

Calculus of Variations

Preliminary program of the course - 2022

- (1) Classic functionals of the Calculus of Variations
 - (a) Euler-Lagrange's equations
 - (b) Du Bois-Reymond's equation
 - (c) Indirect methods and convexity methods
 - (d) Fermat's principle for the geometrical optics
 - (e) Brachistochrone's problem
- (2) Semi-direct methods
 - (a) Functionals depending on the gradient
 - (b) Bounded slope condition
- (3) Direct method of the Calculus of Variations
- (4) Functional in Sobolev spaces
 - (a) Brief review on Sobolev spaces
 - (b) Convexity and lower-semicontinuity in $W^{1,p}$
 - (c) Existence of minimizers in $W^{1,p}$
 - (d) Examples
 - (e) Interior Sobolev regularity
 - (f) Schauder regularity and De Giorgi-Nash theorem (without proofs)
- (5) Plateau's problem
 - (a) The parametric formulation of Douglas-Radò
 - (b) Brief introductions to BV functions and sets with finite perimeter
 - (c) Plateau's problem for sets with finite perimeter
- (6) Γ -convergence and applications to phase transition
 - (a) Relaxation and Γ -limits
 - (b) Convergence of minima and minimizers
 - (c) Applications: Modica-Mortola functional and perimeter
 - (d) De Giorgi's conjecture on the equation $\Delta u = -u(u^2 - 1)$
- (7) The isoperimetric property of the sphere and its applications
 - (a) Steiner's rearrangement
 - (b) Isoperimetric property of the sphere
 - (c) Quantitative version of the isoperimetric inequality
 - (d) The best shape of a drum: minimum eigenvalue for $\Delta u = \lambda u$
 - (e) Schwarz's rearrangement and sharp Sobolev inequalities
 - (f) The Yamabe's equation $\Delta u = -u^{\frac{n+2}{n-2}}$
- (8) Introduction to the theory of optimal transportation
 - (a) Monge's problem
 - (b) Kantorovic's formulation
 - (c) Brenier's theorem
 - (d) Another proof of the isoperimetric inequality

- (9) Introduction to the theory of currents
 - (a) Brief review on exterior algebras. Currents, mass and boundary
 - (b) The Plateau's problem for rectifiable currents
 - (c) Overview on the regularity of minimal surfaces. Simon's cone
 - (d) Holomorphic manifolds are area minimizers

Bibliography

General books on Calculus of Variations:

- 1) B. Dacorogna, Introduction to the Calculus of Variations, Imperial College Press 2015. This is an excellent introductory book. There are many exercises with solutions.
- 2) G. Buttazzo & M. Giaquinta & S. Hildebrandt, One-dimensional Variational Problems, Oxford University Press 2008. This is a complete book on functionals acting on functions of one variable.
- 3) E. Giusti, Direct Methods in the Calculus of Variations, Italian and English editions. I am following this book for the semi-classical theory.

Geometric measure theory:

- 1) F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory, Cambridge 2012. Nice introduction to the theory of sets with finite perimeter, including regularity.
- 2) H. Federer, Geometric Measure Theory, Springer. The bible of GMT, very difficult.
- 3) F. Morgan, Geometric Measure Theory, Academic Press 2008. This is an introduction to Federer's book.
- 4) S. G. Krantz & H. R. Parks, Geometric Integration Theory, Birkhäuser 2008. A reasonable introduction to the theory of currents.

Plateau's problem:

- 1) M. Struwe, Plateau's Problem and the Calculus of Variations, Princeton 1988.

Sobolev and BV functions:

- 1) L. C. Evans & R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC press. This is the most concise and yet rigorous introduction to Sobolev and BV functions.
- 2) L. Ambrosio & N. Fusco & D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press.

Γ -convergence, Optimal transportation, Regularity theory:

- 1) G. Dal Maso, An Introduction to Γ -Convergence, Birkhäuser 1993.
- 2) C. Villani, Topics in Optimal Transportation, AMS.
- 2) L. Ambrosio, E. Bruè, D. Semola, Lectures on Optimal Transport, Springer 2021. These are the lectures held by Ambrosio at the SNS in Pisa.
- 3) L. C. Evans, Partial Differential Equations, AMS.
- 4) M. Giaquinta, L. Martinazzi, An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs, Edizioni SNS.