Introduction
to ordinary differential equations

Roberto Monti

Padova, 19th April 2010

Contents

Introduction

Let $\Omega \subset \mathbb{R}^{n+2}$, $n \in \mathbb{N}$, be an open set and let $F : \Omega \to \mathbb{R}$ be a continuous function. An equation of the form

$$F(x, y, y', ..., y^{(n)}) = 0 \tag{0.0.1}$$

is called ordinary differential equation of order n. Here, x is a real variable, y is a real valued unknown function, and y',...,$y^{(n)}$ are its derivatives.

A function $\varphi \in C^n(I)$ is a solution of the differential equation if:

i) $I \subset \mathbb{R}$ is an open interval;
ii) $(x, y(x), ..., y^{(n)}(x)) \in \Omega$ for all $x \in I$;
iii) $F(x, y(x), ..., y^{(n)}(x)) = 0$ for all $x \in I$.

The main problems concerning ordinary differential equations are:

1) Existence of solutions;
2) Uniqueness of solutions (with suitable initial conditions or boundary value data);
3) Regularity and stability of solutions (e.g. dependence on the initial conditions, large time stability, higher regularity depending on F or on parameters);
4) Computation of solutions.

The existence of solutions can be proved by fixed point theorems, by approximation and compactness, by variational methods (minimization and critical point theorems), by the implicit function theorem in Banach spaces, by Functional Analysis techniques. The problem of uniqueness is typically more difficult. Only in very special cases, it is possible to compute the solutions in some explicit form.

CHAPTER 1

Some methods of resolution

1. First order linear equations

1.1. First order linear equations. Let $I \subset \mathbb{R}$ be an open interval and let $a, b \in C(I)$ be two continuous functions. A first order differential equation of the form

$$y' + a(x)y = b(x), \quad x \in I, \tag{1.1.2}$$

is called linear. In the case $b = 0$, the equation is said to be homogeneous

$$y' + a(x)y = 0, \quad x \in I. \tag{1.1.3}$$

We solve the homogeneous equation, first. Assuming $y \neq 0$, e.g. $y > 0$, the differential equation (1.1.3) has the form $y'/y = -a(x)$. A primitive of y'/y is $\log y$. Then, denoting by A a primitive of a, i.e. $A'(x) = a(x)$ for all $x \in I$, we have

$$-A = \log y + d,$$

for some constant $d \in \mathbb{R}$. It follows that $y = \exp(-d - A)$ and letting $c = e^{-d}$ we find the solution

$$y(x) = ce^{-A(x)}, \quad x \in I. \tag{1.1.4}$$

This function is a solution to the homogeneous equation for any $c \in \mathbb{R}$ (i.e. the restriction $y > 0$ can be dropped).

Now we look for a solution of the form (1.1.4) for the non homogeneous equation (1.1.2), where now $c \in C^1(I)$ is a function that has to be determined (this method is called "Variation of constants"). Plugging $y' = c'e^{-A} - ace^{-A}$ into (1.1.2) we get

$$c'e^{-A} = b, \quad \text{that is} \quad c' = be^{A}.$$

Integrating this equation on some interval $(x_0, x) \subset I$ we get

$$c(x) = c(x_0) + \int_{x_0}^{x} b(t)e^{A(t)}dt,$$

and we find

$$y(x) = \Big(c(x_0) + \int_{x_0}^{x} b(t)e^{A(t)}dt\Big)e^{-A(x)}, \quad x \in I, \tag{1.1.5}$$

where $c(x_0) \in \mathbb{R}$ is a real number and $x_0 \in I$.

Proposition 1.1.1. *Let $x_0 \in I$ and A be a primitive of a. Then the function in* (1.1.5) *is a solution to* (1.1.2)*. Moreover, any solution of* (1.1.2) *is of the form* (1.1.5) *for some $c(x_0) \in \mathbb{R}$.*

PROOF. The first statement is a computation. Let $z \in C^1(I)$ be a solution to (1.1.2) and let

$$w(x) = e^{A(x)} z(x) - \int_{x_0}^{x} b(t) e^{A(t)} dt.$$

For we have

$$w' = (az + z')e^A - be^A = 0,$$

the function w is constant on I, and the second claim is proved.

