Isoperimetric inequality, semilinear equations and regular domains in Grushin spaces

Roberto Monti

Contents

Introduction

Let $\alpha > 0$ be a real number and consider the vector fields in the plane $X = \partial_x$ and $Y = |x|^\alpha \partial_y$. By means of X and Y several analytical and geometrical objects can be defined in $\mathbb{R}^2$. We can define the gradient of a differentiable function u as the vector $D_\alpha u = (Xu, Yu) = (\partial_x u, |x|^\alpha \partial_y u)$. This gradient is "subelliptic" in the sense that it can degenerate on the y axis. The corresponding "subelliptic" second order operator is $\mathcal{L}_\alpha = X^2 + Y^2 = \partial_x^2 + |x|^{2\alpha}\partial_y^2$. In the case $\alpha = 1$, this Laplacian is known as Grushin operator, one of the most simple and better understood elliptic–degenerate operators. The vector fields X and Y can also be used to introduce a notion of weighted perimeter $P_\alpha(E)$ for Lebesgue measurable sets $E \subset \mathbb{R}^2$. P_α is one of the most simple examples of perimeter measure in non–Euclidean metric spaces. Finally, considering X and Y as a possibly degenerating basis for the tangent space at points in the plane, it is possible to define a metric d_α on $\mathbb{R}^2$. The resulting metric space $(\mathbb{R}^2, d_\alpha)$ is then an example of sub–Riemmanian or Carnot–Carathéodory space, sometimes known as Grushin plane.

In this work we study some problems connecting D_α, $\mathcal{L}_\alpha$, P_α and d_α, in the plane and in more general situations. This research is part of a more general research program on Analysis in Metric Spaces, a subject that, in the last years, has raised a great interest in many different areas of mathematics: linear and nonlinear partial differential equations, functional spaces and Sobolev–Poincaré inequalities, quasiconformal mappings, theory of perimeters, rectifiability and currents, sub–Riemannian and Carnot–Carathéodory geometry, differentiability properties of functions. Evidence for this increasing interest is given by the many books recently appeared on related topics [**BR**], [**DS**], [**AT**], [**ASC**], [**HK2**], [**H**].

In Chapter 1 we prove a sharp isoperimetric inequality for the perimeter P_α in the plane. This seems to be the first sharp isoperimetric result in the sub–Riemannian setting. In Chapter 2 we study symmetry properties for critical semilinear equations involving higher dimensional generalizations of $\mathcal{L}_\alpha$. The results are connected with the problem of determining sharp constants and extremal functions for Sobolev inequalities for subelliptic gradients as D_α. Of independent interest are also the tools used, especially the Kelvin transform we have found for Grushin operators. In Chapter 3 we study regular domains in $\mathbb{R}^n$ for Grushin metrics. Our results have interesting applications to the theory of functional inequalities and to the study of the boundary behavior of $\mathcal{L}_\alpha$–harmonic functions.

Carnot–Carathéodory spaces have a metric (Hausdorff) dimension, say $Q > 2$, larger than their topological dimension, and the isoperimetric inequality gives an upper bound for the volume of bounded sets (their Q dimensional Hausdorff measure) in terms of the $Q-1$ dimensional Hausdorff measure of the boundary. Apparently, the first result of this type is due to Pansu in the case of the Heisenberg group [**P**]. In more general Carnot–Carathéodory spaces, isoperimetric inequalities are discussed by Gromov in [**G**]. Relationships between isoperimetric inequalities, Sobolev inequalities and heat kernels are also discussed in various settings in [**VSC**], [**FGaW**], [**CDG1**] and [**GN1**].

In the Grushin plane, the α–perimeter of a measurable set $E \subset \mathbb{R}^2$ can be defined as follows:

$$P_\alpha(E) = \sup\left\{ \int_E (\partial_x\varphi_1 + |x|^\alpha \partial_y \varphi_2)\, dxdy \,:\, \varphi_1, \varphi_2 \in C_0^1(\mathbb{R}^2),\ \sup_{\mathbb{R}^2}(\varphi_1^2 + \varphi_2^2)^{1/2} \le 1 \right\}. \tag{1}$$

This definition is a special case of the one given in [**GN1**] for Carnot–Carathéodory spaces and also of the one introduced in [**A**] for more general metric spaces.

The relation between P_α and the Grushin metric d_α can be described in terms of Minkowski content. Precisely, if E is a bounded open set with regular boundary, then

$$P_\alpha(E) = \mathfrak{M}_\alpha(\partial E) := \lim_{\varepsilon \downarrow 0} \frac{|\{p \in \mathbb{R}^2 : 0 < \mathrm{dist}_\alpha(p; E) < \varepsilon\}|}{\varepsilon},$$

where $|\cdot|$ stands for Lebesgue measure in the plane. This identity holds in general Carnot–Carathéodory spaces (see [**MSC**]). By a general result due to Ambrosio [**A**], α–perimeter also has a representation in terms of Hausdorff measures.

The main result in Chapter 1 is the following theorem.

THEOREM 1. *Let $\alpha > 0$ and $Q = \alpha + 2$. There exists a constant $c(\alpha) > 0$ such that for any measurable set $E \subset \mathbb{R}^2$ with finite measure*

$$|E| \le c(\alpha) P_\alpha(E)^{\frac{Q}{Q-1}}. \tag{2}$$

The constant $c(\alpha)$ is determined by equality in (2) achieved by the isoperimetric set

$$E_\alpha = \left\{ (x, y) \in \mathbb{R}^2 : |y| < \int_{\arcsin|x|}^{\pi/2} \sin^{\alpha+1}(t)\, dt,\ |x| < 1 \right\}. \tag{3}$$

Precisely,

$$c(\alpha) = \frac{\alpha+1}{\alpha+2}\left(2\int_0^\pi \sin^\alpha(t) dt\right)^{-\frac{1}{\alpha+1}}. \tag{4}$$

Isoperimetric sets are unique up to vertical translations and dilations of the form $(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$, $\lambda > 0$.

The number $Q = 2 + \alpha$ is the "homogeneous" dimension of the metric space $(\mathbb{R}^2, d_\alpha)$ with Lebesgue measure. The size of balls B in the metric d_α has been described by Franchi and Lanconelli [**FL**] by means of the boxes

$$\mathrm{Box}((x, y), r) = [x - r, x + r] \times [y - r(|x| + r)^\alpha, y + r(|x| + r)^\alpha].$$

Precisely, there exist constants $0 < c_1 < c_2$ such that $\mathrm{Box}(p, c_1 r) \subset B(p,r) \subset \mathrm{Box}(p, c_2 r)$ for all $p \in \mathbb{R}^2$ and $r > 0$. (Similar estimates play a crucial role in Chapter 3). Therefore, the size of small balls with center away from $x = 0$ is approximately of Euclidean type, whereas the Lebesgue measure of $B((0,y),r)$ is comparable to $r^{2+\alpha}$. The number $Q = 2 + \alpha$ is the isoperimetric dimension of the Grushin plane.

The perimeter P_α is $(Q-1)$–homogeneous with respect to the dilations δ_λ, whereas Lebesgue measure is Q–homogeneous (see Proposition 1.1.2). Using these homogeneity properties, the problem of finding the sharp constant in (2) can be reduced to solving the minimum problem

$$\min \left\{ P_\alpha(E) : \, E \subset \mathbb{R}^2 \text{ measurable set with } |E| = 1 \right\}. \tag{5}$$

A key step in the proof of existence of solutions is to show that the class of admissible sets can be restricted to sets which are symmetric both in the x and in the y direction. In fact, solutions must be symmetric with respect to the y axis. The argument relies upon an adaptation of Steiner symmetrization. After a suitable change of variable Ψ, the α–perimeter of a set E is equal to the Euclidean perimeter of the set $F = \Psi(E)$ (see Proposition 1.1.3). By a result of De Giorgi [**DG**], the Euclidean perimeter of the Steiner symmetrized set F^* is less or equal than that of F. It follows that $P_\alpha(\Psi^{-1}(F^*)) \le P_\alpha(E)$ and the problem is reduced to studying how the map Ψ changes volume (see Theorem 1.2.1).

