Differential Equations 1 Sheet 1

23th February 2011

Exercise 1 Let $f \in L^1(\mathbb{R}^n)$ and let u be the function in $\mathbb{R}^n \times (0,\infty)$ given by the representation formula (2.10). Prove that u solves the Cauchy Problem for the heat equation in the following sense:

1) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and $u_t(x,t) = \Delta u(x,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$;

2) The initial datum is attained in the $L^1(\mathbb{R}^n)$ norm, and namely

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} |u(x,t) - f(x)| dx = 0;$$

3) Moreover, there holds $\|u(\cdot,t)\|_1 \leq \|f\|_1$ for any $t > 0$.

Exercise 2 Let $f \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ be a function such that the following limits exist

$$f(-\infty) = \lim_{x \to -\infty} f(x) \quad \text{and} \quad f(+\infty) = \lim_{x \to +\infty} f(x).$$

Let u be the bounded solution to the Cauchy Problem

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Prove that, for any $x \in \mathbb{R}$, the following limit exists

$$u_\infty(x) = \lim_{t \to \infty} (x,t)$$

and compute it. Show that the convergence is uniform on compact sets.

Exercise 3 Let $X = L^1(\mathbb{R}^n)$ or $X = L^\infty(\mathbb{R}^n)$, and for any $t > 0$, let $T_t : X \to X$ be the operator $T_t(f)(x) = u(x,t)$ where u is given by the representation formula (2.10). Prove that the family of operators $\{T_t\}_{t>0}$ forms a semigroup, and namely

$$T_{t+s}(f) = T_t(T_s(f)), \quad f \in X,$$

for any $s, t > 0$.