Differential Equations 1

Name: Partial exam, 16th March 2011

Problem 1 Let $f \in C(\mathbb{R})$ be a function that is odd and 2π-periodic. Consider the Dirichlet problem

$$\begin{cases} u_t = u_{xx} & \text{in } (0,\pi) \times (0,\infty), \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x) & \text{for } x \in [0,\pi]. \end{cases}$$

i) By separation of variables and superposition, construct a "formal solution" of the form

$$u(x,t) = \sum_{k=1}^{\infty} c_k v_k(x) w_k(t).$$

Determine the functions v_k and w_k. Determine the coefficients $c_k \in \mathbb{R}$ using the Fourier expansion of f in sine functions.

ii) Prove that the function u is in $C^\infty\big([0,\pi] \times (0,\infty)\big)$ and solves the differential equation.

iii) Assume, in addition, that $f \in C^2(\mathbb{R})$. Prove that $u \in C([0,\pi] \times [0,\infty)$ and $u(\cdot,0) = f$.

Problem 2 Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set with regular boundary and let $u \in C^\infty(\bar{\Omega} \times [0,\infty))$ be a solution of the problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times [0,\infty) \\ u = 0 & \text{on } \partial\Omega \times [0,\infty). \end{cases}$$

i) Prove that the function

$$U(t) = \int_\Omega u(x,t)^2 dx, \quad t \in [0,\infty),$$

is decreasing and convex on $[0,\infty)$.

ii) Prove that

$$\lim_{t \to \infty} U(t) = 0.$$

iii) Assume that $U(t) > 0$ for all $t \in [0,\infty)$. Prove that the function $\varphi(t) = \log U(t)$ is convex on $[0,\infty)$.