Besides symmetry, solutions to problem (5) must also be convex. This implies Lipschitz regularity of the boundary of minimum sets, and then, using an integral representation for α–perimeter proved in Theorem 1.1.1, it is possible to write down the Euler–Lagrange equation for problem (5), a simple ordinary differential equation that yields the explicit solutions (3).

A simple corollary of the isoperimetric inequality (2) is the inequality $|E| \le c(\alpha)\mathfrak{M}_\alpha(\partial E)^{\frac{Q}{Q-1}}$ for bounded open sets (Corollary 1.3.1). This is the kind of isoperimetric inequality suggested by Gromov in [**G**] for non–equiregular sub–Riemannian manifolds.

The case $\alpha = 1$ has a special interest in connection with the Heisenberg group. In this particular case the isoperimetric ball

$$E_1 = \left\{ (x,y) \in \mathbb{R}^2 : |y| < \frac{1}{2}\left(\arccos|x| + |x|\sqrt{1-|x|^2} \right), \, |x| < 1 \right\}$$

is bounded by two geodesics for the Grushin metric d_1 which are symmetric with respect to the y axis (see Section 4). The same phenomenon seems to appear in the Heisenberg group, as conjectured by Pansu in [**P**]. Moreover, identifying the Grushin plane with a vertical hyperplane of $\mathbb{R}^3$ and the y axis with the vertical axis of $\mathbb{R}^3$, then by rotating E_1 around the vertical axis one obtains a set which is believed to solve the isoperimetric problem in the Heiseneberg group (see also [**Mo**], [**LM**] and [**DGN2**]). Our interest in the Heisenberg isoperimetric problem was the original motivation to the study of the problems discussed in Chapter 1.

By an argument of Federer and Fleming [**FF**] and Maz'ya [**Ma**], inequality (2) yields a Sobolev–Gagliardo–Nirenberg inequality for the Grushin subelliptic gradient. Precisely,

$$\left(\int_{\mathbb{R}^2} |u|^{\frac{Q}{Q-1}} dxdy\right)^{\frac{Q-1}{Q}} \leq c(\alpha)^{\frac{Q-1}{Q}} \int_{\mathbb{R}^2} |D_\alpha u|\, dxdy$$

for functions $u \in C_0^\infty(\mathbb{R}^2)$. Here $c(\alpha)$ is the constant (4) and the inequality is sharp (Corollary 1.3.2). However, contrary to the Euclidean setting, the isoperimetric inequality does not provide the sharp constant in the Sobolev embedding

$$\left(\int_{\mathbb{R}^2} |u|^{\frac{2Q}{Q-2}} dxdy\right)^{\frac{Q-2}{Q}} \leq c \int_{\mathbb{R}^2} |D_\alpha u|^2 dxdy. \tag{6}$$

Indeed, extremal functions for inequality (6) in the case $\alpha = 1$ have been recently found by Beckner in [**B**]. They are functions of the form $u(x, y) = ((1+x^2)^2+4y^2)^{-1/4}$, and their level sets are not isoperimetric balls.

The results of Chapter 2 are related to the problem of finding extremal functions for inequality (6). We shall study a higher dimensional generalization of the problem. Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $\alpha > 0$ and $n = m + k$, $m, k \geq 1$. In $\mathbb{R}^n$, inequality (6) reads

$$\left(\int_{\mathbb{R}^n} |u|^{2^*} dxdy\right)^{1/2^*} \leq c\left(\int_{\mathbb{R}^n} (|D_x u|^2 + |x|^{2\alpha}|D_y u|^2) dxdy\right)^{1/2}, \tag{7}$$

where

$$2^* = \frac{2Q}{Q-2} \qquad \text{and} \qquad Q = m + k(\alpha + 1).$$

The number Q is the "dimension" and 2^* is the Sobolev conjugate exponent to 2 relatively to this dimension. The natural space for this inequality is $D^1(\mathbb{R}^n)$, the functions u vanishing at infinity having weak partial derivatives satisfying $\|D_\alpha u\|_2 < +\infty$, where this last expression denotes the right hand side of (7). Sobolev inequalities of type (7) are proved in [**FGaW**], [**FGuW**] and [**GN1**].

The first step in the search for extremal functions is to prove some a priori symmetry reducing in this way the class of competitors. In the case $\alpha = 0$, the standard technique is based on rearrangement inequalities. Indeed, the L^2 norm of the (usual) gradient of functions in $\mathbb{R}^n$ having given $L^{\frac{2n}{n-2}}$ norm is minimized in the class of radial functions. Extremal functions for the (usual) Sobolev inequality were determined using this approach by Talenti [**T**] and Aubin [**Au**]. In our case, the natural conjecture is that the class of competitors could be restricted to functions separately radial in the x and y variables. However, the proof of the symmetry in the variable x is not yet known and, in any case, the problem would still remain two dimensional.

A different approach to symmetry is provided by partial differential equations techniques. Extremal positive functions for (7) must satisfy (up to a multiplicative geometric constant) the critical point equation

$$\mathcal{L}u = \Delta_x u + |x|^{2\alpha}\Delta_y u = -u^{2^*-1} \qquad \text{in } \mathbb{R}^n. \tag{8}$$

The exponent $2^*-1=\frac{Q+2}{Q-2}$ is the critical exponent for $\mathcal{L}$. When $\alpha=0$, this equation becomes the well studied Yamabe equation ($n\geq 3$)

$$\Delta u=-u^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^n. \tag{9}$$

Gidas, Ni and Nirenberg proved in [**GNN**], under some assumptions on the behavior at infinity, that every positive solution of problem (9) is radial. These assumptions have been later removed by Caffarelli, Gidas and Spruck in [**CGS**]. The proofs rely on the Maximum Principle and on Alexandrov's moving plane method. A key point in this method is that the reflection of a solution of (9) with respect to any hyperplane is still a solution. This is no longer true for (8), because this equation is not invariant under x–translations. This corresponds to the difficulty in proving the x–radial symmetry in the rearrangement approach.

It is worth noticing that the Yamabe problem has been completely solved in the Heisenberg group by Jerison and Lee [**JL1, JL2**]. In particular, they were able to determine all positive solutions $u=u(x,y,t)$ in $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ of the semilinear equation

$$\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}+y_j\frac{\partial}{\partial t}\right)^2u+\left(\frac{\partial}{\partial y_j}-x_j\frac{\partial}{\partial t}\right)^2u=-u^{\frac{Q+2}{Q-2}}, \tag{10}$$

where $Q=2n+2$. The operator in the left hand side is known as Kohn or Heisenberg Laplacian. This operator acts on functions $u=u(z,t)$ which are radial in the variable $z=(x,y)\in\mathbb{R}^{2n}$ as a Grushin operator with $m=2n$, $k=1$ and $\alpha=1$. The difficult radial symmetry in z for solutions of (10) was established in [**JL2**] by means of some identities of complex analytic character satisfied by solutions. These results have been recently generalized by Garofalo and Vassilev in [**GV2**] to groups of Heisenberg type, but even in this setting the symmetry in the variables of the first layer is still an open problem.

