-space of $\mathbb{C}^2$.

In this section we shall prove the following theorem.

<u>Theorem 1</u> Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a (non-constant) holomorphic manifold. Then the surface $M = \{ F = 0 \}$ is area-minimizing for compact perturbations.

We need some preliminaries on exterior algebras. Let $e_1, \dots, e_4 \in \mathbb{R}^4$ be the canonical basis of $\mathbb{R}^4$ and consider two vectors

$$U = \sum_{i=1}^{4} u_i \, e_i, \qquad u_i \in \mathbb{R},$$

$$V = \sum_{j=1}^{4} v_j \, e_j, \qquad v_j \in \mathbb{R}.$$

The wedge-product $\wedge$ is bilinear and skew-symmetric

$$U \wedge V = \sum_{i=1}^{4} \sum_{j=1}^{4} u_i v_j \;\; e_i \wedge e_j .$$

Let $dx_1 \dots dx_n$ the dual basis of $e_1 \dots e_4$. The 2-differential form $dx_h \wedge dx_k$ acts on $e_i \wedge e_j$ in the following way

$$dx_h \wedge dx_k \,(e_i \wedge e_j) = \delta_{hi}\delta_{kj} - \delta_{hj}\delta_{ki}$$

$$= \det \begin{bmatrix} \langle e_h, e_i \rangle & \langle e_h, e_j \rangle \\ \langle e_k, e_i \rangle & \langle e_k, e_j \rangle \end{bmatrix}$$

By linearity of det on columns:

$$dx_h \wedge dx_k \,(U \wedge V) = \det \begin{bmatrix} \langle e_h, U \rangle & \langle e_h, V \rangle \\ \langle e_k, U \rangle & \langle e_k, V \rangle \end{bmatrix} \tag{7}$$

$$= \det \begin{bmatrix} u_h & v_h \\ u_k & v_k \end{bmatrix} .$$

Now we consider a special 2-forms:

(2) $$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

with $z = x_1 + i x_2$ and $w = x_3 + i x_4$. This forms calibrates <u>any</u> holomorphic manifold.

Using formula (1) we find

$$\omega(U \wedge V) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} + \det \begin{pmatrix} u_3 & v_3 \\ u_4 & v_4 \end{pmatrix}.$$

$$= \langle U, JV \rangle$$

where $J(V) = (v_2, -v_1, v_4, -v_3)$ is the complex structure (multiplication by i).

<u>Lemma 2</u> Let U, V be orthonormal. Then we have

(3) $$|\omega(U \wedge V)| \leq 1$$

with equality if and only if $U = \pm JV$.

Proof. By Cauchy-Schwarz inequality

$$|\omega(U \wedge V)| = |\langle U, JV \rangle| \leq |U| \cdot |JV| = 1$$

with equality if and only $U = \pm JV$.

$\square$

REMARK The condition $U = \pm JV$ means that the real 2-plane $\operatorname{span}_{\mathbb{R}}\{U, V\} \subset \mathbb{R}^4$ is a complex line in $\mathbb{C}^2$.

Now let $M \subset \mathbb{R}^4 \equiv \mathbb{C}^2$ be any 2-dimensional surface and let ω be a 2-form in $\mathbb{R}^4$.

Let U, V be an orthonormal frame of vector-fields tangent to M. The unit "normal" to M is the 2-vector

$$\tau_M = U \wedge V.$$

If M is oriented then τ is globally defined in a unique way up to the sign $\pm \tau_M$.

The integral of ω on M is by definition

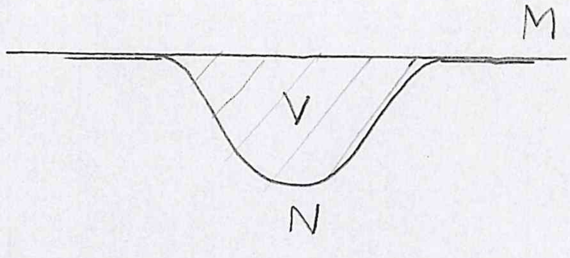
$$(4) \qquad \int_M \omega = \int_M \omega(\tau_M)\, dH^2$$

where H^2 is the 2-Hausdorff measure in $\mathbb{R}^4$.

We are ready to prove Theorem 1. Let $N \subset \mathbb{R}^4$ be a 2-surface of class C^1 such that the exists a 3-surface $V \subset \mathbb{R}^3$ such that:

1) V is bounded with finite measure
2) $\partial V = M \setminus N \cup N / M$ up to a H^2-negligible set,

We claim that



$$(5) \qquad H^2(M \setminus N) \le H^2(N \setminus M).$$

The exterior derivative of the 2-form ω in (2) is zero;

$$d\omega = 0$$

because ω has constant coefficients.

By Stoke's theorem;

$$0 = \int_V d\omega = \int_{\partial V} \omega = \int_{M \setminus N} \omega(\tau_M)\, dH^2 - \int_{N/M} \omega(\tau_N)\, dH$$

Since $\underset{(2w)}{TM}$ is a complex line, by lemma 2 we have $\omega(\tau_M) = 1$ on M, whereas $|\omega(\tau_N)| \le 1$.

So we have

$$H^2(M \setminus N) = \int_{M/N} \omega(\tau_M)\, dH^2 = \int_{N/M} \omega(\tau_N)\, dH^2 \le$$

$$\le H^2(N \setminus M).$$

This is (5).

$\square$

FORMULA DI MONOTONIA

Vogliamo provare il seguente teorema noto come formula di monotonia:

TEOREMA Sia $E \subset \mathbb{R}^n$ un insieme di perimetro localmente finito in un aperto $A \subset \mathbb{R}^n$ e stazionario in A. Sia $0 \in \partial^* E$. Allora per quasi ogni $0 < r < \mathrm{dist}(0, \partial A)$:

$$\frac{d}{dr} \frac{P(E, B_r)}{r^{n-1}} = \frac{d}{dr} \int_{\partial^* E \cap B_r} \frac{\langle \nu_E, x \rangle^2}{|x|^{n+1}} \, dH^{n-1}.$$

In particolare

$$r \longmapsto \frac{P(E, B_r)}{r^{n-1}}$$

è monotona crescente.

COROLLARIO Un insieme stazionario in $\mathbb{R}^n$ non può essere limitato.

Diamo la definizione di insieme stazionario.
Sia $V \in C^\infty(A; \mathbb{R}^n)$ un campo vettoriale e sia ν_E la normale di E. Definiamo la "divergenza tangenziale"

$$\mathrm{div}_E V = \mathrm{div}\, V - \langle \nu_E, (\nabla V) \nu_E \rangle$$

Commenti:

① Dunque la variazione prima del perimetro nella direzione V è:

$$\frac{\partial}{\partial t} P(\Phi_t(E), A)\Big|_{t=0} = \int_{\partial^* E \cap A} \mathrm{div}_E V \, dH^{n-1}$$

② Se E è un minimo del perimetro in A (per variazioni compatte) allora è stazionario.

DEFINIZIONE Sia $H \in L^1(\partial^* E; H^{n-1})$. Diciamo che E ha curvatura media H in A (in senso distribuzionale) se per ogni $V \in C_c^\infty(A; \mathbb{R}^n)$ si ha

$$\int_{\partial^* E \cap A} \mathrm{div}_E V \, dH^{n-1} = \int_{\partial^* E \cap A} H \langle V, \nu_E \rangle \, dH^{n-1}.$$

Dunque: E stazionario equivale a dire $H = 0$.

DEFINIZIONE Diciamo che E è stazionario in un aperto A se si ha

$$\int_{\partial^* E \cap A} \operatorname{div}_E V(x)\, dH^{n-1} = 0$$

per ogni $V \in C_c^\infty(A;\mathbb{R}^n)$.

La motivazione della definizione deriva dalla formula per lo sviluppo di Taylor del perimetro (variazione prima).

Sia $\Phi : A \times (-\delta,\delta) \xrightarrow{C^\infty} A$ un flusso ad un parametro di diffeomorfismi, ovvero

i) $\Phi(\cdot,t) : A \to A$ è un diffeomorfismo;

ii) $\Phi(\cdot,0) = $ Identità;

iii) $\Phi(\cdot,t+s) = \Phi(\cdot,t) \circ \Phi(\cdot,s)$.

Definiamo il campo vettoriale (il generatore del flusso)

$$V(x) = \frac{\partial}{\partial t} \Phi(x,t)\Big|_{t=0}, \quad x \in A.$$

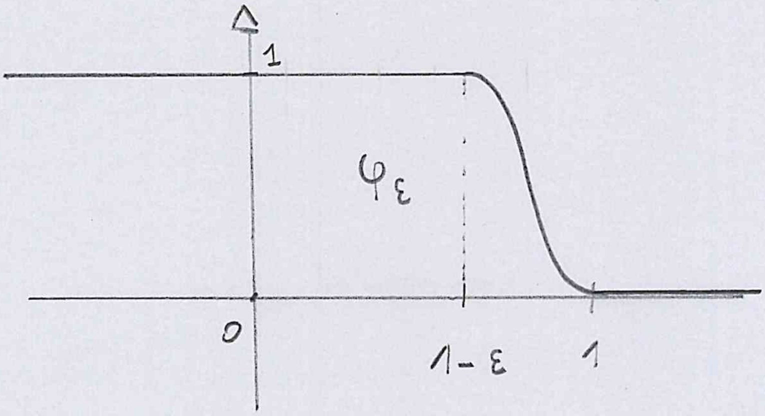
Notazione: $\Phi_t(\cdot) = \Phi(\cdot,t)$.

TEOREMA Sia E con perimetro finito in A. Allora per $t \to 0$

$$P(\Phi_t(E),A) = P(E,A) + t\int_{\partial^* E \cap A} \operatorname{div}_E V\, dH^{n-1} + o(t).$$

Dimostrazione della formula di monotonia

Sia $\varphi_\varepsilon \in C^\infty(\mathbb{R})$, $\varepsilon > 0$, fatta in questo modo:



$0 \leq \varphi_\varepsilon \leq 1$

Consideriamo questo campo vettoriale:

$$V_r(x) = \varphi_\varepsilon\left(\frac{|x|}{r}\right) x$$

Conti:

$$\nabla V_r = \frac{1}{r|x|}\,\varphi_\varepsilon'\left(\frac{|x|}{r}\right) x \otimes x + \varphi_\varepsilon\left(\frac{|x|}{r}\right) Id$$

dove $x \otimes x = (x_i x_j)_{i,j=1,\dots,n}$.

$$\operatorname{div} V_r = \varphi_\varepsilon'\left(\frac{|x|}{r}\right)\frac{|x|}{r} + n\,\varphi_\varepsilon\left(\frac{|x|}{r}\right)$$

$$\begin{aligned}\operatorname{div}_E V_r &= \varphi_\varepsilon'\left(\frac{|x|}{r}\right)\frac{|x|}{r} + n\,\varphi_\varepsilon\left(\frac{|x|}{r}\right) \\ &\quad - \left\langle \nu_E, \frac{1}{r|x|}\,\varphi_\varepsilon'\left(\frac{|x|}{r}\right)\underbrace{(x\otimes x)\nu_E}_{=\langle x, \nu_E\rangle x} + \varphi_\varepsilon\left(\frac{|x|}{r}\right)\nu_E \right\rangle \\ &= (n-1)\,\varphi_\varepsilon\left(\frac{|x|}{r}\right) + \frac{|x|}{r}\,\varphi_\varepsilon'\left(\frac{|x|}{r}\right)\left(1 - \frac{\langle x, \nu_E\rangle^2}{|x|^2}\right)\end{aligned}$$

Siccome E e' stazionario:

$$0 = \int_{\partial^* E \cap A} \mathrm{div}_E \, V_r \, dH^{n-1} = (n-1) \int_{\partial^* E \cap A} \varphi_\varepsilon\left(\frac{|x|}{r}\right) dH^{n-1} +$$

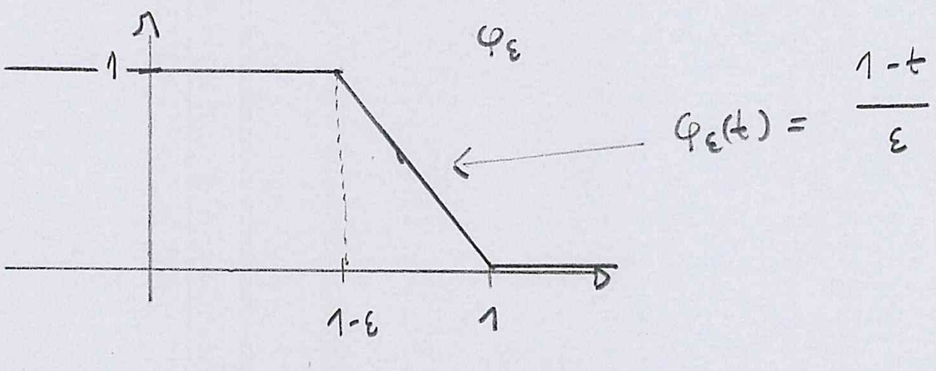
$$+ \int_{\partial^* E \cap A} \frac{|x|}{r} \varphi_\varepsilon'\left(\frac{|x|}{r}\right) \left(1 - \frac{\langle x, \nu_E \rangle^2}{|x|^2}\right) d$$

Riordinata, questa formula diviene

$$A_\varepsilon = (n-1) \int_{\partial^* E} \varphi_\varepsilon\left(\frac{|x|}{r}\right) dH^{n-1} - r \frac{\partial}{\partial r} \int_{\partial^* E} \varphi_\varepsilon\left(\frac{|x|}{r}\right) dH^{n-1} =$$

$$= \int_{\partial^* E} \frac{|x|}{r} \varphi_\varepsilon'\left(\frac{|x|}{r}\right) \frac{\langle x, \nu_E \rangle^2}{|x|^2} dH^{n-1} = B_\varepsilon .$$

Per approssimazione possiamo supporre che φ_ε sia



Con $\varepsilon \to 0^+$

$(\forall r > 0)$

$$\lim_{\varepsilon \to 0^+} \int_{\partial^* E} \varphi_\varepsilon\left(\frac{|x|}{r}\right) dH^{n-1} = \int_{\partial^* E \cap B_r} 1 \, dH^{n-1} = P(E, B_r)$$

(Conv. monotona)

e inoltre per q.o. $r>0$ si ha

$$\lim_{\varepsilon\to 0^+} \frac{\partial}{\partial r}\int_{\partial^* E} \varphi_\varepsilon\left(\frac{|x|}{r}\right) dH^{n-1} = \frac{\partial}{\partial r} P(E, B_r).$$

(Esercizio)

Dunque, con $\vartheta(r) = P(E, B_r)$ si trova per q.o. $r>0$

$$\begin{aligned}\lim_{\varepsilon\to 0^+} A_\varepsilon &= (n-1)\vartheta(r) - r\vartheta'(r) \\ &= -\left[(1-n)\vartheta(r) + r\vartheta'(r)\right] \\ &= -r^n\left[(1-n) r^{-n}\vartheta(r) + r^{1-n}\vartheta'(r)\right] \\ &= -r^n\left(r^{1-n}\vartheta(r)\right)'\end{aligned}$$

Affermiamo ora che per q.o. $r>0$:

$$\lim_{\varepsilon\to 0^+} -\frac{1}{r^n} B_\varepsilon = \frac{\partial}{\partial r}\int_{B_r\cap\partial^* E} \frac{\langle x, \nu_E\rangle^2}{|x|^{n+1}}\, dH^{n-1} = (\psi(r))$$

Sia $r>0$ un punto di differenziabilità di ψ.

$$-\frac{B_\varepsilon}{r^n} = \frac{1}{r^n}\int_{\partial^* E\cap\{1-\varepsilon<\frac{|x|}{r}<1\}} \frac{|x|}{r}\,\frac{1}{\varepsilon}\,\frac{\langle x,\nu_E\rangle^2}{|x|^2}\, dH^{n-1}$$

$$= \frac{1}{r\varepsilon}\int_{\partial^* E\cap(B_r/\overline{B}_{r(1-\varepsilon)})} \frac{\langle x, \nu_E\rangle^2}{r^n |x|}\, dH^{n-1}$$

e quindi

$$-\frac{B_\varepsilon}{r^n} \le \frac{1}{r\varepsilon} \int_{\partial^* E \cap (B_r / \overline{B}_{r(1-\varepsilon)})} \frac{\langle x, \nu_E \rangle^2}{|x|^{n+1}} \, dH^{n-1} \overset{\text{q.o. } r}{=}$$

$$= \frac{1}{r\varepsilon} \int_{\partial^* E \cap (B_r / B_{r(1-\varepsilon)})} \dots \, dH^{n-1}$$

$$= \frac{\psi(r) - \psi(r - r\varepsilon)}{r\varepsilon}$$

$$-\frac{B_\varepsilon}{r^n} \ge \frac{1}{r\varepsilon} \int_{\partial^* E \cap (B_r / \overline{B}_{r(1-\varepsilon)})} (1-\varepsilon)^n \frac{\langle x, \nu_E \rangle^2}{|x|^{n+1}} \, dH^{n-1} =$$

$$\overset{\text{q.o.}}{=} (1-\varepsilon)^n \frac{\psi(r) - \psi(r - r\varepsilon)}{r\varepsilon}$$

AREA AND LENGTH

1. Brief introduction to Hausdorff measures

Let $B_1 = \{ x \in \mathbb{R}^n : |x| < 1 \}$ be the unit ball in $\mathbb{R}^n$, $n \geq 1$.

