Equazioni Differenziali 2

Foglio 4

Nome: Consegna: 21 Maggio 2010

Exercise 1 Compute in implicit form the (maximal) solution $y \in C^\infty(a,b)$, $-\infty \leq a < 1 < b \leq +\infty$, of the Cauchy Problem

$$\begin{cases} y' = \dfrac{y-x}{y+x}, \\ y(1) = 0, \end{cases}$$

and draw a qualitative graph of y. Compute b and show that $a > -\frac{1}{2}e^{-\pi/2}$.

Exercise 2 (Gradient flow) Let $f \in C^2(\mathbb{R}^n)$ be a function such that:

a) The sets $\{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ are compact for all $\lambda \in \mathbb{R}$.

b) $\nabla f(x) = 0$ if and only if $x = 0$.

Consider the Cauchy Problem

$$\begin{cases} \dot\gamma(t) = -\nabla f(\gamma(t)), \quad t \geq 0, \\ \gamma(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}^n$. Prove that:

i) The problem has a unique solution $\gamma_{x_0} \in C^2([0,+\infty))$;

ii) $\lim_{t\to+\infty} \gamma_{x_0}(t) = 0$;

Exercise 3 Let $F \in C^1([0,+\infty))$ be a function such that $F(0) > 0$ and $F'(x) \geq 0$ for all $x \geq 0$. Show that any solution $y \in C^2([0,+\infty))$ to the differential equation

$$y'' + F(x)y = 0, \quad x \geq 0,$$

is bounded.

Hint. Show for $x \geq 0$:

$$F(x)y(x)^2 \leq y'(0)^2 + F(0)y(0)^2 + \int_0^x F'(t)y(t)^2dt := \Phi(x),$$

and then $\dfrac{\Phi'}{\Phi} \leq \dfrac{F'}{F}$.