□

2. Separation of variables

Let $I, J \subset \mathbb{R}$ be two open intervals and let $f \in C(I)$ and $g \in C(J)$ be two continuous functions. We look for solutions to the first order differential equation

$$y' = f(x)g(y). \tag{1.2.6}$$

Let $x_0 \in I$ and $y_0 \in J$. If $g(y_0) = 0$ for some $y_0 \in J$, then the constant function $y(x) = y_0$, $x \in I$, is a solution to the differential equation (1.2.6). Assume that $g(y_0) \neq 0$. Then it is $g \neq 0$ in a neighborhood of y_0 and we can divide the equation be $g(y)$ (separation of variables). We find

$$\frac{y'(x)}{g(y(x))} = f(x). \tag{1.2.7}$$

Let $G \in C^1(J_1)$ be the primitive of $1/g(y)$ (in the variable y), defined in some interval J_1 containing y_0. The function G is strictly monotonic, because $G'(y) \neq 0$, and thus invertible. Moreover, let $F \in C^1(I)$ be a primitive of f. Upon integrating (1.2.7), we get

$$G(y(x)) = F(x) + C, \quad x \in I_1, \tag{1.2.8}$$

for some interval $I_1 \subset I$. Here $C \in \mathbb{R}$ is a real constant. The general solution of the differential equation is then

$$y(x) = G^{-1}(F(x) + C), \quad x \in I_1, \tag{1.2.9}$$

where $G^{-1} : G(J_1) \to J_1$ is the inverse function of G. The constant C is uniquely determined by the initial condition $y(x_0) = y_0$, i.e. $C = G(y_0) - F(x_0)$.

This argument identifies two kinds of solutions to the equation (1.2.6): constant solutions and solutions such that $g(y) \neq 0$. There could be other solutions (see Section 5). If e.g. $g \in C^1(J)$, however, there are no other solutions (see Chapter 2).

EXAMPLE 1.2.1. We look for the solution to the Cauchy Problem

$$\begin{cases} y' = \dfrac{1 + 2x}{\cos y} \\ y(0) = \pi. \end{cases} \tag{1.2.10}$$

The differential equation is of the form $y' = f(x)g(y)$ with $f(x) = 1 + 2x$ and $g(y) = 1/\cos y$. ular, g is defined for $\cos y \neq 0$, i.e. for $y \neq \pi/2 + k\pi$ with $k \in \mathbb{Z}$.

Separating the variables we get $y' \cos y = 1 + 2x$, and integrating we find the general solution in implicit form

$$\sin y = x + x^2 + C,$$

where $C \in \mathbb{R}$ is a constant, which is determined by the initial condition $y(0) = \pi$, i.e. $C = \sin y(0) = 0$. The function

$$z(x) = \arcsin(x + x^2)$$

is not, however, the solution to (1.2.10) because $z(0) = \arcsin(0) = 0$. In order to determine the correct solution, notice the arcsin is the inverse function of sin when restricted to $[-\pi/2, \pi/2]$, whereas y takes values in a neighborhood of π. Letting $w(x) = y(x) - \pi$, we have $w(0) = y(0) - \pi = 0$ and $\sin w = \sin(y - \pi) = -\sin y = -(x + x^2)$. Now we can invert the sine function, obtaining $w = -\arcsin(x + x^2)$ and thus

$$y(x) = \pi - \arcsin(x + x^2).$$

The solution y is defined in a suitable neighborhood of the origin.

3. Equations of homogeneous type

A differential equation of the form

$$y' = f\Big(\frac{y}{x}\Big)$$

is called of homogeneous type. Here $f : I \to \mathbb{R}$ is a (continuous) function in some interval $I \subset \mathbb{R}$. With the change of variable $y = xz$, where z is the new unknown function, we get $y' = z + xz'$ and the differential equation transforms into

$$xz' + z = f(z).$$

This equation can be solved on separating the variables.

For instance, the following differential equation is of homogeneous type (see Exercise **??**)

$$y' = \frac{x^2 + y^2}{xy} = f\Big(\frac{y}{x}\Big), \quad \text{with} \quad f(t) = \frac{1}{t} + t.$$

4. Bernoulli's equations

A differential equation of the form

$$y' + a(x)y = b(x)y^\alpha, \quad x \in I, \tag{1.4.11}$$

where α is a real parameter such that $\alpha \neq 0, 1$ is said to be of Bernoulli type. Letting

$$y = z^{\frac{1}{1-\alpha}}, \quad y' = \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} z',$$

the equations transforms into

$$z' + (1 - \alpha)a(x)z = (1 - \alpha)b(x).$$

This is a linear equation.

5. Exact equations

Let $\Omega \subset \mathbb{R}^2$ be an open set and $f, g \in C(\Omega)$ be continuous function. We look for a solution $y \in C^1(I)$, $I \subset \mathbb{R}$ open interval, of the differential equation

$$f(x,y) + g(x,y)y' = 0, \tag{1.5.12}$$

satisfying the condition $y(x_0) = y_0$ for some $(x_0, y_0) \in \Omega$ with $x_0 \in I$.