We shall study equation (8) in the space $C^2(\mathbb{R}^n)\cap D^1(\mathbb{R}^n)$. The main tool of our investigation is a Kelvin transform for the Grushin operator $\mathcal{L}=\Delta_x+|x|^{2\alpha}\Delta_y$. Let $x\in\mathbb{R}^m$, $y\in\mathbb{R}^k$ and write $z=(x,y)\in\mathbb{R}^n$. Introduce the "norm" (in Chapter 2 we use a different normalization)

$$\|z\|=\left(|x|^{2(\alpha+1)}+(\alpha+1)^2|y|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

The function $\Gamma(z)=\|z\|^{2-Q}$ is a fundamental solution for $\mathcal{L}$ with pole at the origin. In the case $\alpha\in\mathbb{N}$, integral representations formulas for the fundamental solution of $\mathcal{L}$ with pole at arbitrary points in $\mathbb{R}^n$ have been recently computed by Beals, Greiner and Gaveau in [**BGG**]. The norm $\|\cdot\|$ is 1–homogeneous for the group of anisotropic dilations $\delta_\lambda(x,y)=(\lambda x,\lambda^{\alpha+1}y)$, $\lambda>0$. Define the inversion $\mathcal{I}:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n\setminus\{0\}$ by letting $\mathcal{I}(z)=\delta_{\|z\|^{-2}}(z)$. The Kelvin transform of a function $u:\mathbb{R}^n\to\mathbb{R}$ is

$$u^*(z)=\Gamma(z)u(\mathcal{I}(z)), \quad z\neq 0. \tag{11}$$

Equation (8) is invariant for the Kelvin transform.

THEOREM 2. *If* $u \in C^2(\mathbb{R}^n)$ *solves* $\mathcal{L}u = -u^{2^*-1}$ *in* $\mathbb{R}^n$ *then* $\mathcal{L}u^* = -(u^*)^{2^*-1}$ *in* $\mathbb{R}^n \setminus \{0\}$.

The Kelvin transform in the Heisenberg group was discovered by Korànyi [**K**]. The existence of such a transform also characterizes a special subclass of groups of Heisenberg type, and precisely the ones appearing as nilpotent part in the Iwasawa decomposition of simple Lie groups of real rank one (see [**CDKR**]). A Kelvin transform is also known for multiharmonic functions (see [**C**]). Apparently, there are not many other examples. It is well known that in groups of Heisenberg type the subelliptic Laplacian acting on functions which are radial in the first layer of variables is a Grushin operator with $\alpha = 1$. Thus, Theorem (2) yields an improvement of some results proved in [**GV2**] by removing the "Iwasawa assumption".

The proof of Theorem 2 relies on a conformality property for the inversion $\mathcal{I}$ in a suitable metric structure relating the "derivative" of $\mathcal{I}$ to the fundamental solution Γ (see Lemma 2.2.2 and Theorem 2.2.3). Thanks to Theorem 2, we could replace the method of "moving planes" with a method of "moving spheres". The function $\delta_\lambda u(z) = \lambda^{\frac{Q}{2}-1} u(\delta_\lambda(z))$ solves equation (8), if the function u does. Let $u_\lambda(z) = (\delta_\lambda u)^*(z)$ for $\lambda > 0$. Developing Cheng and Liongming's approach to the moving planes method in [**CL**], we prove in Theorem 2.3.6 the following symmetry result.

THEOREM 3. *Let* $u \in C^2(\mathbb{R}^n) \cap D^1(\mathbb{R}^n)$ *be a positive solution of* $\mathcal{L}u = -u^{2^*-1}$. *Then there exists* $\lambda > 0$ *such that* $u = u_\lambda$.

After a rescaling, we can assume $\lambda = 1$ in Theorem 3. The statement then is $u = u^*$, the solution is entirely determined by its values on the set $\|z\| \leq 1$. Since equation (8) is invariant with respect to translations in the variable y, Theorem 3 can be applied to any such translation of a solution u. In Corollary 2.3.7 we show that the solution u must then satisfy

$$u(0,y) = u(0,y_0)(1 + |y - y_0|^2)^{-\beta}, \quad \text{where } \beta = \frac{Q-2}{2(\alpha+1)}, \tag{12}$$

for some $y_0 \in \mathbb{R}^k$. This condition and Theorem 3 yield a hyperbolic radial symmetry for solutions to (8). This phenomenon already appeared in [**B**], where extremal functions for (6) with $\alpha = 1$ are determined by means of a rearrangement technique in the hyperbolic plane.

We describe the hyperbolic symmetry in the case $m = k = 1$. Let $H = \{\zeta = \xi + i\eta \in \mathbb{C} : \xi > 0\}$ be the half plane endowed with the hyperbolic metric, and introduce the following functional change of variable for $\xi > 0$

$$U(\zeta) = \xi^\beta u\big(\xi^{\frac{1}{\alpha+1}}, \eta\big). \tag{13}$$

The reason for introducing such a change of variable is that the equation $\mathcal{L}u = -u^{2^*-1}$ becomes a semilinear equation involving the hyperbolic Laplacian (see (2.4.4)) which is invariant under Moebius transformations. In Proposition 2.4.3, we shall show how to construct the Kelvin transform (11) using (13) and a suitable hyperbolic reflection.

THEOREM 4. *If* $u \in C^2(\mathbb{R}^2) \cap D^1(\mathbb{R}^2)$ *is a positive solution of* $\mathcal{L}u = -u^{2^*-1}$ *with* $u = u^*$ *and* $y_0 = 0$ *in* (12)*, then the function* U *defined in* (13) *is radially symmetric about the point* $(1, 0)$ *for the hyperbolic metric.*

This theorem gives a non trivial symmetry involving simultaneously the variables x and y. A corollary of the higher dimensional version of this result proved in Theorem 2.4.4 is the reduction of equation (8) to an equation involving only x. Precisely, if $u = u^* > 0$ solves $\mathcal{L}u = -u^{2^*-1}$ with $y_0 = 0$ in (12), then the function $v(x) = u(x, 0)$, $x \in \mathbb{R}^m$, solves the problem

$$\begin{cases} \operatorname{div}_x(pD_x v) - qv = -pv^{2^*-1} & |x| < 1 \\ \dfrac{\partial v}{\partial \nu} + \Big(\dfrac{Q}{2} - 1\Big)v = 0 & |x| = 1, \end{cases} \tag{14}$$

where $p = p(x)$ and $q = q(x)$ are suitable radial functions (see Corollary 2.4.5).

We have not yet been able to prove that any positive solution of (14) must be radial in x. However, in the last section of Chapter 2 we prove symmetry and uniqueness in the case $m = k = 1$. In this case, Problem (14) becomes

$$\begin{cases} (pu')' - qu + pu^{2^*-1} = 0, \quad \text{in } (-1, 1) \\ u > 0, \quad \text{in } (-1, 1) \\ \alpha u(1) + 2u'(1) = 0 \\ \alpha u(-1) - 2u'(-1) = 0, \end{cases} \tag{15}$$

where $p(x) = (1 - |x|^{2(\alpha+1)})$ and $q(x) = \alpha(\alpha + 1)|x|^{2\alpha}$.

Using a variant of the energy method introduced by Kwong and Li in [**KL**] we prove in Theorem 2.5.3 that any solution of (15) must be an even function. Then, by a shooting argument we show in Theorem 2.5.5 that the problem has at most one solution.

Chapter 3, the most difficult one, deals with regular domains in $\mathbb{R}^n$ for Grushin metrics. There are several definitions of "regular domain". We recall the notion of domain with the interior (twisted) cone property (or John domain). Consider an open set Ω in a metric space (M, d). Given a rectifiable path $\gamma : [0, 1] \to M$, the cone with core at γ and aperture $\varepsilon > 0$ is the set $C(\gamma, \varepsilon) = \bigcup_{0<t<1} B\big(\gamma(t), \varepsilon \,\text{length}(\gamma|_{(0,t)})\big)$, where B denote metric balls. The set Ω is a John domain with John constant ε and center $x_0 \in \Omega$ if for any $x \in \Omega$ there is a cone $C(\gamma, \varepsilon) \subset \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = x_0$.

The cone property was introduced in the Euclidean setting by John in his paper [**Joh**] on the rigidity of quasiisometric maps in $\mathbb{R}^n$. Besides its importance in geometric function theory, this property plays a central role in the theory of first order Sobolev spaces (see e.g. [**Re**], [**Bes**], [**M**], [**Bo**], [**MP**] and the more recent references [**SS**], [**BK**], [**HK1**], [**KOT**]). The cone property is also related to chaining conditions that are useful in the proof of Sobolev–Poincaré inequalities. This fact was recognized by Jerison in [**J**] and later used by several authors (see [**L**], [**FGuW**], [**GN1**], [**BKL**]).