Then we have

$$\omega_n := \mathcal{L}^n(B_1) \underset{\substack{\uparrow \\ \text{Exercise}}}{=} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$, $s > 0$ is the Γ-function.

Then we define for any $s > 0$:

$$\omega_s := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)} .$$

DEFINITION Let $0 \leq s < \infty$ and $\delta > 0$. For any set $E \subset \mathbb{R}^n$ define:

$$(1) \quad H^s_\delta(E) = \inf \left\{ \sum_{i=1}^\infty \omega_s \left(\frac{\operatorname{diam}(E_i)}{2} \right)^s : E \subset \bigcup_{i=1}^\infty E_i, \ \operatorname{diam}(E_i) < \delta \right.$$

$$(2) \quad H^s(E) = \lim_{\delta \to 0^+} H^s_\delta(E) .$$

EXAMPLE Let $E = S^1 = \{ x \in \mathbb{R}^2 \; ; \; |x| = 1 \}$ be the unit circle. For $\delta > 2$ we have

$$H^1_\delta(E) = 2$$

because $\omega_1 = 2$ and the infimum is realized by the covering of E through itself.

However we will see that

$$H^1(E) = \lim_{\delta \to 0^+} H^1_\delta(E) = 2\pi,$$

the length of the circle.

THEOREM For any $s \geq 0$ H^s is a regular Borel measure on $\mathbb{R}^n$.

The sketch of the proof is the following:

Step 1: H^s is an outer measure

Step 2: Let $\mathcal{M}$ = Carathéodory σ-algebra of measurable sets

Then H^s is a measure on $\mathcal{M}$.

Step 3: The open sets are in $\mathcal{M}$; use Lemma below

Step 4: H^s is regular: $\forall E \in \mathcal{M}$ there is B Borel set such that $E \subset B$ and $H^s(E) = H^s(B)$.

<u>LEMMA</u> Let μ be an outer measure in $\mathbb{R}^n$ such that:

$$\operatorname{dist}(A,B)>0 \implies \mu(A\cup B)=\mu(A)+\mu(B).$$

Then μ is a Borel measure (Borel sets are measurable)

Here is a list of properties of H^s:

1. $H^0(E) = \operatorname{Card}(E)$ is the "counting measure".

2. $H^1 = \mathcal{L}^1$ in $\mathbb{R}$ $(s=n=1)$. This is not difficult

3. $H^n = \mathcal{L}^n$ in $\mathbb{R}^n$ $\forall n$. This is difficult. The proof uses the isodiametric inequality.

4. $H^s(\lambda E) = \lambda^s H^s(E)$ for any $\lambda>0$. Easy.

5. $H^s(T(E)) = H^s(E)$ for any isometry $T:\mathbb{R}^n \to \mathbb{R}^n$ Easy.

We will need the following lemma:

<u>LEMMA</u> Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be L-Lipschitz:

$$|f(x)-f(y)| \leq L|x-y| \quad \text{for all } x,y\in\mathbb{R}^n.$$

Then for any $s\geq 0$ and $E\subset\mathbb{R}^n$ we have

$$H^s(f(E)) \leq L^s H^s(E).$$

Proof. Fix $\delta > 0$ and consider a covering of E

$$E \subset \bigcup_{i=1}^{\infty} C_i \quad \text{with} \quad \operatorname{diam}(C_i) < \delta .$$

Then we have

$$f(E) \subset f\Big(\bigcup_{i=1}^{\infty} C_i\Big) = \bigcup_{i=1}^{\infty} f(C_i)$$

with

$$\operatorname{diam}(f(C_i)) \leq L \operatorname{diam}(C_i) \leq L\delta .$$

It follows that

$$H^s_{L\delta}(f(E)) \leq \sum_{i=1}^{\infty} w_s \left(\frac{\operatorname{diam}(f(C_i))}{2} \right)^s \leq$$

$$\leq L^s \sum_{i=1}^{\infty} w_s \left(\frac{\operatorname{diam}(C_i)}{2} \right)^s .$$

Taking the inf on the coverings in the right hand side:

$$H^s_{L\delta}(f(E)) \leq L^s H^s_{\delta}(E)$$

and with $\delta \to 0^+$ we reach the conclusion:

$$H^s(f(E)) \leq L^s H^s(E),$$

$\square$

EXERCISE The following fact was realized by Archimedes in his work "On Sphere and Cylinder", Axiom IV.

Let $E \subset \mathbb{R}^n$ be a convex set and let $E \subset F$.

Prove that

$$H^{n-1}(\partial E) \leq H^{n-1}(\partial F).$$

In the next pages we give a proof of the Area Formula in the case of C^1 graphs.

2. FORMULA DELL'AREA: GRAFICI C^1

Vogliamo dare una dimostrazione della seguente variante della formula dell'area.

TEOREMA Sia $A \subset \mathbb{R}^n$ un insieme aperto e sia $f \in C^1(A)$. Allora

$$H^n(gr(f)) = \int_A \sqrt{1+|\nabla f(x)|^2}\, dx.$$

DIM. Iniziamo dal caso in cui f sia affine (lineare)

$$f(x) = \langle v, x\rangle, \quad x \in \mathbb{R}^n,$$

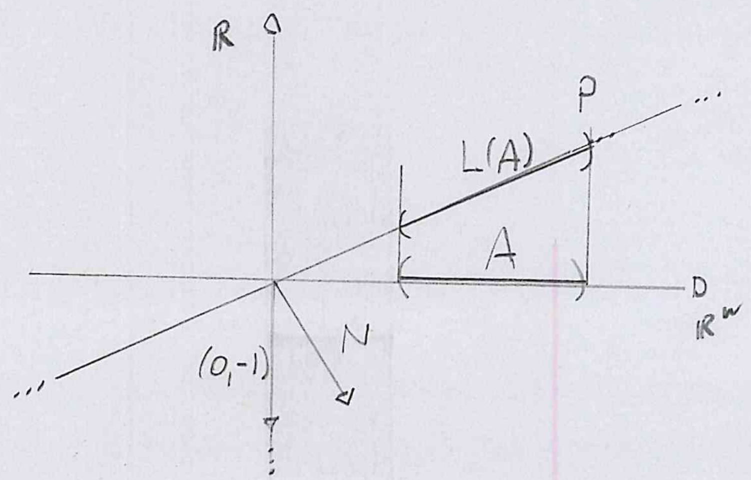
per qualche $v \in \mathbb{R}^n$. Sia $L: \mathbb{R}^n \to \mathbb{R}^{n+1}$ la mappa lineare

$$L(x) = (x, \langle v,x\rangle), \quad x \in \mathbb{R}^n.$$

La normale al piano $P = L(\mathbb{R}^n)$ è

$$N = \frac{(v,-1)}{\sqrt{1+|v|^2}}.$$

Sia $T \in O(n+1)$ la (una) trasformazione ortogonale tale che $T(N) = (0,-1)$



Allora abbiamo, con $S := T \circ L$, dove S è lineare da $\mathbb{R}^n$ a $\mathbb{R}^n$,

$$H^n(L(A)) \underset{T \text{ isometria}}{=} H^n(T \circ L(A))$$

$$= H^n(S(A)) \underset{\substack{\uparrow \\ \text{Teorema} \\ H^n = \mathcal{L}^n \\ \text{su } \mathbb{R}^n}}{=} \mathcal{L}^n(S(A))$$

$$\underset{\substack{\uparrow \\ \text{fatto noto}}}{=} |\det(S)| \, \mathcal{L}^n(A).$$

Inoltre, detta S^* la trasposta di S,

$$|\det(S)| = |\det(S^* S)|^{1/2} = |\det(L^* T^* T L)|^{1/2}$$

$$= |\det(L^* L)|^{1/2} \underset{\substack{\uparrow \\ \text{algebra} \\ \text{lineare}}}{=} \sqrt{1 + |v|^2}$$

e quindi

$$H^n(L(A)) = \int_A \sqrt{1+|v|^2}\, dx = \int_A \sqrt{1 + |\nabla f|^2}\, dx.$$

Sia ora $f \in C^1(A)$, siano $F(x) = (x, f(x))$ e

$$G = \{ F(x) \in \mathbb{R}^{n+1} \; ; \; x \in A \}.$$

Senza perdere generalità possiamo supporre che $Lip(F) < \infty$, questo è vero localmente.

Sia μ la misura di Borel su A definita in questo modo

$$\mu(B) = H^n(F(B)), \quad B \subset A \text{ di Borel}.$$

Abbiamo

$$\mu(B) \leq Lip(F)^n H^n(B) = Lip(F)^n \mathcal{L}^n(B)$$

e quindi $\mu \ll \mathcal{L}^n$. Dunque esiste una funzione $g \in L^1_{loc}(A)$ tale che

$$\mu(B) = \int_B g(x)\, dx$$

e in effetti:

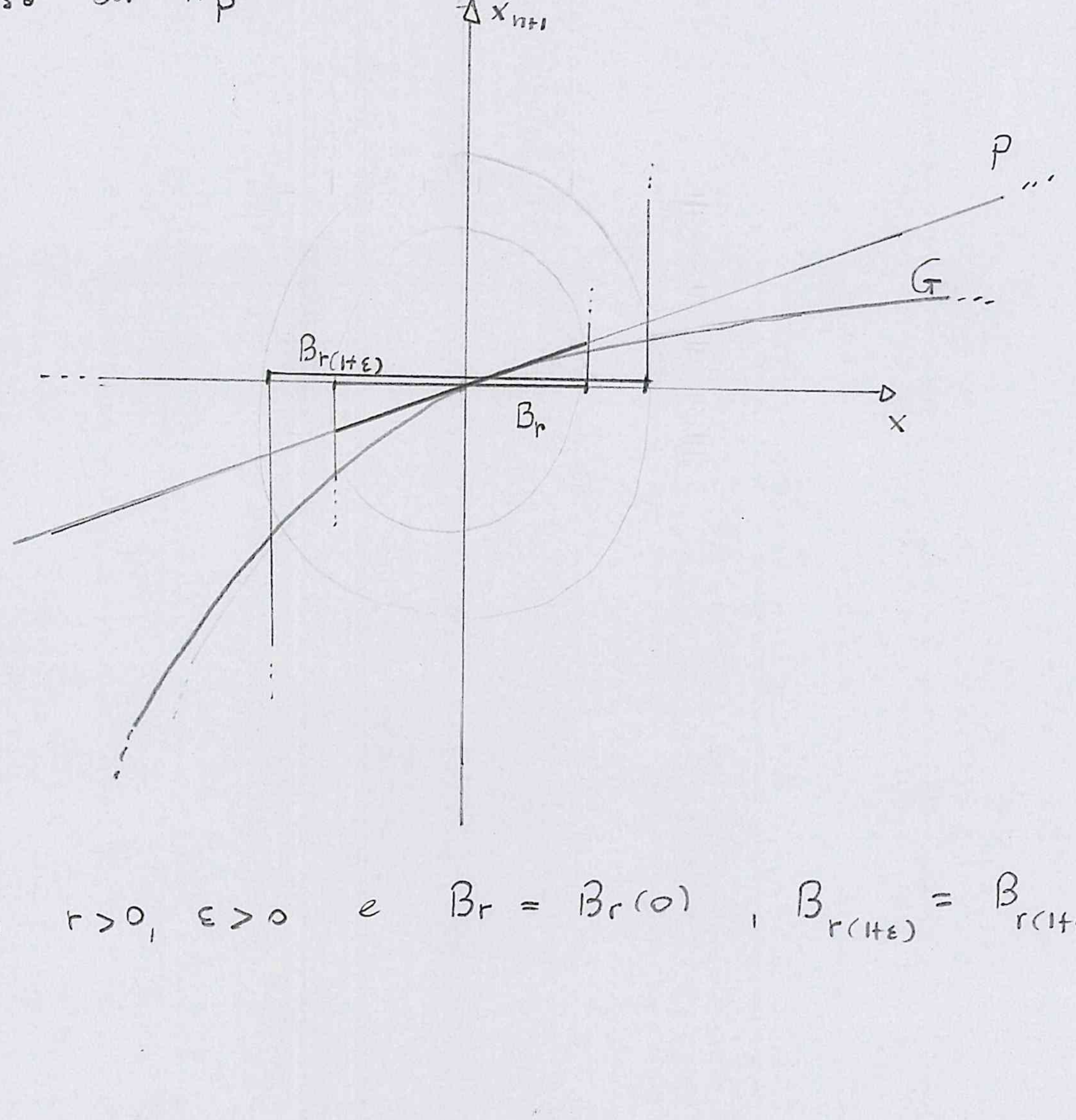
$$g(x) = \lim_{r \to 0^+} \frac{H^n(F(B_r(x)))}{\mathcal{L}^n(B_r(x))},$$

per $\mathcal{L}^n$-q.o. x.

Calcoliamo il limite in ogni punto $x \in A$. Senza perdere di generalità supponiamo $x = 0$ e $f(0) = 0$.

Sia $v = \nabla f(0)$, $L : \mathbb{R}^n \to \mathbb{R}^{n+1}$ sia $L(x) = (x, \langle v, x \rangle)$ e poi $P = L(\mathbb{R}^n)$. Indichiamo con $\pi_P : \mathbb{R}^{n+1} \to P$ la proiezione ortogonale su P.

Detto $G = F(A)$ il grafico di f, sia $\pi_G : \pi_P(G) \to G$ l'inversa di π_P



Siano $r > 0$, $\varepsilon > 0$ e $B_r = B_r(0)$, $B_{r(1+\varepsilon)} = B_{r(1+\varepsilon)}(0)$.

Affermiamo che esiste $r_0 > 0$ tale che per $0 < r < r_0$ si ha:

1) $F(B_r) \subset \pi_G(L(B_{r(1+\varepsilon)}))$,

2) $L(B_r) \subset \pi_p(F(B_{r(1+\varepsilon)}))$.

La proiezione π_p è 1-Lipschitz.
Per $r_0 > 0$ opportuno, la "proiezione" π_G è $(1+\varepsilon)$-Lipsch

Dunque, usando 1):

$$H^n(F(B_r)) \leq H^n(\pi_G(L(B_{r(1+\varepsilon)})))$$

$$\leq (1+\varepsilon)^n H^n(L(B_{r(1+\varepsilon)}))$$

$$= (1+\varepsilon)^n \sqrt{1+|v|^2}\, \mathcal{L}^n(B_{r(1+\varepsilon)})$$

(Prima parte)

$$= (1+\varepsilon)^{2n} \sqrt{1+|v|^2}\, \mathcal{L}^n(B_r)$$

e quindi

$$\limsup_{r \downarrow 0} \frac{H^n(F(B_r))}{\mathcal{L}^n(B_r)} \leq (1+\varepsilon)^{2n} \sqrt{1+|v|^2}.$$

Usando invece 2):

$$\sqrt{1+|v|^2}\,\mathcal{L}^n(B_r) = H^n(L(B_r)) \leq H^n(\Pi_p(F(B_{r(1+\varepsilon)})))$$

$$\leq H^n(F(B_{r(1+\varepsilon)}))$$

che può essere riscritta in questo modo:

$$H^n(F(B_r)) \geq \frac{1}{(1+\varepsilon)^n}\sqrt{1+|v|^2}\,\mathcal{L}^n(B_r)$$

e quindi

$$\lim_{r\to 0^+}\frac{H^n(F(B_r))}{\mathcal{L}^n(B_r)} \geq \frac{1}{(1+\varepsilon)^n}\sqrt{1+|v|^2}.$$

Siccome $\varepsilon>0$ è libero, si trova

$$\lim_{r\to 0^+}\frac{H^n(F(B_r))}{\mathcal{L}^n(B_r)} = \sqrt{1+|v|^2} = \sqrt{1+|\nabla f(0)|^2}.$$

Proviamo la 1). É equivalente a $\Pi_p(F(B_r))\subset L(B_{r(1+\varepsilon)})$

Detta $\pi(x,x_{n+1}) = x$ si tratta di verificare che

$$\pi(\Pi_p(F(B_r))) \subset B_{r(1+\varepsilon)}$$

per $0<r<r_0$.