To this aim, consider the differential form ω in Ω

$$\omega = f(x,y)dx + g(x,y)dy, \tag{1.5.13}$$

where dx, dy is the dual basis of the basis $e_1 = (1,0)$ and $e_2 = (0,1)$.

Definition 1.5.1 (Exact forms). The differential form ω is exact if there exists a function $F \in C^1(\Omega)$ such that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \text{in } \Omega.$$

The function F is called a potential of ω. In this case, the differential equation (1.5.12) is called exact.

Theorem 1.5.2. *Assume that ω is an exact form with potential F such that*

$$\frac{\partial F(x_0, y_0)}{\partial y} \neq 0. \tag{1.5.14}$$

Then the equation $F(x,y) = F(x_0,y_0)$ implicitly defines a function $y \in C^1(I)$ for some open interval I containing x_0 solving the differential equation (1.5.12) *along with the condition $y(x_0) = y_0$. This solution is unique on the interval I.*

Proof. Assume w.l.g. that $F(x_0, y_0) = 0$. By the implicit function theorem, there exist $\delta, \eta > 0$ and $y \in C^1(x_0 - \delta, x_0 + \delta)$ such that

$$\big\{(x,y) \in \Omega : |x - x_0| < \delta, |y - y_0| < \eta, F(x,y) = 0\big\} = \big\{(x, y(x)) \in \Omega : |x - x_0| < \delta\big\}. \tag{1.5.15}$$

Differentiating the identity $F(x, y(x)) = 0$ we get

$$\begin{aligned} 0 = \frac{d}{dx}F(x,y(x)) &= \frac{\partial F(x,y(x))}{\partial x} + \frac{\partial F(x,y(x))}{\partial y}y'(x) \\ &= f(x,y(x)) + g(x,y(x))y'(x), \end{aligned} \tag{1.5.16}$$

i.e. y is a solution of the differential equation, and moreover $y(x_0) = y_0$.

On the other hand, if $z \in C^1(I)$ is a solution to the equation (1.5.12) such that $z(x_0) = y_0$, then the same argument as in (1.5.16) shows that

$$\frac{d}{dx}F(x, z(x)) = 0,$$

and therefore $F(x, z(x)) = F(x_0, z(x_0)) = F(x_0, y_0) = 0$. By (1.5.15) it must by $z = y$. □

DEFINITION 1.5.3 (Closed forms). Assume that $f, g \in C^1(\Omega)$. The differential form $\omega = f dx + g dy$ is closed in Ω if

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x} \quad \text{for all } (x,y) \in \Omega.$$

An exact differential form in an open set Ω with a potential $F \in C^2(\Omega)$ is closed in Ω because mixed derivatives are equal by Schwarz theorem

$$\frac{\partial^2 F}{\partial x \partial y}(x,y) = \frac{\partial^2 F}{\partial y \partial x}(x,y), \quad (x,y) \in \Omega.$$

The converse is also true if Ω is simply connected.

THEOREM 1.5.4. *If $\Omega \subset \mathbb{R}^2$ is a simply connected open set, then any closed differential form in Ω is exact.*

Convex and starshaped open sets are simply connected. In particular, closed forms always have a potential locally.

If $\varphi \in C(\Omega)$ is a function such that $\varphi \neq 0$ in Ω, then the differential equation (1.5.12) and the differential equation

$$\varphi(x,y)\big\{f(x,y) + g(x,y)y'\big\} = 0, \tag{1.5.17}$$

have the same solutions. For a suitable choice of φ, the differential equation (1.5.17) may happen to be exact, even though (1.5.12) is not exact. The function φ is then called integrating factor (or multiplier). If $f, g \in C^1(\Omega)$, a necessary condition for a function $\varphi \in C^1(\Omega)$ to be a multiplier is

$$\frac{\partial}{\partial y}\varphi f = \frac{\partial}{\partial x}\varphi g \quad \text{in } \Omega. \tag{1.5.18}$$

EXAMPLE 1.5.5. The differential equation

$$xy^2 + y - xy' = 0 \tag{1.5.19}$$

is not exact. In fact, with $f = xy^2 + y$ and $g = -x$, we have

$$\frac{\partial f(x,y)}{\partial y} = 2xy + 1 \quad \text{and} \quad \frac{\partial g(x,y)}{\partial x} = -1.$$

We look for a function φ such that (1.5.18) holds. We try with the ansatz $\varphi = \varphi(y)$, i.e. φ depends only on y. We get the necessary condition

$$\varphi'(y)(xy^2 + y) + \varphi(y)(2xy + 1) = -\varphi(y),$$

that is implied by $y\varphi' + 2\varphi = 0$ (this equation does not depend on x). A solution for this linear equation is $\varphi(y) = 1/y^2$. Now the differential equation

$$x + \frac{1}{y} - \frac{xy'}{y^2} = 0$$

is exact, where $y \neq 0$. A potential F for this exact differential equation can be found on solving

[1]

An integration yields

$$F(x,y) = \frac{x^2}{2} + \frac{x}{y} + C, \quad C \in \mathbb{R}.$$

The equation $F(x,y) = 0$ implicitly defines solutions to the differential equation (1.5.19) (for $x \neq 0$ and $y \neq 0$).