In the memoir [**HK2**] by Hajłasz and Koskela, a nice reference on the subject, all these ideas are developed in general metric spaces.

Other classes of domains appear in more refined questions in harmonic analysis, partial differential equations and quasiconformal mappings: uniform domains and non–tangentially accessible domains are the most important examples. The definition of uniform domain (or (ε, δ)–domain) is due to Martio and Sarvas [**MS**] and to Jones [**Jo**] (see Definition 3.5.1). In particular, Jones' extension theorem for Sobolev functions in uniform domains has been generalized to subelliptic Sobolev spaces in [**GN2**], [**VG**] and [**G1**]. Uniform domains also play a special role in the trace problem for Sobolev functions. This theory has been developed in Carnot–Carathéodory spaces by Danielli, Garofalo and Nhieu [**DGN1**] (see also [**MM1**]). A subclass of uniform domains is formed by non–tangentially accessible domains (briefly nta domains) which, in the Euclidean case, were introduced by Jerison and Kenig [**JK**] in connection with the study of the boundary behavior of harmonic functions. The notion of nta domain is purely metric (see Definition 3.5.3) and plays an important role in potential theory, boundary behavior problems for harmonic functions and harmonic measures (see, for instance, [**CG**] and [**FeF**] for the subelliptic case).

In spite of this general theory, not many examples of regular domains are known in metric spaces non bi–Lipschitz equivalent to $\mathbb{R}^n$ with the Euclidean metric. Some results for metrics associated with vector fields, the case we are interested in, can be found in [**HH**], [**CT**], [**VG**], [**CG**], [**CGN**], [**G2**], [**CGP**], [**FeF**], [**DGN1**]. In particular, in [**MM2**] it is shown that any C^2 bounded domain in homogeneous (Carnot) groups of step 2 is non–tangentially accessible. In the same work a sufficient condition for the John property is provided for the step 3 case (Engel group). The difficulty of the problem of finding examples is due to the fact that even the C^∞ regularity of the boundary does not necessarily guarantee the metric regularity of the domain. At *characteristic points* may appear a "cuspidal behavior" of the boundary which destroys regularity (see [**BM**] for a study of the "boundary accessibility" at characteristic points in the Heisenberg group). This was already noticed in [**J**] just in the case of the Grushin plane. Motivated by the need of understanding the role of characteristic points we studied regular domains for Grushin metrics.

We describe the model situation considered in Chapter 3. Let $\alpha_1, \alpha_2 \in \mathbb{N}$ be fixed natural numbers and consider the vector fields in $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1^{\alpha_1} \frac{\partial}{\partial x_2}, \quad X_3 = x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial}{\partial x_3}. \tag{16}$$

These vector fields induce on $\mathbb{R}^3$ a metric d, which is known as control, Carnot–Carathéodory or sub–Riemannian distance associated with (16). The problem is the following: given a bounded open set $\Omega \subset \mathbb{R}^3$, find geometric conditions on $\partial\Omega$ ensuring the John (uniform, nta) property in the metric space $(\mathbb{R}^3, d)$.

A point $x \in \partial\Omega$ is characteristic if X_1, X_2, X_3 are all tangent to the boundary at x. In this case any integral curve of the vector fields starting from x is tangent to the boundary at this point and the cone property becomes critical. On the other hand, if

x is noncharacteristic then there is an integral curve transversal to $\partial\Omega$ starting from x which will be the core of a suitable interior cone. The quantitative understanding of this phenomenon requires a precise knowledge of metric balls.

By the results of [**FL**], balls $B(x,r)$ in the metric d are comparable with the following 3–dimensional boxes (see Theorem 3.1.1)

$$\mathrm{Box}(x,r) = Q(x,r) \times [x_3 - F_3(x,r), x_3 + F_3(x,r)],$$

where $Q(x,r) = [x_1 - r, x_1 + r] \times [x_2 - F_2(x,r), x_2 + F_2(x,r)]$, $F_2(x,r) = r(|x_1| + r)^{\alpha_1}$, and $F_3(x,r) = F_2(x,r)(|x_2| + F_2(x,r))^{\alpha_2}$.

Now consider an open set in $\mathbb{R}^3$ of the form $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 > \varphi(x_1,x_2)\}$ for some function $\varphi \in C^1(\mathbb{R}^2)$. We are going to introduce a definition of "admissible boundary". Assume for a moment that $0 \in \partial\Omega$ is a characteristic point, i.e. $X_1\varphi(0) = \partial_1\varphi(0) = 0$. A curve γ core of a cone $C(\gamma,\varepsilon) \subset \Omega$ with vertex near 0 must be approximately of the form $\gamma(t) = (0,0,t)$, $t > 0$. For $x_3 > 0$ and $r > 0$ small, the box

$$\mathrm{Box}((0,0,x_3),r) = [-r,r] \times [-r^{\alpha_1+1}, r^{\alpha_1+1}] \times [x_3 - r^{(\alpha_1+1)(\alpha_2+1)}, x_3 + r^{(\alpha_1+1)(\alpha_2+1)}]$$

is very large in the first two components with respect to the third one. In fact, the vertical size of the box behaves as $r^{(\alpha_1+1)(\alpha_2+1)} = r^{d_3}$. Therefore, in order the cone property to hold, $X_1\varphi$ and $X_2\varphi$ are expected to vanish fast enough at 0. Quantitatively, this can be formulated in the following way.

The boundary $\partial\Omega$ is said to be *admissible* if there is a constant $C > 0$ such that for all $x = (x_1,x_2) \in \mathbb{R}^2$ and $r > 0$ we have

$$\sum_{i=1,2} \mathrm{osc}(X_i\varphi, Q(x,r)) \le C\Big(r \sum_{i=1,2} |X_i\varphi(x)|^{\frac{d_3-2}{d_3-1}} + \mathrm{osc}(\lambda_3, Q(x,r))\Big). \tag{17}$$

Here, $\lambda_3(y) = y_1^{\alpha_1} y_2^{\alpha_2}$ is the coefficient of X_3. The oscillation of the derivatives of the function φ along the vector fields X_1 and X_2 is bounded by a sum of two terms. The first term in the right hand side vanishes on the characteristic set, while the second one gives an amount of oscillation admitted also at characteristic points. The latter is determined by the oscillation of the function λ_3 on $Q(x,r)$, the section of metric balls in the first two coordinates. This oscillation is also related to the size of metric balls in the vertical direction. The appropriate balance between the two terms is described by the power $\frac{d_3-2}{d_3-1}$ appearing in the first term. This delicate choice is a key point.

In Definition 3.2.6, we generalize (17) and we introduce a class of *domains with admissible boundary* in the $n-$dimensional situation. Condition (17) can be easily checked. For instance, in Theorem 3.3.2 we show that the open set

$$\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : \big(|x_1|^{2(\alpha_1+1)} + x_2^2\big)^{1+\alpha_2} + x_3^2 < 1\}$$

has admissible boundary for the vector fields (16).

Relatively to $\mathbb{R}^3$, the main results in Chapter 3 can be stated as follows (see Theorems 3.4.3 and 3.5.7).

THEOREM 5. *If $\Omega \subset \mathbb{R}^3$ is a domain with admissible boundary, then: (i) it is a John domain in the metric space $(\mathbb{R}^3, d)$; (ii) it is non–tangentially accessible in the metric space $(\mathbb{R}^3, d)$.*

Actually, statement (i) is contained in statement (ii). The proof relies upon a careful construction of cones. The main problem has been to understand how to choose the core γ. This construction is introduced in the proof of Theorem 3.4.3. The reading of Chapter 3 could be difficult. The main steps through it are the following: 1) structure of the boxes (3.1.12); 2) condition (3.2.4) for admissible surfaces; 3) construction of John curves in Theorem 3.4.3; 4) discussion preceding Lemma 3.5.6; 5) Theorem 3.5.7.