Sia $x \in B_r$. Allora $\pi_p(F(x)) = F(x) - \langle F(x), N\rangle$

e quindi

$$\pi(\pi_p(F(x))) = x - \langle F(x), N\rangle \frac{v}{\sqrt{1+|v|^2}}$$

$$= x - \left(\langle x, v\rangle - f(x)\right) \frac{v}{1+|v|^2}$$

dove $f(x) = f(0) + \langle \nabla f(0), x\rangle + o(|x|)$

$$= \langle v, x\rangle + o(|x|)$$

Segue che

$$|\pi(\pi_p(F(x)))| = |x| \, (1+o(1))$$

$$\leq r\,(1+\varepsilon)$$

se $|x| < r$ e $o(1) < \varepsilon$ vero per $r < r_0$.

Lasciamo la verifica di 2) come esercizio. $\square$

$\square$

3. FORMULA DI COAREA. UN CASO MODELLO

TEOREMA Siano $A \subset \mathbb{R}^n$ un insieme aperto ed $f: A \to \mathbb{R}$ una funzione Lipschitziana. Allora

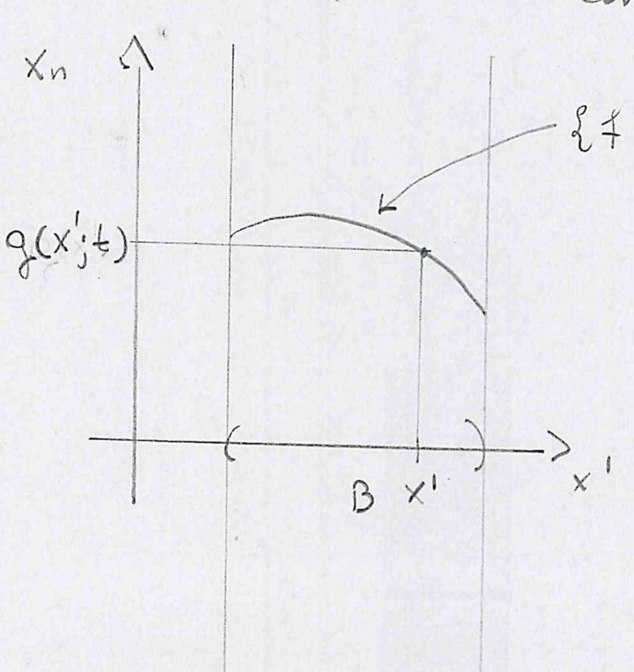
$$\int_A |\nabla f(x)|\,dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}\big(\partial\{x \in A : f(x) > t\}\big)\,dt.$$

É una variante "curvilinea" del teorema di Fubini-Tonelli.

Dimostriamo il teorema nella seguente situazione modello:

$$A = B \times \mathbb{R} \quad \text{con} \quad B \subset \mathbb{R}^{n-1} \text{ aperto}$$

$$f \in C^1(B \times \mathbb{R}) \quad \text{con} \quad \frac{\partial f}{\partial x_n} \neq 0 \text{ su } B \times \mathbb{R}.$$



Per il Teorema della funzione implicita l'insieme

$\{x \in B \times \mathbb{R} \,;\, f(x) = t\}$ è il grafico di una funzione $g(x';t)$

Precisamente

$$\{f = t\} = \{(x', g(x';t)) \in \mathbb{R}^n \,;\, x' \in B\}$$

con $g(\cdot\,;t) \in C^1(B)$. Inoltre

$$f(x', g(x';t)) \equiv t$$

e quindi

$$\begin{cases} \dfrac{\partial f}{\partial x_n}(x', g(x';t)) \, \dfrac{\partial g}{\partial t}(x';t) = 1 \\[2ex] \nabla_{x'} f(x', g(x';t)) + \dfrac{\partial f}{\partial x_n}(x', g(x';t)) \, \nabla_{x'} g(x';t) \end{cases}$$

In particolare

$$|\nabla f(x', g)| = \left(|\nabla_{x'} f(x', g)|^2 + \left| \frac{\partial f}{\partial x_n}(x', g) \right|^2 \right)^{1/2}$$

$$= \left| \frac{\partial f}{\partial x_n}(x', g) \right| \left(1 + |\nabla_{x'} g(x';t)|^2 \right)^{1/2}.$$

Con il cambio di variabile $x = G(\xi, t) := (\xi, g(\xi; t$

con $\xi \in B$ si trova

$$\int_{B \times \mathbb{R}} |\nabla f(x)| \, dx = \int_{B \times \mathbb{R}} |\nabla f(\xi, g(\xi, t))| \left| \frac{\partial g}{\partial t}(\xi; t) \right| d\xi \, d$$

Fubini - Tonelli

$$= \int_{\mathbb{R}} \int_{B} \sqrt{1 + |\nabla_{\xi} g(\xi; t)|^2} \, d\xi \, dt$$

Formula Area

$$= \int_{\mathbb{R}} H^{n-1}(\{ x \in B \times \mathbb{R} ; \ f(x) = t \}) \, dt$$

,

$\square$

4. Length in metric spaces

Let (X,d) be a metric space and let

$\gamma : [0,1] \to X$ be a continuous curve.

DEF The length of γ is

$$L(\gamma) = \sup \left\{ \sum_{i=1}^{N} d(\gamma(t_i), \gamma(t_{i-1})) : 0 \leq t_0 < t_1 < \ldots < t_N = \right.$$
$$N \in \mathbb{N}$$

$\cap$

$[0,\infty)$

If $L(\gamma) < \infty$, the curve γ is said to be rectifiable.

COMMENTS 1) If γ is Lipschitz:

$$d(\gamma(t), \gamma(s)) \leq L\,|s-t| \qquad \text{for } s,t \in [0,1]$$

then γ is rectifiable and $L(\gamma) \leq L$.

2) If γ is rectifiable there exists $\phi : [0, L(\gamma)] \to [0,1]$ homeomorphism ("change of parameter") such that $\tilde{\gamma} := \gamma \circ \phi : [0, L(\gamma)] \to X$ is 1-Lipschitz ("arclength parameterization").

The length does not change: $L(\tilde{\gamma}) = L(\gamma)$.

DEF Let $\gamma : [0,1] \longrightarrow X$ be a curve. If the limit exists, we define the metric derivative of γ at $t \in [0,1]$

$$|\dot{\gamma}|(t) = \lim_{\delta \to 0} \frac{d(\gamma(t+\delta), \gamma(t))}{|\delta|}.$$

THEOREM Let $\gamma : [0,1] \to X$ be Lipschitz. Then the metric derivative $|\dot{\gamma}|(t)$ exists for a.e. $t \in [0,1]$.

Moreover we have

$$L(\gamma) = \int_{[0,1]} |\dot{\gamma}|(t)\, dt.$$

For any set $E \subset X$ we can define its "length" setting

$$H^1(E) = \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i, \ \mathrm{diam}(E_i) < \delta \right\}.$$

Then we have:

THEOREM Let $\gamma : [0,1] \longrightarrow X$ be Lipschitz continuous and injective. Then

$$L(\gamma) = H^1(\gamma([0,1])).$$

For a proof of these theorems see the book:

L. Ambrosio - P. Tilli, Topics on Analysis in Metric Spaces.

CONCLUSIONS:

We have at least three notions of "length" of a curve:

1) Total variation;
2) Length integral;
3) Hausdorff measure H^1.

They are the same for Lipschitz curves.

REMARK The length functional $\gamma \longmapsto L(\gamma)$ is lower semicontinuous for the pointwise convergence.

PROBLEMS OF MINIMAL LENGTH

In $\mathbb{R}^n$ with $n \geq 2$ we fix:

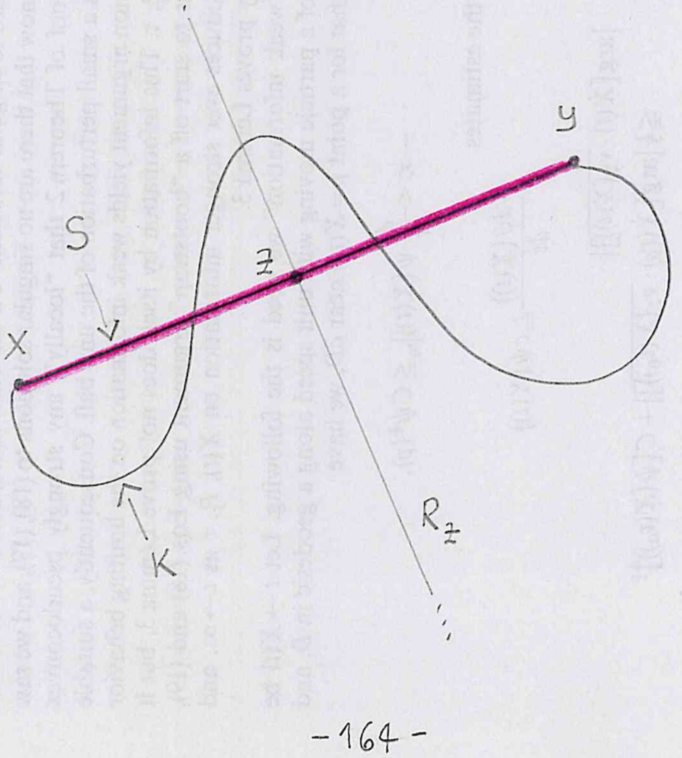
- $C \subset \mathbb{R}^n$ a compact set ("Constraint");
- $M \subset \mathbb{R}^n$ a closed set with $C \subset M$.

In this section we study the following "geodesics problem":

$$(*) \quad \min \left\{ \mathcal{H}^1(K) : K \subset \mathbb{R}^n \ \underline{\text{compact}} \text{ and } \underline{\text{connected}} \right. \left. \text{such that } C \subset K \subset M \right\}.$$

Among all compact connected sets containing C we want to find the one (the ones) with minimal length.

<u>Example</u> Let $M = \mathbb{R}^2$ and $C = \{x, y\}$ with $x \neq y$.
The solution of Problem $(*)$ is the segment $S = [x, y]$:



Let $K \subset \mathbb{R}^2$ be any compact connected set such that $x, y \in K$.

Let $\pi : \mathbb{R}^2 \longrightarrow S$ be the "metric projection" onto S:

$$\pi(p) = q \in S \iff |p-q| = \operatorname{dist}(p; S).$$

For $z \in S$ we denote by R_z the line through z orthogonal to S. Then we have:

$$K \text{ connected} \implies K \cap R_z \neq \emptyset \quad \text{for all } z \in S.$$

Otherwise a line would disconnect K.

It follows that $\pi : K \longrightarrow S$ is surjective. Thus:

$$H^1(S) = H^1(\pi(K)) \leq \underset{\overset{\|}{1}}{\mathrm{Lip}(\pi)}\, H^1(K) = H^1(K).$$

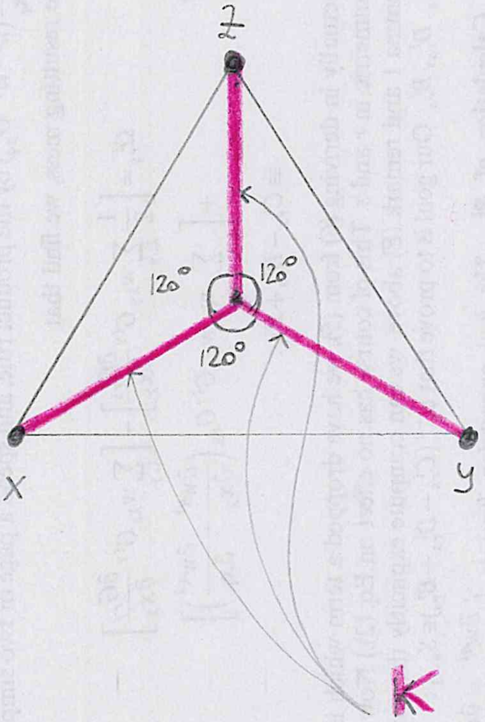
$\square$

PROBLEM Try to prove the following fact:

Let $M = \mathbb{R}^2$ and $C = \{x, y, z\}$ with $x, y, z \in \mathbb{R}^2$ vertices of an equilateral triangle ("equilateral" is not really important).

Then the solution of Problem (*) is the set K

in the picture:



You may use the fact that K is a union of line-segments. □

Our goal in this Section is to prove the existence of solutions to Problem (*).

The strategy is: Direct Method of CV.

We need <u>compactness</u> and <u>lower semicontinuity</u>.

Our ambient space is

$$\mathcal{K} = \{ K \subset \mathbb{R}^n : K \neq \emptyset \text{ compact} \}.$$

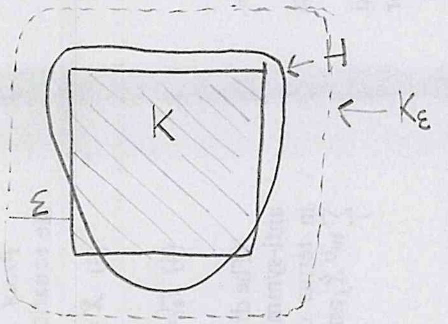
For $K \in \mathcal{K}$ and $\varepsilon > 0$ we let:

$$K_\varepsilon = \{ x \in \mathbb{R}^n : \operatorname{dist}(x; K) < \varepsilon \}$$

This is the ε-neighborhood of K.

<u>DEFINITION</u> The <u>Hausdorff distance</u> on $\mathcal{K}$ is the function $\delta : \mathcal{K} \times \mathcal{K} \longrightarrow [0,\infty)$

$$\delta(K,H) = \inf \{ \varepsilon > 0 : K \subset H_\varepsilon \text{ and } H \subset K_\varepsilon \}.$$



Here: $H \subset K_\varepsilon$

<u>PROPOSITION</u> $(\mathcal{K}, \delta)$ is a metric space.

The proof is left as an exercise for the reader.

<u>THEOREM</u> The metric space $(\mathcal{K}, \delta)$ is complete and locally compact. In particular, closed and bounded subsets of $\mathcal{K}$ are compact.

Proof. Let $(K_m)_{m\in\mathbb{N}}$ be a Cauchy sequence:

$\forall \varepsilon>0 \ \exists \bar{n}\in\mathbb{N} \ \forall m,h\in\mathbb{N}$ with $m,h\geq\bar{n}$: $\delta(K_m, K_h)<\varepsilon$.

This implies that there exists $R>0$ such that

$$K_m \subset B_R := \{x\in\mathbb{R}^n ; \ |x|\leq R\} \quad \forall m\in\mathbb{N}.$$

We shall show that $(K_m)_{m\in\mathbb{N}}$ converges to the following set

$$K := \bigcap_{h=1}^{\infty} \overline{\bigcup_{m\geq h} K_m} .$$

these sets are compact and form a decreasing sequence

We notice that K is compact and $K\neq\emptyset$.

We claim that $K_h \xrightarrow[h\to\infty]{\delta} K$. Fix $\varepsilon>0$.

- We have $K_m \subset (K_h)_\varepsilon$ for all $m,h\geq\bar{n}$. Thus

$$\overline{\bigcup_{m\geq\bar{n}} K_m} \subset (K_h)_{2\varepsilon} \implies K\subset (K_h)_{2\varepsilon} \quad \forall h\geq\bar{n}.$$

- We study the opposite inclusion.

If $x \in K_h$ (with $h \geq \bar{n}$) then $\operatorname{dist}(x, K_m) < \varepsilon \ \forall m \geq \bar{n}$.

It follows that there exists $x_m \in K_m$ such that $|x - x_m| < \varepsilon$

For $m \geq h$ we therefore have

$$x_m \in K_m \subset \overline{\bigcup_{m \geq h} K_m}.$$

The sequence $(x_m)_{m \in \mathbb{N}}$ is bounded. Possibly selecting a subsequence we may assume that it converges to a point $\bar{x} \in \mathbb{R}^n$. It must be $\bar{x} \in K$. Now

$$|x - x_m| < \varepsilon \ \forall m \implies |x - \bar{x}| \leq \varepsilon$$

$$\implies \operatorname{dist}(x; K) < 2\varepsilon.$$

This proves that

$$K_h \subset K_{2\varepsilon} \quad \forall h \geq \bar{n}.$$

This finishes the proof of $K_h \xrightarrow[h \to \infty]{\delta} K$.

Now we show that $(\mathcal{K}, \delta)$ is locally compact. Namely, we show that any bounded sequence in $\mathcal{K}$ has a converging subsequence.

Let $K_h \in \mathcal{K}$ be a bounded sequence: $K_h \subset B_R$ for all $h \in \mathbb{N}$ and for some $R > 0$.

From $(K_h)_{h\in\mathbb{N}}$ we extract a Cauchy sub-sequence. This is the difficult part of the proof.

- For any $k \in \mathbb{N}$ there are $B_1, \dots, B_N$ balls in $\mathbb{R}^n$ with radius $\frac{1}{k}$ such that

$$B_R \subset \bigcup_{i=1}^{N} B_i .$$

The number N depends on k. We define the family

$$\mathcal{B}^k = \{B_1, \dots, B_N\}.$$

Now we start a construction by induction. We begin with the case $k = 1$.