6. Second order linear equations with constant coefficients

Let $f : I \to \mathbb{R}$ be a continuous function on the interval $I \subset \mathbb{R}$ and let $a, b, c \in$ be real numbers such that $a \neq 0$. The differential equation

$$ay'' + by' + cy = f(x), \quad x \in I, \tag{1.6.20}$$

is a second order linear differential equation with constant coefficients. When $f = 0$ the equation is called homogeneous. The general solution of the homogeneous equation

$$ay'' + by' + cy = 0, \quad x \in \mathbb{R}, \tag{1.6.21}$$

is of the form $y_{GH} = C_1 y_1 + C_2 y_2$ where $C_1, C_2 \in \mathbb{R}$ are real numbers and y_1, y_2 are two solutions of (1.6.21) which are linearly independent, i.e. such that for real numbers $\alpha, \beta \in \mathbb{R}$

$$\alpha y_1 + \beta y_2 = 0 \text{ in } \mathbb{R} \quad \Rightarrow \quad \alpha = \beta = 0.$$

The general solution of the inhomogeneous equation (1.6.20) is a function $y \in C^2(I)$ of the form $y = y_{GH} + y_P$, where $y_P \in C^2(I)$ is a particular solution of the inhomogeneous equations. We describe some practical methods to compute y_{GH} and y_P. The general theory is dealt with in Chapter **??**.

6.1. Homogeneous equation. The solutions to the homogeneous equation

$$ay'' + by' + cy = 0 \tag{1.6.22}$$

are a real vector space, i.e. any linear combination of solutions is still a solution. We shall prove in Chapter **??** that this vector space has dimension 2. It is therefore sufficient to find two linearly independent solutions to the equation. We look for solutions of the form $y(x) = e^{\lambda x}$ for some complex number $\lambda \in \mathbb{C}$. Inserting y, y', y'' into (1.6.22) we get $e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$. Since $e^{\lambda x} \neq 0$, the complex number λ must solve the characteristic equation

$$a\lambda^2 + b\lambda + c = 0. \tag{1.6.23}$$

According to the sign of $\Delta = b^2 - 4ac$ we distinguish three cases.

Case 1: $\Delta > 0$. In this case the characteristic equation has two real (simple) solutions

$$\lambda_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

[1]MANCA TESTO

The general solution of the homogeneous equation (1.6.22) is

$$y_{GH}(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

where $C_1, C_2 \in \mathbb{R}$ are real constant.

Case 2: $\Delta = 0$. In this case the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has one real double solution $\lambda = -b/2a$. The ansatz yields only the solution $y_1(x) = e^{\lambda x}$. A direct computation shows that the function $y_2(x) = xe^{\lambda x}$ is also a solution which is linearly independent from the first one. The general solution of the homogeneous equation (1.6.22) is then

$$y_{GH}(x) = e^{\lambda x}(C_1 + C_2 x),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

Case 3: $\Delta < 0$. In this case the characteristic equation has two complex conjugate solutions

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta, \quad \text{where} \quad \alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{|\Delta|}}{2a}.$$

We get the complex valued solutions

$$z_1(x) = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)),$$
$$z_2(x) = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x)),$$

and the real valued solutions

$$y_1(x) = \frac{1}{2}(z_1(x) + z_2(x)) = e^{\alpha x}\cos(\beta x),$$
$$y_2(x) = \frac{1}{2i}(z_1(x) - z_2(x)) = e^{\alpha x}\sin(\beta x).$$

The general solution of the homogeneous equation is

$$y_{GH}(x) = e^{\alpha x}\big(C_1\cos(\beta x) + C_2\sin(\beta x)\big),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

6.2. Inhomogeneous equation. Similar solutions. Consider the inhomogeneous equation (1.6.20), where the right hand side is a function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = e^{\alpha x}\big(P_0(x)\cos(\beta x) + Q_0(x)\sin(\beta x)\big),$$

for some $\alpha, \beta \in \mathbb{R}$ and real polynoms P_0 and Q_0.