Even though we are not going to discuss any application, we would like to illustrate the interest of Theorem 5 with two examples. A corollary of part (i) is the following Sobolev–Poincaré inequality. Let $\Omega \subset \mathbb{R}^3$ be an admissible domain and let $Q = 1 + (\alpha_1 + 1)(\alpha_2 + 2)$. For any $1 \le p < Q$ there exists a constant $C > 0$ such that for all functions $u \in C^1(\Omega)$

$$\Big(\int_\Omega |u(x) - u_\Omega|^{\frac{pQ}{Q-p}} dx\Big)^{\frac{Q-p}{pQ}} \le C\Big(\int_\Omega |D_\alpha u(x)|^p dx\Big)^{\frac{1}{p}},$$

where $|D_\alpha u|^2 = \sum_{i=1}^3 |X_i u|^2$ and u_Ω denotes the average of u over Ω. This inequality is proved for John domains in various metric spaces in [**FGuW**], [**GN1**] and [**HK2**].

A corollary of Theorem 5 part (ii) is the following Besov trace estimate. Let $\Omega \subset \mathbb{R}^3$ be an admissible domain, $1 < p < +\infty$ and $s = 1 - 1/p$. Then there is a constant $C > 0$ such that for all functions $u \in C^1(\Omega) \cap C(\bar{\Omega})$

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p d\mu(x) d\mu(y)}{d(x,y)^{ps}\mu(B(x, d(x,y)))} \le C \int_\Omega |D_\alpha u(x)|^p dx.$$

Here, d is the Grushin metric, B denotes a metric ball, and μ is a surface measure on $\partial\Omega$ depending on X_1, X_2, X_3 (this is the perimeter measure induced by Ω, which can be defined analogously to (1)). Besov estimates of this kind are proved in [**DGN1**] assuming Ω to be a uniform domain with Ahlfors regular boundary in a Carnot–Carathéodory space.

Finally, we describe the author's contribution to the results contained in this work. Chapters 1 and 3 are based on the papers [**Mo**], [**MM3**], [**MM4**], [**MM5**]. The last three are joint work with Daniele Morbidelli. Chapter 2 is entirely new and has been written for this work. The results are part of a research program together with D. Morbidelli on critical semilinear equations. Besides being a friend, Daniele is my favorite coauthor. Actually, it is not possible to determine exactly the contribution of each of us to our research, which is based on a day by day exchange of ideas. Nevertheless, I try to give some indication. The form (3) of Grushin isoperimetric sets and the key symmetry argument for the proof of Theorem 1 have been found by the author. The form (11) of the Kelvin transform for $\mathcal{L}$ and Theorem 2 were originally established by the author, but the elegant and shorter conformal proof based on

Theorem 2.2.3 has been shown to me by D. Morbidelli. It is simply impossible to say who of us discovered condition (17) for regular boundaries in Grushin spaces. However, it was D. Morbidelli who realized how to prove a key technical step towards the uniform condition (this is Lemma 3.5.6). All other results are joint work to which we gave the same contribution.

This Habilitationsschrift collects part of my research work as postdoc at the Mathematisches Institut of Bern University in the years 2002–2003. I would like to acknowledge with gratitude the Institut for the opportunity it gave me to work under the *best* conditions. Especially, I would like to thank M. Reimann and Z. Balogh for their friendly hospitality. With M. Rickly I shared many interesting discussions. His comments helped me to improve Chapter 1.

CHAPTER 1

Isoperimetric inequality in the Grushin plane

1. Perimeter in the Grushin plane

We define the perimeter of a measurable set in the Grushin plane and we study some of its basic properties. In this chapter $\alpha \geq 0$ is a fixed real number. A measurable set $E \subset \mathbb{R}^2$ is a Lebesgue measurable set in the plane and its measure is denote by $|E|$.

Introduce the family of test functions

$$\mathcal{F}(\mathbb{R}^2) = \left\{ \varphi = (\varphi_1, \varphi_2) \in C_0^1(\mathbb{R}^2; \mathbb{R}^2) : \|\varphi\|_\infty \leq 1 \right\},$$

where $\|\varphi\|_\infty = \sup\limits_{\mathbb{R}^2} (\varphi_1^2 + \varphi_2^2)^{1/2}$.

The α–*divergence* of a vector valued function $\varphi \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ is $\mathrm{div}_\alpha \varphi = \partial_x \varphi_1 + |x|^\alpha \partial_y \varphi_2$. Following [**GN1**], we define the α–perimeter of a measurable subset E of $\mathbb{R}^2$ as

$$P_\alpha(E) = \sup_{\varphi \in \mathcal{F}(\mathbb{R}^2)} \int_E \mathrm{div}_\alpha \varphi(x,y)\, dxdy. \tag{1.1.1}$$

Two measurable sets $E, F \subset \mathbb{R}^2$ are said to be equivalent if $|E \setminus F| = |F \setminus E| = 0$. Equivalent sets have the same α–perimeter. Our results are stated and hold up to equivalence of sets. If $P_\alpha(E) < +\infty$, the set E is said to have finite α–perimeter. We shall only consider sets E with finite measure $|E| < +\infty$. In the sequel, when $\alpha = 0$ we shall omit the subscript α, reducing our definitions to the classical (Euclidean) ones.

A key feature of definition (1.1.1) is the following lower semicontinuity property. Let $(E_h)_{h\in\mathbb{N}}$ be a sequence of measurable sets whose characteristic functions are converging in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ to the characteristic function of a set E. Then

$$P_\alpha(E) \leq \liminf_{h\to\infty} P_\alpha(E_h). \tag{1.1.2}$$

Such a lower semicontinuity and a compactness argument will give the existence of isoperimetric sets.

When the set E has regular boundary, its α–perimeter has the following integral representation.

Theorem 1.1.1. *Let $E \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Then*

$$P_\alpha(E) = \int_{\partial E} \left(n_1(x,y)^2 + |x|^{2\alpha} n_2(x,y)^2 \right)^{1/2} d\mathcal{H}^1, \tag{1.1.3}$$

where $n(x,y) = (n_1(x,y), n_2(x,y))$ is the (outward) unit normal to ∂E at the point $(x,y) \in \partial E$, and $\mathcal{H}^1$ is the 1–dimensional Hausdorff measure in the plane.

PROOF. Since ∂E is locally the graph of Lipschitz functions, the normal $n(x,y)$ is defined for $\mathcal{H}^1 -$ a.e. $(x,y) \in \partial E$ and is a $\mathcal{H}^1-$measurable function on ∂E. Let $F \subset \partial E$ be the set of points of ∂E where n is defined.

Fix a test function $\varphi \in \mathcal{F}(\mathbb{R}^2)$ and recall that $\|\varphi\|_\infty \le 1$. Using the divergence theorem and the Cauchy–Schwarz inequality we get

$$\int_E \mathrm{div}_\alpha \varphi \, dxdy = \int_{\partial E} (n_1\varphi_1 + |x|^\alpha n_2 \varphi_2) \, d\mathcal{H}^1 \le \int_{\partial E} \left(n_1^2 + |x|^{2\alpha} n_2^2\right)^{1/2} d\mathcal{H}^1 := I.$$

The inequality $P_\alpha(E) \le I$ follows by taking the supremum over all test functions.

We have to prove the converse inequality $I \le P_\alpha(E)$. Note first that the set $G = \{(x,y) \in F : \, x = 0 \text{ and } n_1(x,y) = 0\}$ is at most countable, because it is discrete. Fix a number $\varepsilon > 0$. By Lusin theorem there exists a compact set $K \subset F \setminus G$ such that $n_{|K}$ is continuous on K and $\mathcal{H}^1(\partial E \setminus K) \le \varepsilon$. Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, $Q = [-1,1] \times [-1,1]$. Fix a homeomorphism $g : B \to Q$.

