Differential Equations 1

Sheet 2

5th March 2011

Exercise 1 Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set with regular boundary (say, $\partial\Omega$ is of class C^1). Let $u \in C^2(\bar{\Omega} \times (0,\infty))$ be a function solving the problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where $f \in C(\bar{\Omega} \times (0,\infty))$ is a function such that

$$\lim_{t\to\infty} \int_\Omega |f(x,t)|^2 dx = 0.$$

Prove that

$$\lim_{t\to\infty} \int_\Omega |u(x,t)|^2 dx = 0.$$

Hints. Letting

$$U(t) = \int_\Omega |u(x,t)|^2 dx \quad \text{and} \quad F(t) = \int_\Omega |f(x,t)|^2 dx,$$

arrive at the inequality $U'(t) + \beta U(t) \leq \varepsilon F(t)$ for suitable constants $\beta, \varepsilon > 0$. You need the Poincarè inequality: there exists a constant $C = C(\Omega)$ such that

$$\int_\Omega |v(x)|^2 dx \leq C \int_\Omega |\nabla v(x)|^2 dx,$$

for all functions $v \in C^1(\bar{\Omega})$ such that $v = 0$ on $\partial\Omega$.

Exercise 2 Let ϑ be a continuous function of the variables $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$ with $0 < s < t$ that satisfies

$$|\vartheta(x,t;s)| \leq \frac{1}{|t-s|^\beta}, \quad x \in \mathbb{R}^n,\ 0 < s < t,$$

for some $\beta \in (0,1)$. Prove that the function

$$\varphi(x,t) = \int_0^t \vartheta(x,t;s) ds$$

is continuous for $x \in \mathbb{R}^n$ and $t > 0$.

Exercise 3 Let $u \in C^2(\mathbb{R}^n \times (0,T)) \cap L^2(\mathbb{R}^n \times (0,T))$, $T > 0$, be a solution to the problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ \lim_{t \downarrow 0} \int_{\mathbb{R}^n} |u(x,t)|^2 dx = 0. \end{cases}$$

i) Prove that $u = 0$.

ii) Deduce that the problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot, 0) = f \in L^2(\mathbb{R}^n) \end{cases}$$

has a unique solution in the class $L^2(\mathbb{R}^n \times (0,T))$.

Hints. Let $\zeta \in C_c^1(\mathbb{R}^n)$ be a continuously differentiable function with compact support. Multiply the differential equation by $u\zeta^2$, integrate over $\mathbb{R}^n \times (0,t)$, $0 < t < T$, and arrive at the inequality

$$\frac{1}{2} \int_{\mathbb{R}^n} u(x,t)^2 \zeta(x)^2 dx \leq \int_0^t \int_{\mathbb{R}^n} u(x,s)^2 |\nabla \zeta(x)|^2 dx ds, \quad 0 < t < T.$$

Now choose a suitable sequence of test functions ζ to conclude.