We describe a practical method to find a particular solution y_P to the differential equation

$$ay'' + by' + cy = f(x). \tag{1.6.24}$$

A more systematic method is given in the next section.

Consider the real or complex number $\lambda = \alpha + i\beta$ and denote by $m \in \{0, 1, 2\}$ the multiplicity of λ as a solution of the characteristic equation (1.6.23). The case $m = 0$ means that λ is not a solution, the case $m = 1$ means that λ is a simple solution, the case $m = 2$ means that λ is a double solution.

It is always possible to find a particular solution y_P for the inhomogeneous equation (1.6.24) of the form

$$y_P(x) = x^m e^{\alpha x}\big(P(x)\cos(\beta x) + Q(x)\sin(\beta x)\big),$$

where P and Q are unknown polynoms with degree equal to the maximum of the degrees of P_0 and Q_0. These polynoms can be determined inserting y_P, y_P', y_P'' into (1.6.24) and comparing the coefficients of the trigonometric functions in the left and right hand side.

6.3. Inhomogeneous equation. Variation of constants. We look for a particular solution of the inhomogeneous equation (1.6.20) when $f : I \to \mathbb{R}$ is any continuous function on some interval $I \subset \mathbb{R}$.

The general solution of the homogeneous equation (1.6.21) is of the form

$$y = C_1 y_1 + C_2 y_2, \tag{1.6.25}$$

where C_1, C_2 are two real constants and y_1 and y_2 are two linearly independent solutions of the homogeneous equation. These solutions are known.

The method of the variation of constants consists in letting C_1, C_2 be functions of the variable x. We look for a particular solution y of (1.6.20) of the form (1.6.25), where now C_1 and C_2 are functions. We have to determine C_1 and C_2. On differentiating y, we get

$$y' = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2'. \tag{1.6.26}$$

We impose on C_1' and C_2' the condition

$$C_1' y_1 + C_2' y_2 = 0. \tag{1.6.27}$$

On differentiating y' we find

$$y'' = C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2''. \tag{1.6.28}$$

Plugging (1.6.25), (1.6.26), (1.6.28) into the inhomogeneous equation (1.6.20) we find

$$a(C_1' y_1' + C_2' y_2') = f(x). \tag{1.6.29}$$

We also used (1.6.27) and the fact that y_1, y_2 are solution to the homogeneous equation.

As y_1 and y_2 are linearly independent, it is

$$\det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0. \tag{1.6.30}$$

This fact follows from Exercise **??**. Then it is possible to solve the system of equations

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x)/a. \end{cases} \tag{1.6.31}$$

Finally, the functions C_1 and C_2 can be computed on integrating C_1' and C_2'.

7. Euler's second order equations

Let $f : I \to \mathbb{R}$ be a continuous function on the interval $I \subset \mathbb{R}^+$ and let $a, b, c \in$ be real numbers such that $a \neq 0$. The differential equation

$$ax^2y'' + bxy' + cy = f(x), \quad x \in I, \tag{1.7.32}$$

is a second order differential equation of Euler's type. We consider only the case $f = 0$, i.e. the homogeneous equation

$$ax^2y'' + bxy' + cy = 0, \quad x \in \mathbb{R}^+. \tag{1.7.33}$$

The differential equation is singular at $x = 0$ because the coefficient of y'' vanishes. We look for solutions on the half line $\mathbb{R}^+ = (0, +\infty)$. For the differential equation is linear, solutions are a two dimensional vector space. We look for two linearly independent solutions of the form

$$y(x) = x^\lambda = e^{\lambda \log(x)} = e^{(\alpha+i\beta)\log x} = x^\alpha(\cos(\beta \log x) + i \sin(\beta \log x)),$$

where $\lambda = \alpha + i\beta$ is a complex parameter. Plugging y, $y' = \lambda x^{\lambda-1}$, and $y'' = \lambda(\lambda-1)x^{\lambda-2}$ into (1.7.33) we get $x^\lambda(a\lambda(\lambda-1) + b\lambda + c) = 0$. Because $x^\lambda \neq 0$, λ must solve the characteristic equation

$$a\lambda^2 + (b-a)\lambda + c = 0. \tag{1.7.34}$$

According to the sign of $\Delta = (b-a)^2 - 4ac$ we distinguish three cases.

Case 1: $\Delta > 0$. In this case the characteristic equation has two real (simple) solutions $\lambda_1, \lambda_2 \in \mathbb{R}$ and the general solution of the homogeneous equation (1.7.33) is

$$y(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2},$$

where $C_1, C_2 \in \mathbb{R}$ are real constant.