The function $\nu : K \to B$ defined by

$$\nu(x,y) = \frac{(n_1(x,y), |x|^\alpha n_2(x,y))}{\left[n_1(x,y)^2 + |x|^{2\alpha} n_2(x,y)^2\right]^{1/2}}, \quad (x,y) \in K,$$

is continuous on K. The map $g \circ \nu : K \to Q$ can be extended to a continuous function from $\mathbb{R}^2$ to Q with compact support (this can be seen by applying Tietze–Urysohn theorem to both its components). Taking the composition of this function with g^{-1} we find a continuous function $\psi \in C_0(\mathbb{R}^2; B)$ such that $\psi = \nu$ on K. Write

$$I = \int_{\partial E} (n_1\psi_1 + |x|^\alpha n_2\psi_2) \, d\mathcal{H}^1 - \int_{\partial E \setminus K} \Big(n_1\psi_1 + |x|^\alpha n_2 \psi_2 - \left(n_1^2 + |x|^{2\alpha} n_2^2\right)^{1/2} \Big) \, d\mathcal{H}^1.$$

Since $\mathcal{H}^1(\partial E \setminus K) \le \varepsilon$, $\|n\|_\infty \le 1$ and $\|\psi\|_\infty \le 1$, there exists a constant C depending on α and E such that

$$\int_{\partial E \setminus K} \Big| n_1\psi_1 + |x|^\alpha n_2\psi_2 - \left(n_1^2 + |x|^{2\alpha} n_2^2\right)^{1/2} \Big| \, d\mathcal{H}^1 \le C\varepsilon.$$

Then it follows that

$$\int_{\partial E} (n_1\psi_1 + |x|^\alpha n_2\psi_2) \, d\mathcal{H}^1 \ge I - C\varepsilon.$$

Let $(J_\eta)_{\eta > 0}$ be a family of mollifiers and define $\psi_\eta = J_\eta * \psi$. Then $\psi_\eta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$, $\|\psi_\eta\|_\infty \le 1$ and $\psi_\eta \to \psi$ uniformly as $\eta \to 0$. Choosing $\varphi = \psi_\eta$ with $\eta > 0$ small enough we get

$$\int_E \mathrm{div}_\alpha \varphi \, dxdy = \int_{\partial E} (n_1\varphi_1 + |x|^\alpha n_2\varphi_2) \, d\mathcal{H}^1 \ge I - 2C\varepsilon,$$

and since $\varphi \in \mathcal{F}(\mathbb{R}^2)$ we have $P_\alpha(E) \ge I - 2C\varepsilon$. But $\varepsilon > 0$ is arbitrary. Then the claim $P_\alpha(E) \ge I$ is proved. □

Consider the real number $Q = 2 + \alpha$. Lebesgue measure and α–perimeter are respectively Q–homogeneous and $(Q-1)$–homogeneous with respect to the dilations $(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$.

PROPOSITION 1.1.2. *Let $E \subset \mathbb{R}^2$ be a measurable set. Then for all $\lambda > 0$*
(i) $|\delta_\lambda(E)| = \lambda^Q |E|$;
(ii) $P_\alpha(\delta_\lambda(E)) = \lambda^{Q-1} P_\alpha(E)$.

PROOF. We prove (ii). Let $\varphi \in \mathcal{F}(\mathbb{R}^2)$ and write

$$\begin{aligned}\int_{\delta_\lambda(E)} \mathrm{div}_\alpha \varphi(x, y)\, dxdy &= \int_{\delta_\lambda(E)} \Big(\partial_x \varphi_1(x, y) + |x|^\alpha \partial_y \varphi_2(x, y)\Big)\, dxdy \\ &= \int_E \Big(\frac{1}{\lambda} \partial_\xi \varphi_1(\lambda\xi, \lambda^{\alpha+1}\eta) + \lambda^\alpha |\xi|^\alpha \frac{1}{\lambda^{\alpha+1}} \partial_\eta \varphi_2(\lambda\xi, \lambda^{\alpha+1}\eta)\Big)\, \lambda^Q d\xi d\eta \\ &= \lambda^{Q-1} \int_E \mathrm{div}_\alpha(\varphi \circ \delta_\lambda)(\xi, \eta)\, d\xi d\eta \le \lambda^{Q-1} P_\alpha(E),\end{aligned}$$

because $\varphi \circ \delta_\lambda \in \mathcal{F}(\mathbb{R}^2)$. Taking the supremum over test functions gives $P_\alpha(\delta_\lambda(E)) \le \lambda^{Q-1} P_\alpha(E)$. The converse inequality is obtained in the same way. $\square$

We introduce a change of variable that transforms the α–perimeter of a set into the Euclidean perimeter of the transformed set. Consider the functions $\Phi, \Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi(\xi, \eta) = \Big(\mathrm{sgn}(\xi) |(\alpha+1)\xi|^{\frac{1}{\alpha+1}}, \eta\Big), \quad \Psi(x, y) = \left(\mathrm{sgn}(x) \frac{|x|^{\alpha+1}}{\alpha+1}, y\right). \tag{1.1.4}$$

Clearly, Ψ is a homeomorphism and Φ is its inverse. Notice that $|\det J\Phi(\xi, \eta)| = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}}$ for $\xi \neq 0$.

PROPOSITION 1.1.3. *Let $E \subset \mathbb{R}^2$ be a measurable set and define $F = \Psi(E)$. Then $P(F) = P_\alpha(E)$.*

PROOF. Take a test function $\varphi \in \mathcal{F}(\mathbb{R}^2)$. A short computation gives

$$\begin{aligned}\int_E \mathrm{div}_\alpha \varphi(x, y) dxdy &= \int_E [\partial_x \varphi_1(x, y) + |x|^\alpha \partial_y \varphi_2(x, y)]\, dxdy \\ &= \int_F [\partial_\xi(\varphi_1 \circ \Phi)(\xi, \eta) + \partial_\eta(\varphi_2 \circ \Phi)(\xi, \eta)]\, d\xi d\eta.\end{aligned}$$

Note that the function $\partial_\xi(\varphi_1 \circ \Phi)(\xi, \eta) = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} (\partial_1 \varphi_1)(\Phi(\xi, \eta))$ is in $L^1(\mathbb{R}^2)$, because $\partial_1 \varphi_1$ is bounded and with compact support, and the singular term $|\xi|^{-\frac{\alpha}{\alpha+1}}$ is locally integrable. The same happens for $\partial_\eta(\varphi_2 \circ \Phi)$.

By known density theorems for Sobolev spaces

$$\begin{aligned}P(F) &= \sup_{\psi \in \mathcal{F}(\mathbb{R}^2)} \int_F \mathrm{div}\psi(\xi, \eta) d\xi d\eta \\ &= \sup \left\{ \int_F \mathrm{div}\psi(\xi, \eta) d\xi d\eta :\ \psi_1, \psi_2 \in \mathrm{W}^{1,1}(\mathbb{R}^2),\ \psi_1^2 + \psi_2^2 \le 1 \text{ a.e.} \right\}.\end{aligned}$$

Then it follows that

$$\int_E \mathrm{div}_\alpha \varphi(x,y)dxdy \le P(F).$$

Taking the supremum over test functions we find $P_\alpha(E) \le P(F)$. The converse inequality can be achieved by the same argument, using the function Ψ instead of Φ. □

2. Isoperimetric inequality

We prove the isoperimetric inequality in the Grushin plane. First we need a theorem that reduces the problem to convex and symmetric sets. To this aim we introduce some definitions concerning geometrical properties of sets. A set $E \subset \mathbb{R}^2$ is *x–symmetric* if $(x,y) \in E$ implies $(-x,y) \in E$. E is *y–symmetric* if $(x,y) \in E$ implies $(x,-y) \in E$. Finally, E is said to be *symmetric* if it is both x– and y–symmetric.