- We claim that there exists $(K_h^1)_{h\in\mathbb{N}}$ subsequence of $(K_h)_{h\in\mathbb{N}}$ and there exists $\underset{\neq\emptyset}{Q^1} \subset \mathcal{B}^1$ such that

$$Q^1 = \{B \in \mathcal{B}^1 : B \cap K_h^1 \neq \emptyset\} \text{ for all } h \in \mathbb{N}.$$

Proof: there exists $B \in \mathcal{B}^1$ that intersects infinitely many K_h (otherwise $\bigcup_{i=1}^{N} B_i = \bigcup_{B\in\mathcal{B}^1} B$ would intersect finitely many K_h. Impossible.)

Now $(K_h^1)_{h\in\mathbb{N}}$ is chosen and we can define Q^1 as above.

• Now we proceed by induction. We claim that there exist $(K_h^m)_{h\in\mathbb{N}}$ subsequence of $(K_h^{m-1})_{h\in\mathbb{N}}$ and $\mathcal{Q}^m \subset \mathcal{B}^m$ such that

$$\mathcal{Q}^m = \{B \in \mathcal{B}^m : K_h^m \cap B \neq \emptyset\} \quad \forall h \in \mathbb{N}.$$

The proof is as in the previous step.

• Now we let

$$F^m = \bigcup_{B\in\mathcal{Q}^m} B \subset \mathbb{R}^n$$

and we notice that

$$F^m \subset (K_h^m)_{2/m} \quad \forall h \in \mathbb{N},$$

because each $B \in \mathcal{Q}^m$ intersects K_h^m and $\operatorname{diam} B = \frac{2}{m}$.

On the other hand, by the definition of $\mathcal{Q}^m$:

$$K_h^m \subset \bigcup_{B\in\mathcal{Q}^m} B_{2/m} \subset F^m_{2/m}.$$

The conclusion of this step is

$$\delta(K_h^m, F^m) < \frac{2}{m} \quad \text{for all } h \in \mathbb{N}.$$

• Conclusion. By the diagonal selection procedure, we finally consider the subsequence $(K_h^h)_{h\in\mathbb{N}}$.

For $h, l \in \mathbb{N}$, by the triangle inequality we obtain

$$\delta(K^h_h, K^l_l) \leq \underbrace{\delta(K^h_h, F^h)}_{\frac{2}{h}} + \underbrace{\delta(F^h, F^l)}_{2\left(\frac{1}{h}+\frac{1}{l}\right)} + \underbrace{\delta(F^l, K^l_l)}_{\frac{2}{l}}$$

$$\leq 4\left(\frac{1}{h}+\frac{1}{l}\right) < \varepsilon \quad \text{for } h, l \geq \bar{n} = \bar{n}(\varepsilon)$$

$\square$

The compactness is now established.

We need the lower semicontinuity. Here there is a problem.

<u>EXAMPLE</u> Let $K = [0,1] \subset \mathbb{R}$ and for $n \in \mathbb{N}$

$$K_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1\right\} \subset K.$$

Then we have $\delta(K, K_n) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$.

However

$$0 = \liminf_{n\to\infty} \underbrace{H^1(K_n)}_{\substack{\parallel \\ 0}} < 1 = H^1(K).$$

Hausdorff measures are NOT lower-semicontinuous with respect to the Hausdorff distance.

$\square$

Luckly, the lowersemicontinuity holds when the sets are connected, but only for the measure H^1.

<u>THEOREM</u> (Gołąb) Let $K_h \subset \mathbb{R}^n$, $h \in \mathbb{N}$, be a sequence of <u>connected</u> compact sets such that

$$K_h \xrightarrow[h \to \infty]{\delta} K \in \mathcal{K}.$$

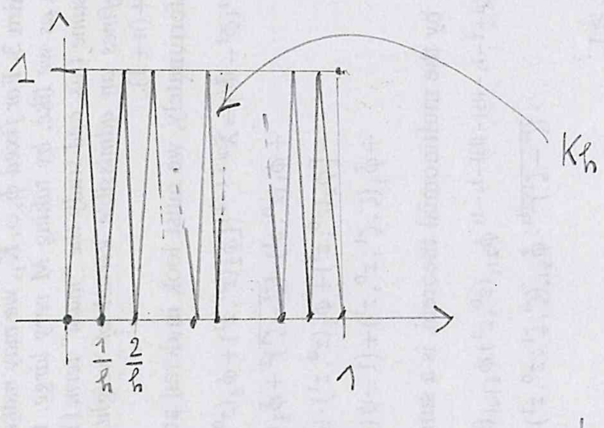
Then K is connected and

$$H^1(K) \leq \liminf_{h \to \infty} H^1(K_h).$$

The proof is omitted: see Falconer, Geometry of Fractal Sets, Cambridge.

<u>REMARK</u> Gołąb's theorem does not hold for H^2.

Consider $K_h \subset \mathbb{R}^2$ as follows:



Then $K_h \xrightarrow[h \to \infty]{\delta} = [0,1] \times [0,1] := K$, but

$$H^2(K) = 1 > 0 = \liminf_{h \to \infty} \underbrace{H^2(K_h)}_{\underset{0}{\parallel}}.$$

Existence of solutions for the minimum length problem

Let
$$\mathcal{A} = \{ K \subset \mathbb{R}^n : K \text{ compact and connected } C \subset K \subset M \}$$

where $C \subset \mathbb{R}^n$ compact and $M \subset \mathbb{R}^n$ closed are fixed, $\neq \emptyset$

We assume: there is $K_0 \in \mathcal{A}$ such that $\mathcal{H}^1(K_0) < \infty$.

Otherwise the problem is empty.

Let $K_h \in \mathcal{A}$ be a minimizing sequence:

$$\lim_{h \to \infty} \mathcal{H}^1(K_h) = \inf \{ \mathcal{H}^1(K) : K \in \mathcal{A} \}.$$

We can assume that: $\exists R > 0$ such that $K_h \subset \{ |x| \leq R \}$ for all $h \in \mathbb{N}$ (Exercise).

By compactness there exists a subsequence - still denoted K_h - and there exists $K \in \mathcal{K}$ such that

$$K_h \xrightarrow[h \to \infty]{\delta} K.$$

The convergence implies that $C \subset K \subset M$.

By Golab's theorem K is connected, and hence $K \in \mathcal{A}$, and moreover

$$\mathcal{H}^1(K) \leq \liminf_{h \to \infty} \mathcal{H}^1(K_h).$$

So K is a minimizer. $\square$

When we try to study the structure of the minimizer K the following rectifiability theorem is very useful:

THEOREM Let $K \subset \mathbb{R}^n$ be a compact connected set such that $H^1(K) < \infty$. Then there exist a Borel set $N \subset \mathbb{R}^n$ and countably many Lipschitz curves $\gamma_i : [0,1] \to \mathbb{R}^n$, i such that:

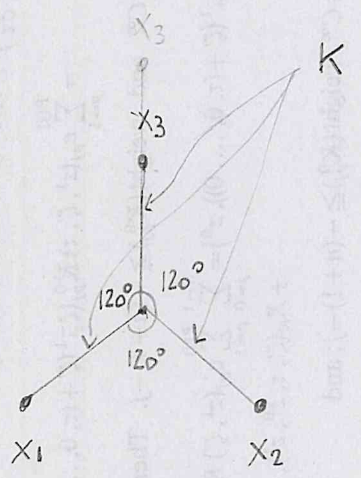
1) $H^1(N) = 0$;

2) $K = N \cup \bigcup_{i=1}^{\infty} \gamma_i([0,1])$.

For the proof see again Falconer's book.

EXERCISE Let $C = \{x_1, x_2, x_3\} \subset \mathbb{R}^2$ be the set vertices of an equilateral triangle in the plane.

Show that the minimizer K is as in the picture



"Triple point".

Γ - CONVERGENZA

1) RILASSAMENTO

2) Γ - LIMITI

3) CONVERGENZA DEI MINIMI E DEI VALORI MINIMI

1) Rilassamento.

(X, τ) spazio toplogico

$$\mathcal{I}(x) = \{ U \subset X : U \text{ intorno di } x \} , \quad x \in X$$

DEF (Semicontinuità inferiore) Una funzione $F : X \longrightarrow (-\infty, \infty]$ è semicontinua inferiormente su X (sci) se per ogni $x \in X$ si ha

$$F(x) \leq \sup_{U \in \mathcal{I}(x)} \inf_{y \in U} F(y) .$$

COMMENTI

1) É equivalente richiedere ; $F(x) = \sup\limits_{U \in \mathcal{I}(x)} \inf\limits_{y \in U} F(y)$.

2) Se X è uno Spazio Metrico (oppure uno

spazio topologico N_I) allora F è sci se $\forall x \in X$ e per ogni $x_h \xrightarrow[h\to\infty]{} x$ si ha

$$F(x) \leq \liminf_{h\to\infty} F(x_h).$$

<u>DEF (Inviluppo semicontinuo inferiore)</u> Data $F: X \to (-\infty$ chiamiamo la funzione $F^{sci}: X \to [-\infty, \infty]$

$$F^{sci}(x) = \sup_{U \in I(x)} \inf_{y \in U} F(y)$$

l'inviluppo semicontinuo inferiore di F.

<u>COMMENTI</u>

1) $F^{sci}(x) = \sup \left\{ G(x) \; ; \; \begin{matrix} G: X \to [-\infty, \infty], \; G \leq F \\ G \text{ sci} \end{matrix} \right\}$

2) Negli Spazi Metrici:

$$F^{sci}(x) = \inf \left\{ \liminf_{h\to\infty} F(x_h) \; ; \; x_h \xrightarrow[h\to\infty]{} x \right\}.$$

3) F^{sci} è sci su X.

2) Γ-limiti

(X, τ) Spazio Topologico

$F_h : X \to (-\infty, \infty]$, $h \in \mathbb{N}$

<u>DEFINIZIONE</u> Definiamo le funzioni $F^-, F^+ : X \to [-\infty, \infty]$

$$F^-(x) = \Gamma\text{-}\liminf_{h\to\infty} F_h(x) = \sup_{U \in \mathcal{I}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y) \, ,$$

$$F^+(x) = \Gamma\text{-}\limsup_{h\to\infty} F_h(x) = \sup_{U \in \mathcal{I}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y) \, ,$$

per $x \in X$. Se $F^- = F^+ = F$ diremo che esiste il Γ-limite

$$F(x) = \Gamma\text{-}\lim_{h\to\infty} F_h(x) \, , \quad x \in X.$$

Negli Spazi metrici il Γ-limite si descrive in modo sequenziale.

<u>TEOREMA</u> Sia (X, d) uno spazio metrico, $F, F_h : X \to (-\infty, \infty]$ con $h \in \mathbb{N}$. Sono equivalenti:

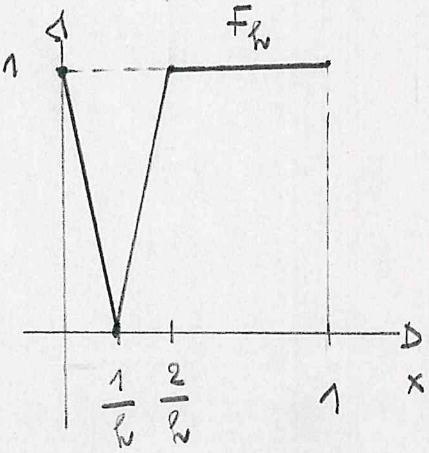
A) $F = \Gamma\text{-}\lim_{h\to\infty} F_h$;

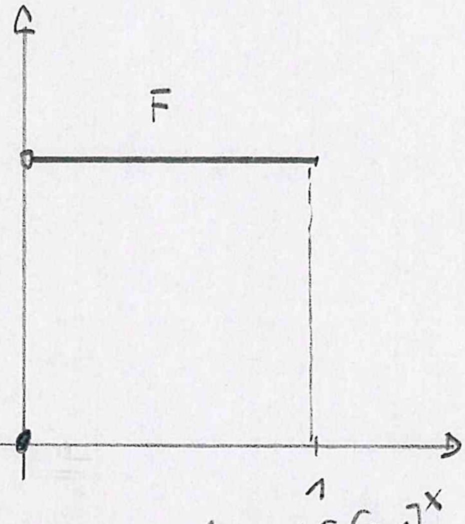
B) i) $\forall x \in X$ e $\forall\, x_h \xrightarrow[h\to\infty]{} x$: $F(x) \leq \liminf_{h\to\infty} F_h(x_h)$;

ii) $\forall x \in X \; \exists\, x_h \xrightarrow[h\to\infty]{} x$; $F(x) \geq \limsup_{h\to\infty} F_h(x_h)$.

Non proveremo il teorema e useremo B) come definizione di Γ-limite negli spazi metrici.

<u>ESEMPIO</u> Siano $F, F_h : [0,1] \longrightarrow \mathbb{R}$ le funzioni in figura:





$$F(x) = \begin{cases} 1 & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

Allora si ha

$$F = \Gamma\text{-}\lim_{h \to \infty} F_h .$$

Controlliamo i) e ii) nel punto $x = 0$:

i) $0 = F(0) \leq \liminf_{h \to \infty} \underbrace{F_h(x_h)}_{\geq 0} \qquad \forall x_h \to 0$

ii) Esiste $x_h \to 0$ tale che $\quad 0 = F(0) \geq \limsup_{h \to \infty} F_h(x_h)$.

Basta scegliere $x_h = \frac{1}{h}$.

$\square$

3) Convergenza dei minimi.

(X, d) Spazio Metrico

$F, F_h : X \to (-\infty, \infty]$, $h \in \mathbb{N}$

LEMMA $F = \Gamma\text{-}\lim_{h \to \infty} F_h$ e' sci su X.

DIM. Siano $x \in X$ e $x_h \to x$. Per ogni $h \in \mathbb{N}$ esiste $x_{k,h} \xrightarrow[k \to \infty]{} x_h$ tale che

$$F(x_h) \geq \limsup_{k \to \infty} F_k(x_{k,h})$$

e quindi $\forall h \; \exists k_h$ tale che

$$F(x_h) \geq F_k(x_{k,h}) - \frac{1}{h}, \qquad \forall k \geq k_h.$$

$$|x_{k,h} - x_h| < \frac{1}{h},$$

Dunque, con $k = k_h$

$$\liminf_{h \to \infty} F(x_h) \geq \liminf_{h \to \infty} F_{k_h}(x_{k_h,h}) \geq F(x)$$

in quanto $x_{k_h,h} \to x$,

$\square$

TEOREMA Sia X compatto e sia $F_h \geq C > -\infty$ $\forall h$. Se esiste $F = \Gamma\text{-}\lim_{h\to\infty} F_h$ allora F ha minimo su X e inoltre

$$\min_X F = \lim_{h\to\infty} \inf_X F_h .$$

DIM. Dall'ipotesi $F_h \geq C > -\infty$ deduciamo che $F(x) > -\infty$ per ogni $x \in X$. Siccome F è sci su X, possiede minimo; esiste $x_0 \in X$ tale che

$$F(x_0) = \min_{x\in X} F(x) .$$

Esiste $x_h \to x_0$ tale che

$$(1) \qquad F(x_0) \geq \limsup_{h\to\infty} F_h(x_h) \geq \limsup_{h\to\infty} \inf_X F_h .$$

D'altra parte esiste $x_h \in X$ tale che

$$a_h = F_h(x_h) \leq \inf_X F_h + \frac{1}{h} .$$

Affermiamo che per ogni s.s. $(a_{h_k})_{k\in\mathbb{N}}$ esiste una ulteriore s.s. $(a_{h_{k_j}})_{j\in\mathbb{N}}$ tale che

$$F(x_0) \leq \liminf_{j\to\infty} a_{h_{k_j}} . \qquad (*)$$

Siccome X è compatto, la successione $(x_{h_k})_{k \in \mathbb{N}}$ ha una sottosuccessione convergente

$$x_{h_{k_j}} \xrightarrow[j \to \infty]{} \bar{x} \in X .$$

Siccome $F = \Gamma\text{-}\lim\limits_{h \to \infty} F_h$ si ha

$$F(x_0) \leq F(\bar{x}) \leq \liminf_{j \to \infty} F_{h_{k_j}}(x_{h_{k_j}}) .$$ Questo prova (*).

Dalla affermazione in "} segue che

$$(2) \qquad F(x_0) \leq \liminf_{h \to \infty} F_h(x_h) \leq \liminf_{h \to \infty} \inf_X F_h .$$

Da (1) e (2) deriva la tesi:

$$F(x_0) = \lim_{h \to \infty} \inf_X F_h .$$

□

<u>TEOREMA</u> Sia (X,d) uno spazio metrico e siano $F, F_h : X \longrightarrow (-\infty, \infty]$ funzioni tali che $F = \Gamma\text{-}\lim\limits_{h \to \infty} F_h$.
Supponiamo esistano punti $x, x_h \in X$ tali che

i) $F_h(x_h) = \min\limits_X F_h$;

ii) $x_h \xrightarrow[h \to \infty]{} x$.