Case 2: $\Delta = 0$. In this case the characteristic equation has one real double solution $\lambda \in \mathbb{R}$ and we get the solution $y_1(x) = x^\lambda$. A direct computation shows that the function $y_2(x) = x^\lambda \log x$ is also a solution which is linearly independent from the first one. The general solution of the homogeneous equation (1.7.33) is then

$$y(x) = x^\lambda(C_1 + C_2 \log x),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

Case 3: $\Delta < 0$. In this case the characteristic equation has two complex conjugate solutions

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta.$$

We get the complex valued solutions

$$z_1(x) = x^{\alpha+i\beta} = x^\alpha\big(\cos(\beta \log x) + i \sin(\beta \log x)\big),$$
$$z_2(x) = x^{\alpha-i\beta} = x^\alpha\big(\cos(\beta \log x) - i \sin(\beta \log x)\big),$$

and the real valued solutions

$$y_1(x) = \frac{1}{2}(z_1(x) + z_2(x)) = x^\alpha \cos(\beta \log x),$$
$$y_2(x) = \frac{1}{2i}(z_1(x) - z_2(x)) = x^\alpha \sin(\beta \log x).$$

The general solution of the homogeneous equation is

$$y(x) = x^\alpha \big(C_1 \cos(\beta \log x) + C_2 \sin(\beta \log x)\big),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

CHAPTER 2

Existence and uniqueness in the Lipschitz case

1. Banach fixed point theorem

DEFINITION 2.1.1 (Contraction). Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists $0 < \lambda < 1$ such that $d(T(x), T(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

THEOREM 2.1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction. Then there exists a unique $x \in X$ such that $x = T(x)$.*

PROOF. Let $x_0 \in X$ be a given point and define $x_n = T^n(x_0) = T \circ ... \circ T(x_0)$, n-times. The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. In fact, by the triangle inequality we have for all $n, k \in \mathbb{N}$

$$\begin{aligned} d(x_{n+k}, x_n) &\leq \sum_{h=1}^{k} d(x_{n+h}, x_{n+h-1}) = \sum_{h=1}^{k} d(T^{n+h}(x_0), T^{n+h-1}(x_0)) \\ &\leq d(T(x_0), x_0) \sum_{h=1}^{k} \lambda^{n+h-1} \leq \lambda^n d(T(x_0), x_0) \sum_{h=1}^{\infty} \lambda^{h-1}. \end{aligned}$$

The series converges and $\lambda^n \to 0$ as $n \to \infty$, because $\lambda < 1$. For X is complete, there exists $x \in X$ such that $x = \lim_{n\to\infty} T^n(x_0)$.

We show that $x = T(x)$. The mapping T is continuous and so we have

$$x = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T(T^{n-1}(x_0)) = T(\lim_{n\to\infty} T^{n-1}(x_0)) = T(x).$$

Finally, we prove that the fixed point is unique. Let $\bar{x} \in X$ be such that $\bar{x} = T(\bar{x})$. Then we have

$$d(x, \bar{x}) = d(T(x), T(\bar{x})) \leq \lambda d(x, \bar{x}) \quad \Rightarrow \quad d(x, \bar{x}) = 0,$$

because $\lambda < 1$, and thus $x = \bar{x}$.

□

THEOREM 2.1.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that T^n is a contraction, for some $n \in \mathbb{N}$. Then there exists a unique $x \in X$ such that $x = T(x)$.*

PROOF. There exists a unique $x \in X$ such that $T^n(x) = x$. Then we have for some $0 \leq \lambda < 1$

$$d(x, T(x)) = d(T^n(x), T(T^n(x))) = d(T^n(x), T^n(T(x))) \leq \lambda d(x, T(x)),$$

and thus $d(x, T(x)) = 0$, which is equivalent to $T(x) = x$.

Now assume that for some $y \in X$ it is $y = T(y)$. Then we also have $y = T^n(y)$ and thus $x = y$, because the fixed point of T^n is unique. $\square$

2. Excursus. Other fixed point theorems

Some fixed point theorems are based on the notion of convexity.

THEOREM 2.2.1 (Brouwer). *Let $K \subset \mathbb{R}^n$ be a closed ball and let $T : K \to K$ be continuous. Then there exists $x \in K$ such that $T(x) = x$.*

For an analytical proof, see Evans, *Partial Differential Equations*, p.441. Brouwer's theorem extends to the infinite dimensional case.

THEOREM 2.2.2 (Schauder I). *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, convex, and compact set. Any continuous mapping $T : K \to K$ has at least one fixed point in K, i.e., there exists $x \in K$ tale che $T(x) = x$.*

See Evans, *Partial Differential Equations*, p.502. The assumption on K to be compact can be be transferred to the mapping T.