Given a set $E \subset \mathbb{R}^2$ define for every $x, y \in \mathbb{R}$

$$E^x = \{y \in \mathbb{R} : (x,y) \in E\}, \quad E^y = \{x \in \mathbb{R} : (x,y) \in E\}.$$

A set $E \subset \mathbb{R}^2$ is *x–convex* if E^y is an (open or empty) interval for all $y \in \mathbb{R}$. E is *y–convex* if E^x is an (open or empty) interval for all $x \in \mathbb{R}$. Finally, E will be said to be *separately convex* if it is both x– and y–convex.

Theorem 1.2.1. *Let $E \subset \mathbb{R}^2$ be a measurable set with $P_\alpha(E) < +\infty$ and $0 < |E| < +\infty$. There exists a symmetric, convex set $E^* \subset \mathbb{R}^2$ such that $P_\alpha(E^*) \le P_\alpha(E)$ and $|E^*| = |E|$. Moreover, in case $\alpha > 0$, if E is not (equivalent to) an x–symmetric and convex set, then the strict inequality $P_\alpha(E^*) < P_\alpha(E)$ holds.*

Proof. Let $E \subset \mathbb{R}^2$ be a measurable set with positive and finite measure and finite perimeter. Define $F = \Psi(E)$, where Ψ is the map introduced in (1.1.4). By Proposition 1.1.3, $P(F) = P_\alpha(E) < +\infty$. Moreover, letting

$$\mu(F) = \int_F |(\alpha+1)\xi|^\beta d\xi d\eta, \quad \beta = -\frac{\alpha}{\alpha+1},$$

we find

$$|E| = \int_{\Phi(F)} dxdy = \int_F |\det J\Phi(\xi,\eta)| d\xi d\eta = \mu(F).$$

Let F_1 be the Steiner symmetrization of F in the η–direction. Precisely,

$$F_1 = \left\{(\xi,\eta) \in \mathbb{R}^2 : |\eta| < \frac{1}{2}|F^\xi|\right\}.$$

Here, $|\cdot|$ stands for 1 dimensional Lebesgue measure. By [**DG**, Teorema II], (see also [**T**, Section 3.8]), $P(F_1) \le P(F)$, where the inequality is strict if F is not (equivalent to) an η–convex set. Moreover, by Fubini–Tonelli Theorem

$$\begin{aligned}\mu(F) &= \int_F |(\alpha+1)\xi|^\beta d\xi d\eta = \int_{-\infty}^{+\infty} |(\alpha+1)\xi|^\beta |F^\xi|\, d\xi \\ &= \int_{-\infty}^{+\infty} |(\alpha+1)\xi|^\beta |F_1^\xi|\, d\xi = \mu(F_1),\end{aligned}$$

because $|F^\xi| = |F_1^\xi|$ for all $\xi \in \mathbb{R}$.

Let F_2 be the Steiner symmetrization of F_1 in the ξ–direction. Precisely,

$$F_2 = \left\{ (\xi, \eta) \in \mathbb{R}^2 : |\xi| < \frac{1}{2}|F_1^\eta| \right\}.$$

Then, as above, $P(F_2) \le P(F_1) \le P(F)$. Consider the volume

$$\mu(F_2) = \int_{F_2} |(\alpha+1)\xi|^\beta d\xi d\eta = \int_{-\infty}^{+\infty} \left(\int_{F_2^\eta} |(\alpha+1)\xi|^\beta \, d\xi \right) d\eta.$$

In order to estimate the last term, we use the following elementary fact. Given a measurable set $I \subset \mathbb{R}$ with finite measure, denote by $I^* = (-|I|/2, |I|/2))$ its symmetrized set. Since the number β is negative, we have $|\xi|^\beta \ge (|I|/2)^\beta$ if $\xi \in I^*$, and $|\xi|^\beta \le (|I|/2)^\beta$ if $\xi \in I \setminus I^*$. Thus

$$\begin{aligned} \int_I |\xi|^\beta d\xi &= \int_{I\cap I^*} |\xi|^\beta d\xi + \int_{I\setminus I^*} |\xi|^\beta d\xi \le \int_{I\cap I^*} |\xi|^\beta d\xi + \left(\frac{|I|}{2}\right)^\beta |I \setminus I^*| \\ &= \int_{I\cap I^*} |\xi|^\beta d\xi + \left(\frac{|I|}{2}\right)^\beta |I^* \setminus I| \le \int_{I^*} |\xi|^\beta d\xi. \end{aligned}$$

The inequality is strict if and only if I is not equivalent to I^*.

From the above considerations it follows that $\mu(F_2) \ge \mu(F_1)$ with equality if and only if F_1 is (equivalent to) an x–symmetric and x–convex set.

F_2 is a symmetric, separately convex open set. Moreover, ∂F_2 is the union of the image of four 1–Lipschitz curves. This can be easily visualized by looking at the set after a rotation of 45 degrees. More precisely, for all $s \in \mathbb{R}$ such that the set written below is nonempty, define the function

$$\vartheta(s) = \sup \left\{ t > |s| : \left(\frac{t+s}{\sqrt{2}}, \frac{t-s}{\sqrt{2}} \right) \in F_2 \right\}.$$

F_2 is separately convex and then ϑ is 1–Lipschitz. Moreover, $\partial F_2 \cap \{\xi > 0, \eta > 0\}$ is a graph of the form $t = \vartheta(s)$ in the variables $s = (\xi - \eta)/\sqrt{2}$ and $t = (\xi + \eta)/\sqrt{2}$. From a well known characterization of Euclidean perimeter, it follows that $P(F_2) = \mathcal{H}^1(\partial F_2)$.

Let $F_3 = \mathrm{co}(F_2)$ be the convex hull of F_2. Since $F_2 \subset F_3$, it follows that $\mu(F_2) \le \mu(F_3)$ with strict inequality if F_2 is not a convex set. Write $\partial F_3 = (\partial F_3 \cap \partial F_2) \cup (\partial F_3 \setminus \partial F_2)$, where $\partial F_3 \setminus \partial F_2$ is the disjoint union of an at most countable family of line segments $I_n = (p_n, q_n) \subset \mathbb{R}^2$, $n \in \mathbb{N}$. Analogously, $\partial F_2 = (\partial F_2 \cap \partial F_3) \cup (\partial F_2 \setminus \partial F_3)$, where $\partial F_2 \setminus \partial F_3$ is the disjoint union of an at most countable family of rectifiable curves γ_n, $n \in \mathbb{N}$. After a relabelling, we can assume that γ_n connects p_n and q_n. Then the length of γ_n is greater than that of I_n, and therefore $P(F_3) = \mathcal{H}^1(\partial F_3) \le \mathcal{H}^1(\partial F_2) = P(F_2)$.

Define $E^* = \delta_\lambda(\Phi(F_3))$, where $\lambda > 0$ is chosen in order to ensure $|E^*| = |E|$ (it turns out that $\lambda \le 1$, see below). The set E^* is symmetric because Φ preserves symmetry. We show that E^* is also convex. Since the map δ_λ is linear, it is sufficient to show that $\Phi(F_3)$ is convex. Let $(x_0, y_0), (x_1, x_1) \in \Phi(F_3)$ and write $(x_i, y_i) = \Phi(\xi_i, \eta_i)$,

$(\xi_i, \eta_i) \in F_3$, $i = 0, 1$. $\Phi(F_3)$ is symmetric and separately convex and therefore we can without loss of generality assume $x_0, x_1 \geq 0$. Clearly, $\Phi(\tau(\xi_0, \eta_0) + (1-\tau)(\xi_1, \eta_1)) \in \Phi(F_3)$, $\tau \in [0, 1]$, because F_3 is convex. From the concavity inequality

$$\tau\xi_0^{\frac{1}{\alpha+1}} + (1-\tau)\xi_1^{\frac{1}{\alpha+1}} \leq (\tau\xi_0 + (1-\tau)\xi_1)^{\frac{1}{\alpha+1}}, \quad \tau \in [0,1],\ \xi_0, \xi_1 \geq 0,$$

and from x–symmetry, x– and y–convexity of $\Phi(F_3)$, it follows that $\tau\Phi(\xi_0, \eta_0) + (1-\tau)\Phi(\xi_1, \eta_1) \in \Phi(F_3)$ for all $\tau \in [0, 1]$.