Allora si ha $F(x) = \min\limits_X F$.

DIM. Da un lato si ha:

$$F(x) \leq \liminf_{h \to \infty} F_h(x_h) = \liminf_{h \to \infty} \min_X F_h .$$

D'altra parte, per ogni $y \in X$ esiste $y_h \to y$ tale che

$$F(y) \geq \limsup_{h \to \infty} F_h(y_h) \geq \limsup_{h \to \infty} \min_X F_h \geq$$

$$\geq \liminf_{h \to \infty} F_h(x_h) \geq F(x).$$

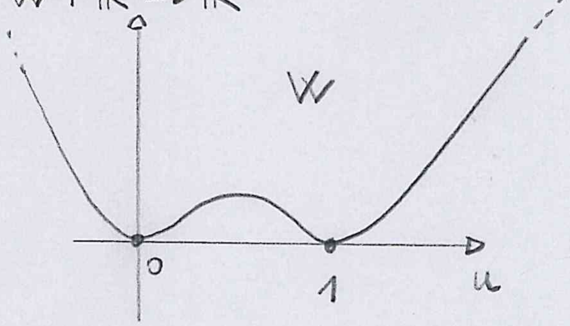
□

Funzionale di Modica-Mortola

Sia $A \subset \mathbb{R}^n$, $n \geq 1$, un insieme aperto e limitato.

Consideriamo il "potenziale" $W : \mathbb{R} \to \mathbb{R}$

$$W(u) = u^2 (1-u)^2$$



Fissiamo $0 < V < \mathcal{L}^n(A)$.

Consideriamo il problema di minimo

$$\min \left\{ \int_A W(u(x))\,dx \; : \; u \in L^1(A), \; \|u\|_1 = V \right\}.$$

Vogliamo separare la fase 0 (olio) dalla fase 1 (acqua).

Le soluzioni sono della forma $u = \chi_E$ con $\mathcal{L}^n(E) = V$.

Ci sono troppe soluzioni. Occorre un criterio di selezione.

Sia $\varepsilon > 0$ un parametro. Definiamo $F_\varepsilon : L^1(A) \to [0, \infty]$

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_A \left\{ \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right\} dx & \text{se } u \in H^1(A) \\ \infty & \text{se } u \in L^1(A) \setminus H^1(A). \end{cases}$$

Ricordiamo che $H^1(A) = \{ u \in L^2(A) \; ; \; |\nabla u| \in L^2(A) \}$.

gradiente debole

(Eventualmente: $F_\varepsilon(u) = \infty$ se $\|u\|_{L^1} \neq V$)

Poi definiamo $F : L^1(A) \longrightarrow [0,\infty]$

$$F(u) = \begin{cases} 2\alpha\, P(E; A) & \text{se } u = \chi_E, \\ \infty & \text{altrimenti}, \end{cases}$$

dove

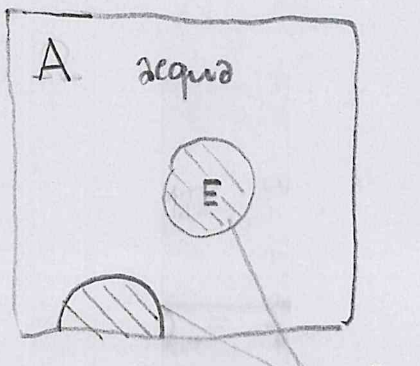
$$\alpha = \int_0^1 \sqrt{W(u)}\, du = \frac{1}{6}.$$

(Eventualmente: $F(u) = \infty$ se $\mathcal{L}^n(E) \neq V$)

<u>TEOREMA</u> Si ha $F = \Gamma\text{-}\lim_{\varepsilon \downarrow 0} F_\varepsilon$ in $L^1(A)$.

<u>COMMENTO</u> I punti di minimo di F_ε convergono per $\varepsilon \downarrow 0$ ai punti di minimo del perimetro (eventualmente: con vincolo di volume).

Le gocce di olio nell'acqua hanno forma sferica.



PREPARAZIONE EURISTICA Consideriamo il problema 1-dimensionale

$$\min \left\{ F_\varepsilon(\chi) \ : \ \chi : \mathbb{R} \to \overset{\mathbb{R}}{[0,1]} \ , \ \chi' \in L^2(\mathbb{R}) \quad \chi(-\infty) = 0 \quad \chi(\infty) = 1 \right\}$$

dove

$$F_\varepsilon(\chi) = \int_{-\infty}^{\infty} \left\{ \varepsilon \, \chi'^2 + \frac{1}{\varepsilon} W(\chi) \right\} dt \, .$$

Vogliamo andare da 0 a 1 con energia minima.

L'equazione di Eulero - Lagrange associata è

$$-2\varepsilon \chi_\varepsilon'' + \frac{1}{\varepsilon} W'(\chi_\varepsilon) = 0 \qquad \text{su } \mathbb{R}$$

Moltiplicando per χ_ε':

$$-\varepsilon (\chi_\varepsilon'^2)' + \frac{1}{\varepsilon} \left(W(\chi_\varepsilon) \right)' = 0$$

e integrando

$$-\varepsilon \chi_\varepsilon'^2 + \frac{1}{\varepsilon} W(\chi_\varepsilon) = \text{costante} = 0 \, .$$

Che debba essere costante $= 0$ si vede con $t \mapsto \pm\infty$.

In definitiva

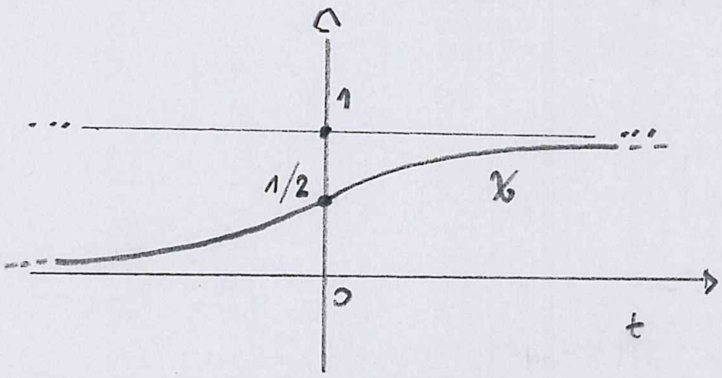
$$\chi_\varepsilon' = \frac{1}{\varepsilon} \sqrt{W(\chi_\varepsilon)} \, .$$

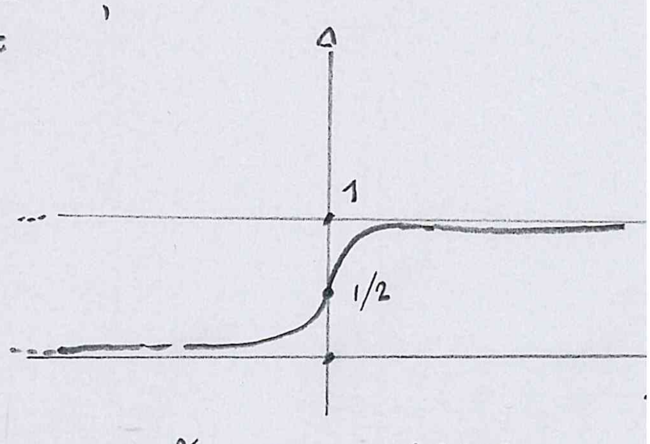
In effetti si ha $\chi_\varepsilon(t) = \chi(t/\varepsilon)$ dove

$$\begin{cases} x' = \sqrt{W(x)} \quad \text{su } \mathbb{R} \\ x(-\infty) = 0, \ x(\infty) = 1 \ \left(\Leftarrow \ x(0) = 1/2 \right). \end{cases}$$

La soluzione è:

$$x(t) = \frac{e^t}{1+e^t}, \quad t \in \mathbb{R}$$





x_ε con $0 < \varepsilon << 1$.

x_ε separa 0 da 1 in modo rapido quando $0 < \varepsilon << 1$.

DIM. Per semplicità ignoriamo il vincolo di volume.

Notazione: $\varepsilon = \varepsilon_h = 1/h$ con $\varepsilon \to 0^+ \Leftrightarrow h \to \infty$.

i) Siano $u \in L^1(A)$ e $u_\varepsilon \in L^1(A)$ tali che $u_\varepsilon \xrightarrow{L^1} u$.

Vogliamo provare che

$$F(u) \leq \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon).$$

Possiamo supporre che $\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) < \infty$.

Possiamo anche supporre che: $u_\varepsilon(x) \xrightarrow[\varepsilon \to 0^+]{} u(x)$ q.o.

Per il Lemma di Fatou:

$$\int_A W(u)\, dx \leq \liminf_{\varepsilon \to 0^+} \int_A W(u_\varepsilon)\, dx \leq$$

$$\leq \liminf_{\varepsilon \to 0^+} \varepsilon F_\varepsilon(u_\varepsilon) = 0 .$$

Quindi $W(u) = 0$ q.o. $\Rightarrow u \in \{0,1\}$ q.o.

$\Rightarrow u = \chi_E$.

Possiamo anche supporre che $0 \leq u_\varepsilon \leq 1$.
Infatti il troncamento continua a convergere ad u in $L^1(A)$ e inoltre l'energia diminuisce.

Per $t \in [0,1]$ definiamo

$$\varphi(t) = \int_0^t \sqrt{W(u)}\, du$$

e quindi definiamo

$$w(x) := \varphi(u(x)) = \alpha\, u(x) ,$$

$$w_\varepsilon(x) := \varphi(u_\varepsilon(x)) .$$

Con $L = \sup_{0 \leq u \leq 1} \sqrt{W(u)}$ si ha $|w_\varepsilon - w| \leq L |u_\varepsilon - u|$

e quindi $w_\varepsilon \xrightarrow[\varepsilon \to 0^+]{L^1(A)} w$.

Per la semicontinuità inferiore della variazione totale:

$$\alpha P(E;A) = \|\nabla w\|(A) \leq \liminf_{\varepsilon \to 0^+} \int_A |\nabla w_\varepsilon| \, dx =$$

$$= \liminf_{\varepsilon \to 0^+} \int_A \sqrt{W(u_\varepsilon)} \, |\nabla u_\varepsilon| \, dx$$

$$\leq \frac{1}{2} \liminf_{\varepsilon \to 0^+} \int_A \left\{ \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right\} dx =$$

$$= \frac{1}{2} \liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon).$$

Questo prova la prima condizione della Γ-convergenza.

ii) Ora sia $u \in L^1(A)$ e proviamo che esistono $u_\varepsilon \in L^1(A)$ tali che

$$F(u) \geq \limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon).$$

Basta considerare $u = \chi_E$ con $P(E;A) < \infty$.

In effetti basta considerare il caso

$$\partial E \cap A \text{ è } C^\infty \quad \left(e \quad H^{n-1}(\partial A \cap \partial E) = 0 \right).$$

Consideriamo la funzione distanza

$$\varrho(x) = \begin{cases} \operatorname{dist}(x; \partial E) & \text{se } x \in E \cap A \\ -\operatorname{dist}(x; \partial E) & \text{se } x \in A \setminus E \end{cases},$$

É noto che $|\nabla \varrho| = 1$ q.o. e $\varrho \in C^\infty$ in un intorno di ∂E.

Definiamo le funzioni

$$u_\varepsilon(x) = \chi_\varepsilon(\varrho(x)) = \chi(\varrho(x)/\varepsilon), \quad x \in A,$$

dove $\chi \in C^\infty(\mathbb{R})$ e' $\chi(t) = \dfrac{e^t}{1+e^t}$.

Usando la Formula di Coarea e $|\nabla \varrho| = 1$:

$$F_\varepsilon(u_\varepsilon) = \int_A \left\{ \varepsilon \left| \chi'(\varrho/\varepsilon) \frac{\nabla \varrho}{\varepsilon} \right|^2 + \frac{1}{\varepsilon} W(\chi(\varrho/\varepsilon)) \right\} dx$$

$$= \int_A \left\{ \frac{1}{\varepsilon} |\chi'(\varrho/\varepsilon)|^2 + \frac{1}{\varepsilon} W(\chi(\varrho/\varepsilon)) \right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\varepsilon} \left\{ |\chi'(t/\varepsilon)|^2 + W(\chi(t/\varepsilon)) \right\} \int_{\{x \in A \,:\, \varrho(x) = t\}} d\mathcal{H}^{n-1} \, dt$$

In definitiva si ottiene

$$F_\varepsilon(u_\varepsilon) = \int_{-\infty}^{\infty} \{ \chi'^2 + W(\chi) \} \, H^{n-1}(\{ \rho = \tau\varepsilon \} \cap A) \, d\tau .$$

É possibile provare (omettiamo la dimostrazione) che

$$\lim_{\varepsilon \to 0^+} H^{n-1}(\{ \rho = \tau\varepsilon \} \cap A) = H^{n-1}(\partial E \cap A)$$
$$= P(E;A),$$

e quindi

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) = P(E;A) \int_{-\infty}^{\infty} \{ \chi'^2 + W(\chi) \} \, d\tau$$

$$= P(E;A) \cdot 2 \int_{-\infty}^{\infty} W(\chi) \, d\tau \qquad (\chi(\tau) = u$$

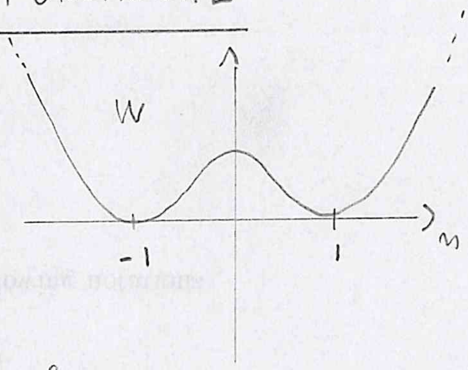
$$= P(E;A) \; 2 \int_0^1 \sqrt{W(u)} \, du .$$

$\square$

16

A CONJECTURE ON THE GINZBURG-LANDAU FUNCTIONAL

Consider the double-well potential



$$W(u) = \frac{1}{4}(1-u^2)^2$$

And the Ginzburg-Landau functional

$$F(u) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx .$$

If $u \in C^2(\mathbb{R}^n)$, $n \geq 2$, is a minimizer for compact variations, then it solves the Euler-Lagrange equation

$$\circledast \quad \Delta u = u^3 - u \quad \text{in } \mathbb{R}^n$$

(Entire solution.)

QUESTION (De Giorgi 1978) Let $u \in C^2(\mathbb{R}^n)$ be a solution to $\circledast$ such that

1) $|u| \leq 1$;

2) $\frac{\partial u}{\partial x_n} > 0$.

Is it true that for $n \leq 8$ the level sets $\{x \in \mathbb{R}^n ; u(x) = \lambda\}$ are hyperplanes for any $\lambda \in (0,1)$?

COMMENTS 1) The hyperplanes must be parallel and so if $N \in \mathbb{R}^n$ is the normal, we have

$$u(x) = \phi(\langle x, N \rangle)$$

and ϕ can be computed.

2) Hyperplanes are the unique entire minimal surfaces (minimal graphs) only if $n \leq 8$. (Simon's theorem)

So De Giorgi conjecture was motivated by Simon's theorem and by the Γ-convergence theorem that relates minimal surfaces and solutions to ⊛

ANSWERS:

1998 Yes if $n = 2$ (Ghoussoub - Gui)

2000 Yes if $n = 3$ (Ambrosio - Cabré)

2009 If $n \leq 8$ the answer is Yes with the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$$

(Savin)

2011 No if $n \geq 9$ (Del Pino Kowalczyk - Wei)

Introduction to Optimal Trasportation

Let μ, ν be two finite Borel measures on $\mathbb{R}^n$, with $n \geq 1$, and

1) $$\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) = 1.$$

A map $T : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \geq 1$, is Borel if

$$B \subset \mathbb{R}^m \text{ Borel} \Rightarrow T^{-1}(B) \subset \mathbb{R}^n \text{ Borel}.$$

Given a Borel measure μ on $\mathbb{R}^n$, we can define the push forward measure $T_\# \mu$ on $\mathbb{R}^m$ letting

2) $$T_\# \mu(B) = \mu(T^{-1}(B)) \quad \text{for } B \subset \mathbb{R}^m \text{ Borel}.$$

This is a Borel measure.

Exercise For any Borel function $f : \mathbb{R}^m \to [0, \infty)$ we have the change of variable formula

3) $$\int_{\mathbb{R}^m} f(y)\, dT_\#\mu(y) = \int_{\mathbb{R}^n} f(T(x))\, d\mu(x).$$

When $f = \chi_B$ with $B \subset \mathbb{R}^m$ Borel we have $f(T(x)) = 1$ if and only if $T(x) \in B$ if and only if $x \in T^{-1}(B)$. This means $f \circ T = \chi_{T^{-1}(B)}$ and formula 3) reduces to 2). The general formula 3) follows by approximation with simple functions.

Now we fix $n = m$ and we consider μ, ν as in 1).