THEOREM 2.2.3 (Schauder II). *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, convex, closed and bounded set. Let $T : K \to K$ be a mapping such that:*

(i) *T is continuous;*
(ii) *T is compact, i.e., $T(K) \subset K$ is precompact.*

Then there exists $x \in K$ such that $T(x) = x$.

Tarki's Fixed Point theorem relies upon the notion of partial order.

THEOREM 2.2.4 (Tarski). *Let $(X, \leq)$ be a partially ordered set such that any subset $Y \subset X$ has a supremum. Let $T : X \to X$ be an order preserving mapping, i.e. a mapping such that*

$$x \leq y \;\Rightarrow T(x) \leq T(y).$$

Then there exists $x \in X$ such that $x = T(x)$.

The proof of Tarki's Lemma is an exercise.

3. Cauchy Problem. Introduction

In $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, we introduce the coordinates $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $f \in C(\Omega; \mathbb{R}^n)$ be a continuous function. Given a point $(x_0, y_0) \in \Omega$ we consider the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \tag{2.3.35}$$

A function $y \in C^1(I; \mathbb{R}^n)$ is a solution to the problem if:

i) $I \subset \mathbb{R}$ is an interval such that $x_0 \in I$;

ii) $(x, y(x)) \in \Omega$ for all $x \in I$;
iii) $y'(x) = f(x, y(x))$ for all $x \in I$ (the differential equations is solved);
iv) $y(x_0) = y_0$ (the initial datum is attained).

We are interested in the following questions:

a) Existence of solutions;
b) Uniqueness of solutions;
c) Dependence of solutions from x_0, y_0, and f (regularity and stability).

Integrating the differential equation $y' = f(x, y)$ on the interval with end-points x_0 and x we get the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt = Ty(x), \tag{2.3.36}$$

where $y \mapsto Ty$ is a mapping defined on a suitable functional space. A solution to the Cauchy Problem is then a fixed point of the mapping T. On the other hand, if a continuous function y solves the fixed point equation (2.3.36) then y is of class C^1 and solves the Cauchy Problem (2.3.35).

We fix the functional space. For a $\delta > 0$ consider the real vector space

$$V = C([x_0 - \delta, x_0 + \delta]; \mathbb{R}^n). \tag{2.3.37}$$

Endowed with the norm

$$\|y\| = \max_{x \in [x_0-\delta, x_0+\delta]} |y(x)|, \quad y \in V, \tag{2.3.38}$$

the vector space V is a Banach space, because any Cauchy sequence in V converges to a function in V. For any $\varepsilon > 0$, the subset X of V

$$X = \big\{ y \in V : y(x_0) = y_0,\ \|y - y_0\| \le \varepsilon \big\} \tag{2.3.39}$$

is closed because both conditions $y(x_0) = y_0$ and $\|y - y_0\| \le \varepsilon$ are preserved by the uniform (pointwise) convergence. Then the metric space (X, d) is complete w.r.t. the metric $d(y, z) = \|y - z\|$.

We shall see that for a suitable choice of δ and ε the mapping $T : X \to X$

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2.3.40}$$

is well defined, i.e. it is $Ty \in X$ for all $y \in X$.

4. Local existence and uniqueness under the Lipschitz condition

Definition 2.4.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. We say that a function $f \in C(\Omega; \mathbb{R}^n)$ has the local Lipschitz property in y if for any compact set $K \subset \Omega$ there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2| \tag{2.4.41}$$

for all $(x, y_1), (x, y_2) \in K$.

THEOREM 2.4.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, $(x_0, y_0) \in \Omega$, and let $f \in C(\Omega; \mathbb{R}^n)$ be a function with the local Lipschitz property in y. Then there exists $\delta > 0$ such that the Cauchy Problem (2.3.35) has a unique solution $y \in C^1(I; \mathbb{R}^n)$ in the interval $I = [x_0 - \delta, x_0 + \delta]$.*

PROOF. Let $\delta > 0$ and $\varepsilon > 0$ be such that $K = [x_0 - \delta, x_0 + \delta] \times \{y \in \mathbb{R}^n : |y - y_0| \leq \varepsilon\} \subset \Omega$. Let $H \subset \Omega$ be any compact set such that $K \subset \mathrm{int}(H)$. For f is continuous on H, the number

$$M = \sup_{(x,y)\in H} |f(x, y)| < +\infty$$

is finite. Let X be the set introduced in (2.3.39) and let T be the mapping (2.3.40). For any $y \in X$ we have for $x \in I$

$$|Ty(x) - y_0| \leq \Big| \int_{x_0}^{x} |f(t, y(t))| dt \Big| \leq M|x - x_0| \leq \delta M.$$

In fact it is $(t, y(t)) \in K$ for all $t \in I$. Possibly choosing a smaller $\delta > 0$ (this does not affect M), we can assume that $\delta M \leq \varepsilon$. With such a choice, it is $Ty \in X$ for all $y \in X$. The choice of $\delta > 0$ is independent from x_0 and y_0 as long as $K \subset \mathrm{int}(H)$.

