Differential Equations 1

Name: Partial exam, 2nd March 2011

Problem 1 Let $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ be a function such that the following limits exist

$$f(-\infty) = \lim_{x\to-\infty} f(x) \quad \text{and} \quad f(+\infty) = \lim_{x\to+\infty} f(x).$$

Let u be the bounded solution to the Cauchy Problem

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Prove that, for any $x \in \mathbb{R}$, the following limit exists

$$u_\infty(x) = \lim_{t\to\infty} u(x,t)$$

and compute it. Show that the convergence is uniform on compact sets.

Problem 2 Let $f \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $n \geq 1$. We search for a solution $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ of the quasilinear Cauchy problem

$$\begin{cases} u_t - \Delta u + |\nabla u|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Here, $|\nabla u|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2$ denotes the squared norm of the gradient of u in the x variables.

i) Find a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that the function $w = \varphi(u)$ solves a *linear* partial differential equation. Start from $u = \psi(w)$, where ψ is the inverse of φ.

ii) Determine a representation formula for a solution u of the quasilinear problem.