Definition (Transport map) We say that $T \in \mathcal{T}(\mu, \nu)$ ("T maps μ to ν") if $T : \mathbb{R}^n \to \mathbb{R}^n$ is Borel and $T_\# \mu = \nu$.

Now we fix a "cost function" $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$.
We assume that c is continuous (for existence theorem "lower semicontinuos" is enough).

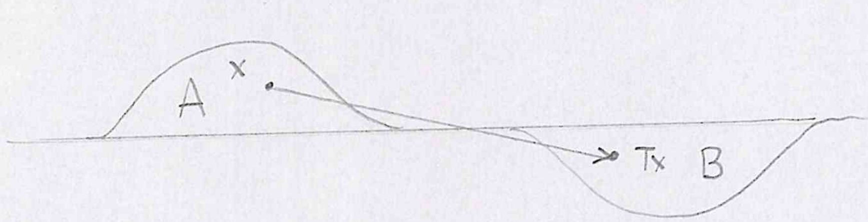
The Monge optimal trasportation problem is the following minimum problem

$$4) \qquad \min \left\{ \int_{\mathbb{R}^n} c(x, T(x)) \, d\mu(x) \; ; \; T \in \mathcal{T}(\mu, \nu) \right\}$$

The original Monge's problem was for the cost = distance case: $c(x,y) = |x-y|$;

$$5) \qquad \min \left\{ \int_A |x - Tx| \, dx \; ; \; T_\# \mathcal{L}^n \llcorner A = \mathcal{L}^n \llcorner B \right\}$$

where $\mu = \mathcal{L}^n \llcorner A$ and $\nu = \mathcal{L}^n \llcorner B$ with $\mathcal{L}^n(A) = \mathcal{L}^n(B)$:



In the following, we will focus on the quadratic cost:

$$6) \qquad c(x,y) = \frac{1}{2} |x-y|^2 .$$

The metric cost $c(x,y) = |x-y|$ is more difficult.

Remark (Relation with Monge-Ampère equation) Assume that

7) $$\mu = f\, \mathcal{L}^n \quad \text{and} \quad \nu = g\, \mathcal{L}^n$$

with $f, g \geq 0$ and $\| f \|_1 = \| g \|_1$. Assume that $\nu = T_{\#\mu}\mu$ where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of class C^1. Then for any Borel set $B \subset \mathbb{R}^n$ we have

$$\int_B f(x)\, dx = \mu(B) = \mu(T^{-1} T B)$$

8) $$= \nu(TB) = \int_{T(B)} g(y)\, dy \qquad \text{Change of variable } y = T(x)$$

$$= \int_B g(T(x))\, |\det JT(x)|\, dx .$$

Choosing $B = B_r(x)$, dividing by $\mathcal{L}^n(B_r)$ and letting $r \to 0^+$ we get

9) $$f(x) = g(T(x))\, |\det JT(x)| \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Now assume that $T(x) = \nabla \varphi(x)$ where $\varphi \in C^2(\mathbb{R}^n)$ is a convex function, $H\varphi(x) \geq 0$, the Hessian is non-negative. In this case equation 9) reads

10) $$\det H\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}, \quad x \in \mathbb{R}^n,$$

at least where $g(\nabla\varphi(x)) > 0$. This is Monge-Ampère equation.

Example 1 Let $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$ be distinct points and let $Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^n$ be distinct points.

Consider the measures

11)
$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \quad \text{sum of Dirac measures}$$
$$\nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i} .$$

Any bi-jection $T : X \to Y$ belongs to $\mathcal{T}(\mu, \nu)$.

For any cost $c : X \times Y \longrightarrow \mathbb{R}$ the problem

12)
$$\min \left\{ \int_X c(x, Tx)\, d\mu \ : \ T \in \mathcal{T}(\mu, \nu) \right\}$$

has a solution because we are minimizing $T \mapsto \int_X c(x, Tx)\, d\mu$ on a finite set.

Example 2 Let $X = \{x\} \subset \mathbb{R}^n$ be a singleton and $Y = \{y_1, y_2\}$ with $y_1 \neq y_2$. Take

13)
$$\mu = \delta_x \quad \text{and} \quad \nu = \frac{1}{2}\left(\delta_{y_1} + \delta_{y_2}\right).$$

Then we have $\mathcal{T}(\mu, \nu) = \emptyset$ because it is not possible for a map to split the mass concentrated on one point.

<u>Kantorovic formulation</u> Let μ, ν be two Borel probability measures on $\mathbb{R}^n$, $n \geq 1$. On $\mathbb{R}^n \times \mathbb{R}^n$ we consider the projections:

$$14) \qquad \begin{aligned} p^1(x,y) &= x \\ p^2(x,y) &= y, \end{aligned}$$

$p^1, p^2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

<u>Definition</u> We define the set of transport plans linking μ and ν as the (convex) set

$$15) \qquad \Pi(\mu,\nu) = \left\{ \pi : \begin{array}{l} \pi \text{ is a Borel probability measure} \\ \text{on } \mathbb{R}^n \times \mathbb{R}^n \text{ such that } p^1_{\#}\mu \text{ and } p^2_{\#}\nu \end{array} \right\}$$

<u>Remark</u> We always have $\Pi(\mu,\nu) \neq \emptyset$ because $\mu \otimes \nu \in \Pi(\mu,\nu)$

For a continuous cost $c : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ we define the functional

$$16) \qquad I(\pi) = \int_{\mathbb{R}^n \times \mathbb{R}^2} c(x,y)\, d\pi .$$

Kantorovic minimum problem is:

$$17) \qquad \min \left\{ I(\pi) \ ; \ \pi \in \Pi(\mu,\nu) \right\}.$$

The advantages are:

1) We look for a minimizer in the larger set of measures where we have nice compactness theorems;
2) the constraints are linear.

<u>Remark</u> For $T \in \mathcal{T}(\mu,\nu)$ consider the Borel map $Id \times T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ given by

18) $$Id \times T\,(x) = (x, Tx).$$

Then we have $\pi := (Id \times T)_\# \mu \in \Pi(\mu,\nu)$ and $spt(\pi) \subset gr(T)$, i.e.,

19) $$\pi\,(\mathbb{R}^n \times \mathbb{R}^n \setminus gr(T)) = 0.$$

As an exercise check these claims. We shall prove the converse:

<u>Lemma</u> If $\pi \in \Pi(\mu,\nu)$ and there exists a Borel map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $spt(\pi) \subset gr(T)$ then $T_\# \mu = \nu$, i.e., $T \in \mathcal{T}(\mu,\nu)$.

Proof. Let $B \subset \mathbb{R}^n$ be a Borel set:

$$T_\#\mu\,(B) = \mu(T^{-1}B) \qquad \text{we use } \mu = p^1_\# \pi$$

20) $$= p^1_\#\pi\,(T^{-1}B)$$

$$= \pi\,(T^{-1}B \times \mathbb{R}^n)$$

$$= \pi\left(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\ Tx \in B \text{ and } y \in \mathbb{R}^n\}\right)$$

we use $spt(\pi) \subset gr(T)$

$$= \pi\left(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n:\ Tx \in B \text{ and } y = Tx\}\right)$$

$$\overset{spt(\pi) \subset gr(T)}{=} \pi\left(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n:\ x \in \mathbb{R}^n \text{ and } y \in B\}\right)$$

$$= \pi\,(\mathbb{R}^n \times B)$$

$$= p^2_\#\pi\,(B) = \nu(B).$$

□

<u>Example</u> Let $X = \{x_1, \dots, x_N\}$ and $Y = \{y_1, \dots, y_N\}$ as above and fix $p_i, q_i \in (0,1)$ such that

21) $$\sum_{i=1}^{N} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{N} q_i.$$

The measures

22) $$\mu = \sum_{i=1}^{N} p_i \delta_{x_i} \quad \text{and} \quad \nu = \sum_{i=1}^{N} p_i \delta_{y_i}$$

are Borel probability measures. A measure π on $X \times Y$ is uniquely defined by the matrix $(\pi_{ij})_{i,j=1,\dots,N}$

23) $$\pi_{ij} = \pi(\{(x_i, y_j)\}) \in [0,\infty)$$

The condition $p^1_\# \pi = \mu$ is equivalent to

24) $$p_i = \mu(\{x_i\}) = p^1_\# \pi(\{x_i\}) = \pi(\{x_i\} \times Y)$$
$$= \sum_{j=1}^{N} \pi_{ij}$$

and $p^2_\# \pi = \nu$ is equivalent to

25) $$q_j = \sum_{i=1}^{N} \pi_{ij}.$$

The set $\Pi(\mu, \nu) \subset M_n(\mathbb{R})$ is (convex and) compact. The mapping $I : \Pi(\mu,\nu) \to [0,\infty)$

26) $$I(\pi) = \int_{X \times Y} c(x,y)\, d\pi = \sum_{i,j=1}^{N} c(x_i, y_j) \pi_{ij}$$

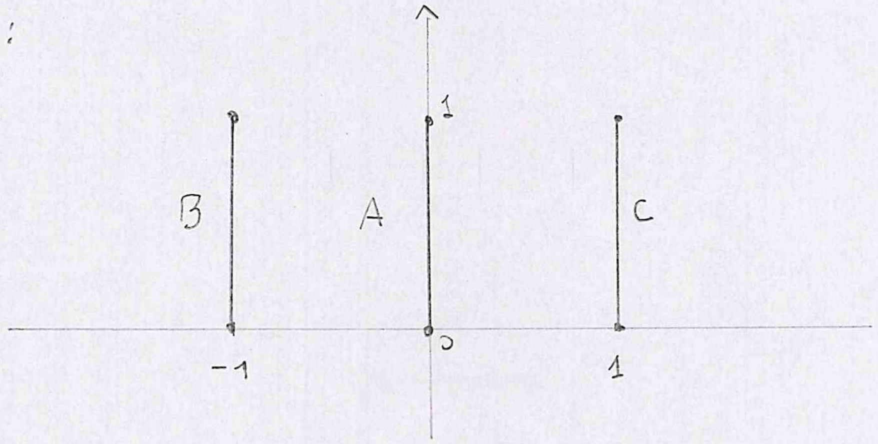
is linear (continuous) for any cost $c : X \times Y \to \mathbb{R}$. The minimum

27) $$\min\{ I(\pi) ;\ \pi \in \Pi(\mu,\nu)\}$$

is attained by elementary arguments.

$\square$

Example In $\mathbb{R}^2$ let A, B, C be the three sets as in the picture:



Consider the measures

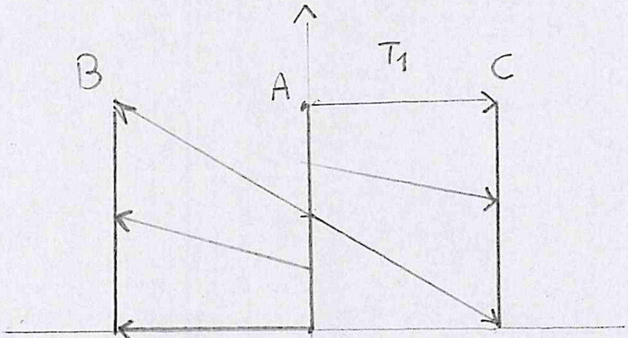
28) $$\mu = H^1 \llcorner A \quad \text{and} \quad \nu = \frac{1}{2}\left(H^1 \llcorner B + H^1 \llcorner C\right)$$

For any $T \in \mathcal{T}(\mu, \nu)$ we have

29) $$\int_A |x - Tx|^2 \, dH^1 \geq 1$$

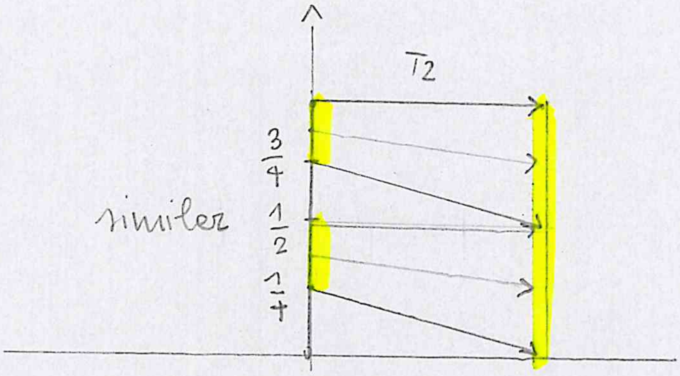
because $|x - Tx| \geq 1$ $\forall x \in A$, as $Tx \in B \cup C$.

Let $T_1 : A \to B \cup C$ be as in the picture



Then we have $|x - T_1 x|^2 \leq 1 + \frac{1}{4}$.

Iterate the construction in the following way:



Now we have $|x - T_2 x|^2 \leq 1 + \frac{1}{4^2}$.

In general, we construct Borel maps $T_n : A \to BUC$ such that

30) $$|x - T_n x|^2 \leq 1 + \left(\frac{1}{2^n}\right)^2$$

It follows that

31) $$\inf \left\{ \int_A |x - Tx|^2 \, dH^1 : T \in \mathcal{T}(\mu,\nu) \right\} = 1.$$

But there exists no $T \in \mathcal{T}(\mu,\nu)$ such that

32) $$\int_A \underbrace{|x - Tx|^2}_{\geq 1} \, dH^1 = 1.$$

This would imply $|x - Tx| = 1$ for all $x \in A$, with $Tx \in BUC$, and this is not possible.

Exercise On $\mathbb{R}^2 \times \mathbb{R}^2$ let $\pi_n := (Id \times T_n)_{\#}\mu$, $n \in \mathbb{N}$. Compute the limit (weak$*$ limit of Radon measures):

$$\lim_{n\to\infty} \pi_n.$$

Remark. Assume that $\pi = (Id \times T)_{\#}\mu$, then we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d\pi = \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d(Id \times T)_{\#}\mu \tag{33}$$

$$= \int_{\mathbb{R}^n} c(x,Tx)\, d\mu.$$

Corollary We always have

34) Kantorovic infimum $\leq$ Monge infimum.

Moreover, if the Kantorovic minimum exists and is of the form $\pi = (Id \times T)_{\#}\mu$ then it is also a Monge minimum

Theorem (Existence) Let μ, ν be two Borel probability measures on $\mathbb{R}^n$ with compact support and let $c: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be continuous. Then the Kantorovic minimum exists (and is finite).

When the measures do not have compact support, further assumptions are needed to ensure compactness for the problem.

The proof relies upon the weak-$*$ compactness theorem for Radon measures.

Theorem (Weak-* compactness) Let $(\mu_h)_{h\in\mathbb{N}}$ be a sequence of Borel measures in $\mathbb{R}^n$ such that

$$35) \qquad \sup_{h\in\mathbb{N}} \mu_h(\mathbb{R}^n) < \infty .$$

Then there exist a subsequence $(\mu_{h_k})_{k\in\mathbb{N}}$ and a Borel measure μ such that $\mu_{h_k} \xrightarrow[k\to\infty]{*} \mu$, i.e.,

$$36) \qquad \lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x)\, d\mu_{h_k} = \int_{\mathbb{R}^n} \varphi(x)\, d\mu$$

for all $\varphi \in C_c(\mathbb{R}^n)$.

See e.g. Evans - Gariepy book.

Remark The following are equivalent:

A) $\mu_h \xrightarrow[h\to\infty]{*} \mu$

B) i) $\liminf_{h\to\infty} \mu_h(A) \geq \mu(A) \quad \forall A \subset \mathbb{R}^n$ open

ii) $\limsup_{h\to\infty} \mu_h(K) \leq \mu(K) \quad \forall K \subset \mathbb{R}^n$ compact

C) $\lim_{h\to\infty} \mu_h(B) = \mu(B) \quad \forall B \subset \mathbb{R}^n$ Borel with $\mu(\partial B) = 0$.

Proof of Existence Theorem Let H, K be compact sets such that $\mathrm{spt}(\mu) \subset H$ and $\mathrm{spt}(\nu) \subset K$. If $\pi \in \Pi(\mu,\nu)$ then we have $\mathrm{spt}(\pi) \subset H \times K$.

Let $\pi_h \in \Pi(\mu,\nu)$ be a minimizing sequence for Kantorovic problem;

$$\lim_{h\to\infty} I(\pi_h) = \lim_{h\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\, d\pi_h$$

37)
$$= m := \inf\{ I(\pi) : \pi \in \Pi(\mu,\nu)\} < \infty .$$

By weak-* compactness there exists a sub-sequence - still denoted π_h - such that

38)
$$\pi_h \xrightarrow[h\to\infty]{*\text{-weak}} \pi$$

for some Borel measure π. We have

39)
$$\left.\begin{array}{l} spt(\pi_h) \subset H\times K \ \forall h \\ \pi_h \xrightarrow{*} \pi \end{array}\right\} \Rightarrow spt(\pi) \subset H\times K .$$

Without loss of generality we can assume that $c \in C_c(\mathbb{R}^n\times\mathbb{R}^n)$ has compact support. By weak-* convergence we have

40)
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\, d\pi = \lim_{h\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\, d\pi_h = m .$$

<u>Exercise</u> Show that $p^1_{\#}\pi = \mu$ and $p^2_{\#}\pi = \nu$.