We prove that the mapping $T : X \to X$ has a unique fixed point. It is enough to show that, for some $k \in \mathbb{N}$, the iterated mapping T^k is a contraction. Let $y, \bar{y} \in X$ and $x \in I$. We have (with e.g. $x \geq x_0$)

$$\begin{aligned}
|Ty(x) - T\bar{y}(x)| &= \Big| \int_{x_0}^{x} \big(f(t, y(t)) - f(t, \bar{y}(t))\big) dt \Big| \\
&\leq \int_{x_0}^{x} |f(t, y(t)) - f(t, \bar{y}(t)| dt \\
&\leq L \int_{x_0}^{x} |y(t) - \bar{y}(t)| dt \leq L|x - x_0| \cdot \|y - \bar{y}\|.
\end{aligned}$$

Here, L is the Lipschitz constant for f relative to the compact set H. Analogously, it is (e.g. $x \geq x_0$)

$$\begin{aligned}
|T^2y(x) - T^2\bar{y}(x)| &= \Big| \int_{x_0}^{x} (f(t, Ty(t)) - f(t, T\bar{y}(t)) dt \Big| \\
&\leq L \int_{x_0}^{x} |Ty(t) - T\bar{y}(t)| dt \\
&\leq L^2 \|y - \bar{y}\| \int_{x_0}^{x} (t - x_0) dt \leq L^2 \frac{(x - x_0)^2}{2} \|y - \bar{y}\|.
\end{aligned}$$

By induction, we get for any $k \in \mathbb{N}$ and $x \in I$

$$|T^k y(x) - T^k \bar{y}(x)| \leq \frac{L^k |x - x_0|^k}{k!} \|y - \bar{y}\|,$$

which implies

$$\|T^k y - T^k \bar{y}\| \leq \frac{(L\delta)^k}{k!} \|y - \bar{y}\|.$$

Because

$$\lim_{k \to +\infty} \frac{(L\delta)^k}{k!} = 0,$$

there exists $k \in \mathbb{N}$ such that $\frac{(L\delta)^k}{k!} < 1$. For such a k, the mapping T^k is a contraction. Then T has a unique fixed point $y \in X$. Then it is $y \in C^1([x_0 - \delta, x_0 + \delta]; \mathbb{R}^n)$ and y solves the Cauchy Problem (2.3.35). □

5. Peano's example

Consider the Cauchy problem

$$\begin{cases} y'(x) = 2\sqrt{|y(x)|}, & x \in \mathbb{R}, \\ y(0) = 0 \end{cases} \tag{2.5.42}$$

The function $f(x, y) = 2\sqrt{|y|}$ is not locally Lipschitz in the variable y. The Lipschitz property (2.4.41) fails in a neighborhood of $y = 0$ and the assumptions of Theorem 2.4.2 are not fulfilled. The Cauchy Problem could have more than one solution.

In fact, a solution is the constant function $y = 0$. A second solution can be found separating the variables: $2 = y'/\sqrt{|y|}$. Integrating this equation on the interval between 0 and $x \in \mathbb{R}$ we get

$$2x = \int_0^x \frac{y'(t)}{\sqrt{|y(t)|}} dt = \int_0^{y(x)} \frac{1}{\sqrt{|z|}} dz = \begin{cases} 2\sqrt{y(x)}, & \text{if } y(x) > 0 \\ -2\sqrt{-y(x)}, & \text{if } y(x) < 0. \end{cases}$$

In the change of variable $z = y(t)$ we used the initial datum $y(0) = 0$. Then we find the solution $y \in C^1(\mathbb{R})$

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

On the other hand, for all real numbers $\alpha \leq 0 \leq \beta$, the function

$$y_{\alpha\beta}(x) = \begin{cases} (x - \beta)^2 & \text{if } x \geq \beta, \\ 0 & \text{if } \alpha < x < \beta, \\ -(x - \alpha)^2 & \text{if } x \leq \alpha \end{cases}$$

is of class $C^1(\mathbb{R})$ and solves the Cauchy Problem (2.5.42). So there is a continuum of solutions.