Hint: Use $\pi_h(B) \longrightarrow \pi(B)$ for B Borel with $\pi(\partial B)=0$.

The Exercise implies that π is a Kantorovic minimum. □

Kantorovic - Rubinstein duality

For a fixed cost function $c = c(x,y)$, consider the set of functions

41) $$\Phi = \left\{ (\varphi,\psi) : \varphi \in L^1(\mathbb{R}^n;\mu),\ \psi \in L^1(\mathbb{R}^n;\nu) \right.$$
$$\left. \varphi(x) + \psi(y) \leq c(x,y) \text{ for } x,y \in \mathbb{R}^n \right\}.$$

Define the functional $J : \Phi \to \mathbb{R}$

42) $$J(\varphi,\psi) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu + \int_{\mathbb{R}^n} \psi(y)\, d\nu .$$

For $\pi \in \Pi(\mu,\nu)$ we have the identity

43) $$J(\varphi,\psi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\varphi(x) + \psi(y))\, d\pi(x,y) .$$

It follows that

44) $$\sup_{(\varphi,\psi)\in\Phi} J(\varphi,\psi) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d\pi = I(\pi)$$

and thus

45) $$\sup_{(\varphi,\psi)\in\Phi} J(\varphi,\psi) \leq \inf_{\pi\in\Pi(\mu,\nu)} I(\pi) ,$$

In fact we have equality.

Theorem (Kantorovic - Rubinstein) Siano μ, ν due misure di Borel con $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) = 1$ e sia $c: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ continua. Allora

46) $$\sup_{(\varphi,\psi)\in\Phi} J(\varphi,\psi) = \inf_{\pi\in\Pi(\mu,\nu)} I(\pi).$$

The proof is omitted.

Quadratic cost

From now on we assume that $c: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is the qudratic cost

47) $$c(x,y) = \frac{1}{2}|x-y|^2.$$

In this case we have

$$I(\pi) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \left(|x|^2 - 2\langle x,y\rangle + |y|^2\right) d\pi$$

48)

$$= \frac{1}{2}\int_{\mathbb{R}^n} |x|^2 d\mu + \frac{1}{2}\int_{\mathbb{R}^n} |y|^2 d\nu - \int_{\mathbb{R}^n\times\mathbb{R}^n} \langle x,y\rangle\, d\pi.$$

The quantity

49) $$M := \frac{1}{2}\int_{\mathbb{R}^n} |x|^2 d\mu + \frac{1}{2}\int_{\mathbb{R}^n} |y|^2 d\nu$$

does not depend on π.

<u>Remark</u> For a general μ we need to assume

50) $$\int_{\mathbb{R}^n} |x|^2 \, d\mu < \infty ,$$

(and the same for ν). In this case we say that μ has finite 2-moment.

We transform J in the following way:

$$J(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(x) \, d\mu + \int_{\mathbb{R}^n} \psi(y) \, d\nu$$

51) $$= M - \int_{\mathbb{R}^n} \left(\frac{|x|^2}{2} - \varphi(x) \right) d\mu - \int_{\mathbb{R}^n} \left(\frac{|y|^2}{2} - \psi(y) \right) d\nu$$

$$= M - \int_{\mathbb{R}^n} \bar{\varphi}(x) \, d\mu - \int_{\mathbb{R}^n} \bar{\psi}(y) \, d\nu .$$

The new functions $\bar{\varphi}$ and $\bar{\psi}$ satisfy:

52) $$\bar{\varphi}(x) + \bar{\psi}(y) = \frac{1}{2}|x|^2 - \varphi(x) + \frac{1}{2}|y|^2 - \psi(y) \geq \frac{1}{2}\left(|x|^2 + |y|^2 \right) - \frac{1}{2}|x-y|^2 = \langle x, y \rangle ,$$

<u>Corollary</u> Let μ, ν, c be as in K-R Theorem. Then formula 46) reads:

53) $$\inf_{\substack{(\bar\varphi,\bar\psi)\\ \bar\varphi(x)+\bar\psi(y)\geq\langle x,y\rangle}} J(\bar\varphi,\bar\psi) = \sup_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n} \langle x,y\rangle \, d\pi .$$

<u>Remark</u>. The sup in the RHS is a maximum, at least when μ,ν have compact support.

<u>Elements of convex analysis</u>

Let $\varphi: \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be any function, $f \not\equiv +\infty$. Its Legendre transform is $\varphi^*: \mathbb{R}^n \to (-\infty,+\infty]$

54) $$\varphi^*(y) = \sup_{x\in\mathbb{R}^n} \langle y,x\rangle - \varphi(x).$$

This is a convex and LSC function.

If there are $y\in\mathbb{R}^n$ and $\varphi_0\in\mathbb{R}$ such that $\varphi(x) \geq \langle y,x\rangle - \varphi_0$ then $\varphi^*(y) \leq \varphi_0 < \infty$ and thus $\varphi^* \not\equiv +\infty$.

The function $\varphi^{**} = (\varphi^*)^*$ satisfies (Fenchel-Moreau Theorem)

i) $\varphi^{**} \leq \varphi$

ii) $\varphi^{**} = \varphi \iff \varphi$ convex and LSC.

We have the universal inequality

55) $$\varphi(x) + \varphi^*(y) \geq \langle x,y\rangle \quad \forall x,y\in\mathbb{R}^n.$$

The domain of φ is $D(\varphi) = \{ x \in \mathbb{R}^n ; \varphi(x) < \infty \}$.

For $x \in D(\varphi)$ define the sub-differential

56) $$\partial\varphi(x) = \{ z \in \mathbb{R}^n ; \quad \varphi(y) \geq \varphi(x) + \langle z, y-x \rangle \quad \forall y \in \mathbb{R}^n \}.$$

If φ is convex then $\underset{x \in \operatorname{int}(D(\varphi))}{\partial\varphi(x) \neq \phi}$ is convex and compac

If φ is convex and differentiable at $x \in \operatorname{int}(D(\varphi))$

57) $$\partial\varphi(x) = \{ \nabla\varphi(x) \}.$$

Conversly, for φ convex;

58) $\partial\varphi(x)$ singlton $\Rightarrow$ φ differentiable at x.

— * — * —

Let $(\bar{\varphi}, \bar{\psi})$ be a pair of funcions such that

59) $$\bar{\varphi}(x) + \bar{\psi}(y) \geq \langle x, y \rangle$$

We may assume $\bar{\varphi}$ bounded. Consider its Legendre transform

60) $$\bar{\varphi}^*(y) = \sup_{x \in \mathbb{R}^n} \langle x, y \rangle - \bar{\varphi}(x)$$

By 59) we have

61) $$\bar{\varphi}^*(y) \leq \bar{\psi}(y).$$

It follows that

62) $$\int_{\mathbb{R}^n} \bar{\varphi}(x)\, d\mu + \int_{\mathbb{R}^n} \bar{\psi}(y)\, d\nu = J(\bar{\varphi}, \bar{\psi}) \geq J(\bar{\varphi}, \bar{\varphi}^*) \geq$$
$$\geq J(\bar{\varphi}^{**}, \bar{\varphi}^*)$$

because $\bar{\varphi} \geq \bar{\varphi}^{**}$. So the inf in 53) is optimized by pairs of convex functions conjugate to each other.

Now we have this Lemma:

<u>Lemma</u> The infimum

63) $$\inf_{(\bar{\varphi}^*, \bar{\varphi}^{**})} J(\bar{\varphi}^*, \bar{\varphi}^{**})$$

is a minimum.

Proof: See Villani's book. It relies upon Ascoli-Arzelà theorem.

Let $\varphi := \bar{\varphi}^*$ be a convex function realizing the min in 63).

Let $\pi \in \Pi(\mu, \nu)$ be an optimal transport plan.

Then identity 53) reads

64) $$\int_{\mathbb{R}^n \times \mathbb{R}^n} (\varphi(x) + \varphi^*(y))\, d\pi = \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle x, y\rangle\, d\pi,$$

that is

65) $$\int_{\mathbb{R}^n \times \mathbb{R}^n} \underbrace{(\varphi(x) + \varphi^*(y) - \langle x, y\rangle)}_{\geq 0 \quad \forall x, y} d\pi = 0.$$

We deduce that for π-a.e. $(x,y) \in \mathbb{R}^n$

66) $$\langle x, y\rangle = \varphi(x) + \varphi^*(y) \geq \varphi(x) + \langle y, z\rangle - \varphi(z). \quad \forall z \in \mathbb{R}^n$$

This is equivalent to:

67) $$\varphi(z) \geq \varphi(x) + \langle y, z-x \rangle \qquad \forall z \in \mathbb{R}^n$$

which means: $y \in \partial\varphi(x)$.

The conclusion is that there exists $N \subset \mathbb{R}^n \times \mathbb{R}^n$ with

i) $\pi(N) = 0$

ii) $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus N \Rightarrow y \in \partial\varphi(x)$.

The function $\varphi : \mathbb{R}^n \to \mathbb{R}$ (we are assuming: $D(\varphi) = \mathbb{R}^n$) is differentiable $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$. So there is $N_1 \subset \mathbb{R}^n$ such that

j) $\mathcal{L}^n(N_1) = 0$

jj) $x \in \mathbb{R}^n \setminus N_1 \Rightarrow \partial\varphi(x) = \{\nabla\varphi(x)\}$.

Finally, assume that $\mu << \mathcal{L}^n$. In this case:

68) $$0 = \mu(N_1) = p^1_{\#}\pi(N_1) = \pi(N_1 \times \mathbb{R}^n).$$

The conclusion is:

69) $$(x,y) \notin \underbrace{N \cup N_1 \times \mathbb{R}^n}_{\text{has } 0 \ \pi\text{-measure}} \Rightarrow y = \nabla\varphi(x).$$

The mapping $T : \mathbb{R}^n \to \mathbb{R}^n$

70) $$T(x) = \begin{cases} \nabla\varphi(x) & x \in \mathbb{R}^n \setminus N_1 \\ 0 & \text{otherwise} \end{cases}$$

is Borel.

Condition 63) means that $y = T(x)$ for π-a.e. $(x,y) \in \mathbb{R}^n$, i.e., $spt(\pi) \subset gr(T)$. We know that in this case:

$$\pi = (Id \times T)_\# \mu .$$

We summarize our discussion in the following theorem.

Theorem (Brenier Knott-Smith) Let μ, ν be two Borel probability measures with compact support (with finite 2-momen Assume that $\mu \ll \mathcal{L}^n$. Then Monge problem

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |x - Tx|^2 \, d\mu \; : \; T_\# \mu = \nu \right\}$$

has a solution $T = \nabla \varphi$ where $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex LSC function.

(Brenier)

TEOREMA Siano μ, ν due misure di Borel in $\mathbb{R}^n$ con supporto compatto, $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) < \infty$ e $\mu \ll \mathcal{L}^n$. Allora esiste una funzione convessa $\varphi : \mathbb{R}^n \to \mathbb{R}$ tale che $T = \nabla \varphi$ realizza il minimo

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |x - Tx|^2 \, d\mu \; : \; T_\# \mu = \nu \right\}.$$

APPLICAZIONE ALLA DISUGUAGLIANZA ISOPERIMETRICA

Sia $B = \{ x \in \mathbb{R}^n : |x| < 1 \}$ la palla unitaria e sia $A \subset \mathbb{R}^n$ un insieme limitato e aperto.

Consideriamo le misure

$$\mu = \mathcal{L}^n \llcorner A$$

$$\nu = \mathcal{L}^n \llcorner B$$

con l'ipotesi $\mathcal{L}^n(A) = \mathcal{L}^n(B)$.

Esiste $\varphi : \mathbb{R}^n \to \mathbb{R}$ tale che $T = \nabla \varphi$ verifica $T_\# \mu = \nu$ (ed è un minimo con costo quadratico)

ASSUMIAMO che T sia un diffeomorfismo di classe C^1 da A in B.

Per provarlo occorre la teoria della regolarità.

Allora avremo: (JT = Jacobiano di T)

1) $T(x) \in B$ per ogni $x \in A$

2) $|\det JT(x)| = 1$ per ogni $x \in A$.

Possiamo dire che $\det JT(x) = +1$ per $x \in A$.

Ora useremo la disuguaglianza:

$$\circledast \quad \det(M)^{\frac{1}{n}} \leq \frac{1}{n}\,\mathrm{tr}(M)$$

per ogni matrice M simmetrica $n \times n$.

(È la disuguaglianza media geom. $\leq$ media aritm.)

Inoltre se c'è $=$ in $\circledast$ allora gli autovalori di M sono tutti uguali fra loro ed in particolare M è diagonale.

Dunque:

$$\mathcal{L}^n(A) = \int_A \det JT(x)^{\frac{1}{n}}\,dx \overset{\circledast}{\leq} \frac{1}{n}\int_A \mathrm{div}\,T(x)\,dx \leq$$

$$\leq \frac{1}{n}\int_{\partial A} \langle T(x), \nu(x)\rangle\, dH^{n-1} \leq$$

$|T(x)| \leq 1$

ν = Normale Esterna a ∂A

$$\leq \frac{1}{n} H^{n-1}(\partial A).$$

Per la palla sappiamo che $\mathcal{L}^n(B) = \frac{1}{n} \mathcal{H}^{n-1}(\partial B)$,

Deduciamo che

$$\mathcal{H}^{n-1}(\partial B) \leq \mathcal{H}^{n-1}(\partial A).$$

Commento Euristico:

Se poi è $\mathcal{H}^{n-1}(\partial B) = \mathcal{H}^{n-1}(\partial A)$ allora sopra nel punto ⊛ c'è un "=" e quindi

$$1 = \det JT(x)^{\frac{1}{n}} = \frac{1}{n} \operatorname{tr}(JT(x))$$

per $x \in A$. Deduciamo che $JT(x) = Id = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

ovvero $T(x) = x_0 + x$, per $x_0 \in \mathbb{R}^n$.

In altri termini T è una traslazione e quindi A è una palla.

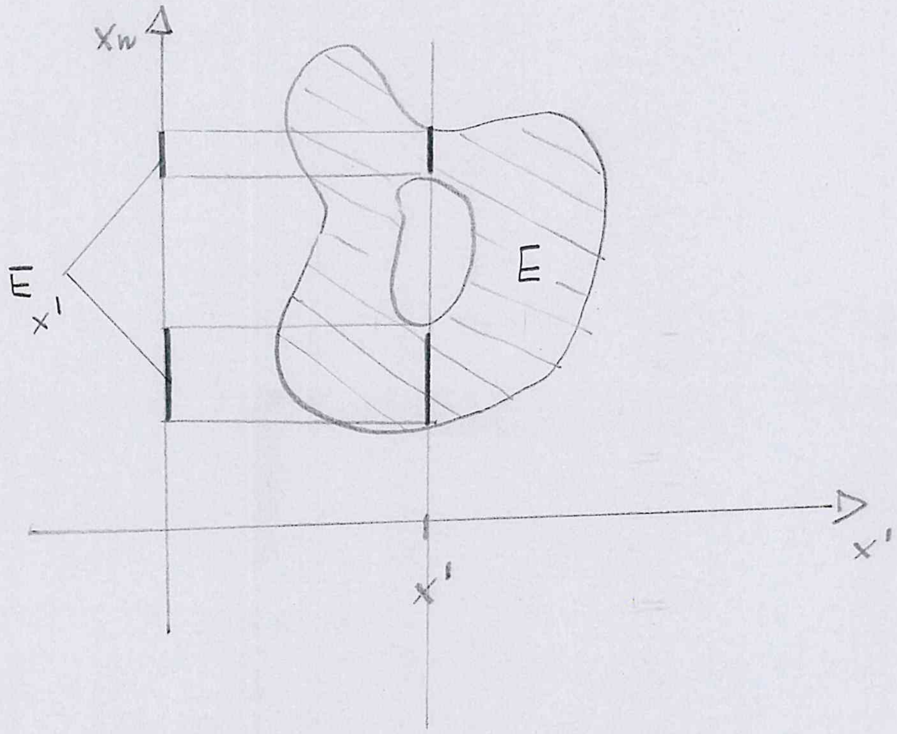
Tuttavia $JT(x) = Id$ solo a meno di una isometria che dipende dal punto x.

SIMMETRIZZAZIONE DI STEINER

Coordinate in $\mathbb{R}^n$, $n \geq 2$: $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Per $E \subset \mathbb{R}^n$ ed $x' \in \mathbb{R}^{n-1}$ definiamo le sezioni:

$$E_{x'} = \{ x_n \in \mathbb{R} : (x', x_n) \in E \}$$



Sia $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-misurabile con $\mathcal{L}^n(E) < \infty$.

Definiamo la funzione $\vartheta : \mathbb{R}^{n-1} \to [0, \infty)$

$$\vartheta(x') = \begin{cases} \mathcal{L}^1(E_{x'}) & \text{se } \mathcal{L}^1(E_{x'}) < \infty \\ 0 & \text{altrimenti.} \end{cases}$$

Fubini-Tonelli $\Rightarrow$ $\vartheta \in L^1(\mathbb{R}^{n-1})$.

<u>DEFINIZIONE</u> Dato $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-misurabile con $\mathcal{L}^n(E) < \infty$, l'insieme

$$E^* = \left\{ x = (x', x_n) \in \mathbb{R}^n : |x_n| < \frac{1}{2} \vartheta(x') \right\}$$

si chiama riarrangiamento di Steiner di E.

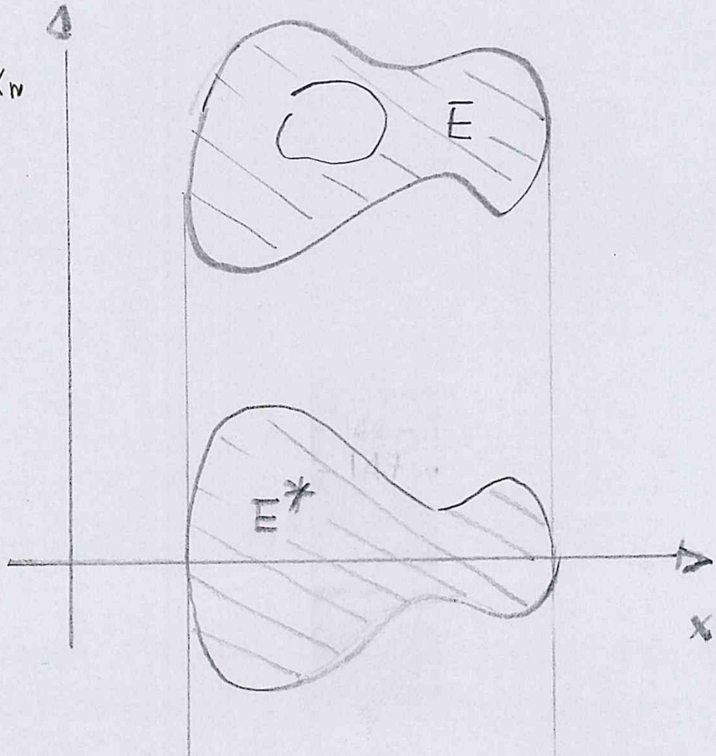
<u>OSSERVAZIONI</u>

(1) E^* è $\mathcal{L}^n$-misurabile ed $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$

(2) E^* è x_n-simmetrico:

$$(x', x_n) \in E^* \iff (x', -x_n) \in E^*.$$

(3) E^* è x_n-normale, ovvero gli insiemi $E_{x'} \subset \mathbb{R}$ sono intervalli.



<u>ESERCIZIO</u> Provare che $\operatorname{diam}(E^*) \leq \operatorname{diam}(E)$.

Proveremo il seguente teorema.

<u>TEOREMA</u> Sia $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-misurabile con misura finita. Allora:

$$P(E^*) \leq P(E).$$

Inoltre, se $P(E^*) = P(E)$ allora E è equivalente ad un insieme x_n-normale.

Dim. Per $i = 1, \dots, n$ ed $A \subset \mathbb{R}^n$ aperto definiamo i perimetri parziali:

$$P_i(E; A) = \sup\left\{ \int_E \frac{\partial \varphi}{\partial x_i}(x)\,dx : \varphi \in C_c^1(A), \|\varphi\|_\infty \leq 1 \right\}$$

Esistono $\mu_1, \dots, \mu_n$ misure di Borel finite su $\mathbb{R}^{n-1}$ tali che per ogni $A \subset \mathbb{R}^{n-1}$ aperto si abbia

$$\mu_i(A) = P_i(E; A \times \mathbb{R}), \quad i = 1 \dots n.$$

In modo analogo esistono misure di Borel μ_i^*, $i=1,\dots,n$, tali che

$$\mu_i^*(A) = P_i(E^*; A\times\mathbb{R})$$

con $A\subset\mathbb{R}^{n-1}$ aperto.

<u>CLAIM:</u> $\mu_i^*(A) \le \mu_i(A)$ $\forall i=1,\dots,n$
$\forall A\subset\mathbb{R}^{n-1}$ aperto.

Partiamo dal caso $i=n$. Abbiamo:

$$\mu_n(A) = P_n(E; A\times\mathbb{R}) \le P(E; A\times\mathbb{R}) \le$$
$$\le P(E;\mathbb{R}^n) \underset{\substack{\uparrow\\ \text{Possiamo supporre}}}{<} \infty$$

Poi si ha:

$$\mu_n(A) = \sup_{\substack{\varphi\in C_c^1(A\times\mathbb{R})\\ |\varphi|\le 1}} \int_{(A\times\mathbb{R})\cap E} \frac{\partial\varphi}{\partial x_n}(x', x_n)\,dx$$

$(*)$

$$= \sup_{''} \int_A \int_{E_{x'}} \frac{\partial\varphi}{\partial x_n}(x', x_n)\,dx_n\,dx' \ge$$

$$\overset{(*)}{\underset{(=)}{\geq}} \int_A \left(\sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \leq 1}} \int_{E_{x'}} \frac{\partial \varphi}{\partial x_n}(x_n)\, dx_n \right) dx'.$$

ESERCIZIO Provare la disuguaglianza (*) (e dedurre quindi che è un =).

Fare così:

(1) Provare (*) quando χ_E è sostituita da $\chi \in C^\infty_c(\mathbb{R}^n)$

(2) Approssimare χ_E con funzioni $C^\infty_c(\mathbb{R}^n)$ e passare al limite.

Ora affermiamo che per $\mathcal{L}^{n-1}$-q.o. $x' \in A$ si ha

$$\underbrace{\sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \leq 1}} \int_{E_{x'}} \frac{\partial \varphi}{\partial x_n}(x_n)\, dx_n}_{= P(E_{x'})} \overset{(**)}{\geq} \underbrace{\sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \leq 1}} \int_{E^*_{x'}} \frac{\partial \varphi}{\partial x_n}(x_n)\, d}_{= P(E^*_{x'})}$$

Da (*) e (**) segue che $\mu_n(A) \geq \mu_n^*(A)$.

Proviamo (**). Per $\mathcal{L}^{n-1}$-q.o. $x' \in A$ si ha

$$\mathcal{L}^1(E_{x'}) < \infty \quad e$$

$$P(E_{x'}) < \infty .$$

Quindi (ESERCIZIO) $E_{x'}$ è equivalente ad una unione finita di intervalli:

$$E_{x'} = \bigcup_{j=1}^{k} (a_j', b_j') \subset \mathbb{R}$$

con $k \geq 0$ e dunque $P(E_{x'}) = 2k$.

Se $\mathcal{L}^1(E_{x'}) > 0$ deve essere $k \geq 1$.

In questo caso $P(E^*_{x'}) = 2$ (altrimenti $= 0$)

Questo prova (**).

Ora proviamo il CLAIM quando $i = 1, 2, \dots, n-1$.

Abbiamo

$$\mu_i(A) = P_i(E; A\times\mathbb{R}) = \sup_{\substack{\varphi\in C^1_c(A\times\mathbb{R})\\ \|\varphi\|_\infty\le 1}} \int_E \frac{\partial\varphi}{\partial x_i}(x)\,dx$$

$$\geq \sup_{\substack{\varphi\in C^1_c(A)\\ \|\varphi\|_\infty\le 1}} \int_A \vartheta(x')\,\frac{\partial\varphi}{\partial x_i}(x')\,dx', = (\ast)$$

dove $\vartheta(x') = \mathcal{L}^1(E_{x'})$. Deduciamo in particolare che $\vartheta \in BV(A)$. Supponendo per un attimo che $\vartheta\in C^1(A)$ si trova:

$$(\ast) = \int_A \left|\frac{\partial\vartheta}{\partial x_i}(x')\right| dx' = \text{[Formula di Coarea]}$$

$$= \int_0^\infty \left(\int_{A\cap\partial\{\vartheta>t\}} d\lambda_i\right) dt \qquad \lambda_i = \text{misura perimetro parziale } i\text{-esim}$$

$$= \int_0^\infty \left(\sup_{\substack{\varphi\in C^1_c(A)\\ \|\varphi\|_\infty\le 1}} \int_{\{x'\in\mathbb{R}^{n-1}:\,\vartheta(x')>t\}} \frac{\partial\varphi}{\partial x_i}(x')\,dx'\right) dt$$

Questi passaggi possono essere "formalizzati" quando $\vartheta\in BV(A)$. Deduciamo che $(t = 2x_n)$

$$(\ast) \geq 2 \sup_{\substack{\varphi \in C^1_c(A\times\mathbb{R}) \\ \|\varphi\|_\infty \leq 1}} \int_0^\infty \int_{\{x'\in\mathbb{R}^{n-1}:\frac{1}{2}\vartheta(x')>x_n\}} \frac{\partial\varphi}{\partial x_i}(x', x_n)\, dx'\, dx_n$$

$$= 2 P_i(E^*; A\times(0,\infty)) = P_i(E^*; A\times\mathbb{R})$$

$$= \mu_i^*(A).$$

Questo termina la prova del Claim.

Per terminare la dimostrazione abbiamo bisogno del seguente fatto.

Sia $\mu = (\mu_1, \ldots, \mu_n)$ una misura di Borel in $\mathbb{R}^{n-1}$ a valori in $\mathbb{R}^n$. La variazione totale $|\mu|$ di μ è la misura di Borel

$$|\mu|(A) = \sup\left\{ \sum_{k=1}^\infty |\mu(A_k)| : A = \bigcup_{k=1}^\infty A_k \text{ disgiunta} \right\}$$

$A \subset \mathbb{R}^{n-1}$ di Borel $\qquad A_k \subset \mathbb{R}^{n-1}$ di Borel

<u>ESERCIZIO</u> Provare che nel nostro caso si ha

$$|\mu|(\mathbb{R}^{n-1}) = P(E).$$

Siccome nel nostro caso $\mu^* \leq \mu$ per componenti si deduce che

$$P(E^*) = |\mu^*|(\mathbb{R}^{n-1}) \leq |\mu|(\mathbb{R}^{n-1}) = P(E).$$

Supponiamo ora che sia $P(E^*) = P(E)$.

Allora:

$$|\mu^*|(\mathbb{R}^{n-1}) = |\mu|(\mathbb{R}^{n-1}) \overset{\text{(facile)}}{\Longrightarrow} |\mu^*|(A) = |\mu|(A) \quad \forall A \subset \mathbb{R}^{n-1} \text{ Borel}$$

ESERCIZIO $\longrightarrow$ $\Downarrow$

(Usare Radon-Nycodim)

$$\mu^* = \mu$$

In particolare $\mu^*_n = \mu_n$ e quindi

$$P(E^*_{x'}) = 2 \Longrightarrow P(E_{x'}) = 2$$

per $\mathcal{L}^{n-1}$-q.o. $x' \in \mathbb{R}^{n-1}$.

Ovvero: E è equivalente ad un insieme x_n-normale.

$\square$

TEOREMA ISOPERIMETRICO

Consideriamo il problema di trovare l'insieme di $\mathbb{R}^n$, $n \geq 2$, che ha frontiera di area minima per volume racchiuso fissato:

$$\min \{ P(E) : E \subset \mathbb{R}^n \ \mathcal{L}^n\text{-misurabile con } \mathcal{L}^n(E) = 1 \}.$$

Siccome $\mathcal{L}^n(\lambda E) = \lambda^n \mathcal{L}^n(E)$ e $P(\lambda E) = \lambda^{n-1} P(E)$, determinare il minimo precedente è equivalente a determinare

$$\min \left\{ \frac{P(E)}{\mathcal{L}^n(E)^{\frac{n-1}{n}}} ; E \subset \mathbb{R}^n \ \mathcal{L}^n\text{-mis. con } 0 < \mathcal{L}^n(E) < \infty \right\}.$$

<u>TEOREMA</u> Il minimo è raggiunto. Gli insiemi minimi sono esattamente le palle Euclidee (a meno di misura nulla).

Una formulazione più precisa del teorema è la seguen

<u>TEOREMA</u> Sia $n \geq 2$. Per ogni insieme $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-misurabile si ha

$$\min \{ \mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E) \} \leq c_n P(E)^{\frac{n}{n-1}}, \qquad (*)$$

dove $\mathcal{L}^n(B) = c_n P(B)^{\frac{n}{n-1}}$ con $B \subset \mathbb{R}^n$ palla.
L'uguaglianza in $(*)$ implica che E oppure $\mathbb{R}^n \setminus E$ è una palla.

Dimostrazione del Teorema isoperimetrico

Dim. Si parte da questo fatto, che non dimostriamo.

Per $n \geq 2$ vale:

$$P(E) < \infty \quad \Rightarrow \quad \mathcal{L}^n(E) < \infty \text{ oppure } \mathcal{L}^n(\mathbb{R}^n \setminus E) < \infty .$$

Supponiamo dunque $\mathcal{L}^n(E) < \infty$.

Supponiamo anche che E sia limitato. Fissiamo $k > 0$ tale che $E \subset B_k(0)$ e consideriamo la famiglia di insiemi

$$\mathcal{F} = \{ F \subset \overline{B_k(0)} \; ; \; F \text{ è } \mathcal{L}^n\text{-misurabile e } \mathcal{L}^n(F) = \mathcal{L}^n(E) \}$$

Fissiamo su $\mathcal{F}$ la topologia L^1 delle funzioni caratteristiche. Per il Teorema di immersione compatta di $BV(A)$ in $L^1(A)$ con $A \subset \mathbb{R}^n$ aperto limitato regolare e per la semicontinuità inferiore del perimetro per la convergenza L^1 esiste $F \in \mathcal{F}$ tale che

$$P(F) = \min \{ P(G) \; ; \; G \in \mathcal{F} \}.$$

Sia F^* il riarrangiamento di Steiner di F in x_n. Allora si ha ancora $F^* \in \mathcal{F}$ e inoltre $P(F^*) \leq P(F)$.

Dalla minimalità segue che $P(F^*) = P(F)$.

Quindi: F è x_n-normale.

Siccome la scelta del sistema di coordinate è arbitraria:

F è normale rispetto ad ogni iperpiano di $\mathbb{R}^n$ che passa per $0 \in \mathbb{R}^n$.

Quindi F è un insieme convesso e dunque

$$F = \{ x \in \mathbb{R}^n : \ f_1(x') < x_n < f_2(x'), \ x' \in D \}$$

con $D \subset \mathbb{R}^{n-1}$ convesso ed $f_1, -f_2 : D \to \mathbb{R}$ funzioni convesse. Inoltre

$$F^* = \{ x \in \mathbb{R}^n : |x_n| < \frac{1}{2} (f_2(x') - f_1(x')) \}.$$

Dato un insieme aperto $A \subset \mathbb{R}^{n-1}$, con $A \subset\subset \text{int}(D)$,

Per la Formula dell' Area:

$$P(F; A \times \mathbb{R}) = \int_A \left(\sqrt{1+|\nabla f_1|^2} + \sqrt{1+|\nabla f_2|^2} \right) dx',$$

$$P(F^*; A \times \mathbb{R}) = 2 \int_A \sqrt{1 + \frac{1}{4} |\nabla (f_2 - f_1)|^2} \ dx'.$$

Dal fatto che $P(F; A \times \mathbb{R}) = P(F^*; A \times \mathbb{R})$ si deduce (Teor. Differenziazione di Lebesgue) che in q.o. punto di D vale

$$\sqrt{1+|\nabla f_1|^2} + \sqrt{1+|\nabla f_2|^2} = 2 \sqrt{1 + \frac{1}{4} |\nabla f_2 - \nabla f_1|^2}$$

che è equivalente a

$$\sqrt{1+|\nabla f_1|^2} \sqrt{1+|\nabla f_2|^2} = \langle (1, \nabla f_1), (1, -\nabla f_2) \rangle$$

che implica $\nabla f_1 = -\nabla f_2$ q.o. in D.

Segue che $f_1 + f_2$ è costante in D e quindi a meno di una traslazione in x_n possiamo supporre che

$$f_1 + f_2 = 0 \quad \text{in} \quad D.$$

Conclusione: A meno di una traslazione F è simmetrico e normale rispetto ad ogni iperpiano di $\mathbb{R}^n$ passante per $0 \in \mathbb{R}^n$.

Dunque: F è una palla.

Abbiamo finito la dimostrazione (nel caso E limitato)

$$\mathcal{L}^n(E) = \mathcal{L}^n(F) = C_n \, P(F)^{\frac{n}{n-1}} \leq C_n \, P(E)^{\frac{n}{n-1}}.$$

Inoltre se c'è un $=$ allora E è una palla.

Caso E non limitato: lasciato al lettore.

Considerare $E \cap B_r$ con r opportuno. $\square$