Notice that $|\Phi(F_3)| = \mu(F_3) \geq \mu(F_2) \geq \mu(F_1) = \mu(F) = |E|$, and then it must be $\lambda \leq 1$, with $\lambda < 1$ if E is not (equivalent to) an x–symmetric, convex set. Moreover, by Propositions 1.1.2 and 1.1.3 it follows that

$$\lambda^{1-Q} P_\alpha(E^*) = P_\alpha(\Phi(F_3)) = P(F_3) \leq P(F_2) \leq P(F) = P_\alpha(E).$$

Hence, $P_\alpha(E^*) \leq P_\alpha(E)$ with strict inequality if E is not (equivalent to) an x–symmetric, convex set. ☐

A measurable set with positive and finite measure minimizing the ratio $P_\alpha(E)^{\frac{Q}{Q-1}}/|E|$ will be called an *isoperimetric set*. The class of isoperimetric sets is invariant under dilations $(x, y) \mapsto \delta_\lambda(x, y)$, $\lambda > 0$, and under vertical translations $(x, y) \mapsto (x, y + h)$, $h \in \mathbb{R}$.

THEOREM 1.2.2. *Let $\alpha > 0$ and $Q = \alpha + 2$. There exists a constant $c(\alpha) > 0$ such that for any measurable set $E \subset \mathbb{R}^2$ with finite measure*

$$|E| \leq c(\alpha) P_\alpha(E)^{\frac{Q}{Q-1}}. \tag{1.2.1}$$

The constant $c(\alpha)$ is determined by equality in (1.2.1) achieved by the isoperimetric set

$$E_\alpha = \left\{ (x, y) \in \mathbb{R}^2 : |y| < \int_{\arcsin|x|}^{\pi/2} \sin^{\alpha+1}(t)\, dt,\ |x| < 1 \right\}. \tag{1.2.2}$$

Precisely,

$$c(\alpha) = \frac{\alpha+1}{\alpha+2} \left(2 \int_0^\pi \sin^\alpha(t) dt \right)^{-\frac{1}{\alpha+1}}. \tag{1.2.3}$$

Isoperimetric sets are unique up to dilations and vertical translations.

PROOF. Consider the following minimum problem

$$\min\{P_\alpha(E) : E \subset \mathbb{R}^2 \text{ measurable set with } |E| = 1\}. \tag{1.2.4}$$

We study the existence of solutions by the direct method of the calculus of variations. By Theorem 1.2.1 the class of admissible sets can be restricted to symmetric and convex sets. Recall that a set is symmetric if it is both x– and y–symmetric. Define

$$\mathcal{A} = \{E \subset \mathbb{R}^2 : E \text{ symmetric, convex set with } |E| = 1 \text{ and } P_\alpha(E) \leq k\}.$$

Here $k > 0$ is any fixed constant large enough to ensure $\mathcal{A} \neq \emptyset$. Such a constant does exist.

We claim that any set $E \in \mathcal{A}$ is contained in the rectangle $[-a,a]\times[-b,b]$, where $a>0$ and $b>0$ depend only on k and α. Fix a number $\varepsilon>0$. Let $\psi_\varepsilon \in C^1(\mathbb{R})$ be an increasing function such that $\psi_\varepsilon(y)=1$ if $y\geq\varepsilon$ and $\psi_\varepsilon(y)=-1$ if $y\leq-\varepsilon$. Take a set $E\in\mathcal{A}$ and let $a=\sup\{x>0:|E^x|>0\}$, $b=\sup\{y>0:|E^y|>0\}$, $a_\varepsilon=\sup\{x>0:|E^x|>2\varepsilon\}$ and $b_\varepsilon=\sup\{y>0:|E^y|>2\varepsilon\}$. The numbers a_ε and b_ε are both finite and tend respectively to a and b, as $\varepsilon\to0$. Choose the test function $\varphi_\varepsilon(x,y)=(0,\vartheta(x,y)\psi_\varepsilon(y))\in\mathcal{F}(\mathbb{R}^2)$, where $\vartheta\in C^1_0(\mathbb{R}^2)$ is a function such that $\chi_E\leq\vartheta\leq1$. We have

$$\begin{aligned} k\geq P_\alpha(E) &\geq \int_E |x|^\alpha \partial_y(\vartheta(x,y)\psi_\varepsilon(y))dxdy = \int_E |x|^\alpha\partial_y\psi_\varepsilon(y)dxdy \\ &= \int_{-a}^{a}|x|^\alpha\int_{E^x}\partial_y\psi_\varepsilon(y)dy\,dx \geq 2\int_{-a_\varepsilon}^{a_\varepsilon}|x|^\alpha dx = 4\frac{a_\varepsilon^{\alpha+1}}{\alpha+1}. \end{aligned} \tag{1.2.5}$$

Since $a_\varepsilon\to a$ when $\varepsilon\to0$, we get $4a^{\alpha+1}\leq k(\alpha+1)$. A similar argument shows that $4b\leq k$. The claim is proved.

Let $(E_h)_{h\in\mathbb{N}}\subset\mathcal{A}$ be a minimizing sequence for problem (1.2.4)

$$\lim_{h\to\infty}P_\alpha(E_h)=\inf\{P_\alpha(E):E\in\mathcal{A}\}.$$

The sets $F_h=\Psi(E_h)$ are contained in the bounded set $\Psi([-a,a]\times[-b,b])$. Moreover, by Proposition 1.1.3, $P(F_h)=P_\alpha(E_h)\leq k$ for all $h\in\mathbb{N}$. The space of functions with bounded variation $\mathrm{BV}(\mathbb{R}^2)$ is compactly embedded in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Therefore, possibly extracting a subsequence, there exists a measurable set $F\subset\Psi([-a,a]\times[-b,b])$ such that $\chi_{F_h}\to\chi_F$ in $L^1(\mathbb{R}^2)$. Letting $E=\Phi(F)$, it follows that $\chi_{E_h}\to\chi_E$ in $L^1(\mathbb{R}^2)$. The set E is (equivalent to) an $x-$ and $y-$symmetric and convex set. This follows from the fact that χ_{E_h} can be also assumed to converge almost everywhere to χ_E. By the lower semicontinuity (1.1.2)

$$P_\alpha(E)\leq\liminf_{h\to\infty}P_\alpha(E_h)=\inf\{P_\alpha(E):E\in\mathcal{A}\}.$$

Thus E is a minimum, because $E\in\mathcal{A}$. By Proposition 1.1.2 this set is also a solution of the problem

$$\min\left\{\frac{P_\alpha(E)^{\frac{Q}{Q-1}}}{|E|}:E\subset\mathbb{R}^2\text{ measurable set with }0<|E|<+\infty\right\}. \tag{1.2.6}$$

The set E is convex and therefore its boundary ∂E is locally the graph of Lipschitz functions. In a neighborhood of the point $(0,b)\in\partial E$, $b>0$, the set ∂E can be written as a Lipschitz graph of the form $y=\varphi(x)$. We are led to the following situation. Let $\delta>0$, $\varphi\in\mathrm{Lip}(-\delta,\delta)$ and assume that $\{(x,\varphi(x)):x\in(-\delta,\delta)\}=\partial E\cap\{(x,y)\in\mathbb{R}^2:-\delta<x<\delta,y>0\}$. Fix a function $\vartheta\in C^1_0\big(-\delta,\delta\big)$. For $|t|<t_0$ let E_t be the set obtained from E by replacing $\partial E\cap\{(x,y)\in\mathbb{R}^2:-\delta<x<\delta,y>0\}$ with $\{(x,\varphi(x)+t\vartheta(x)):x\in(-\delta,\delta)\}$. Denote by (n_1^t,n_2^t) the unit normal to ∂E_t. By

Theorem 1.1.1 and by the length formula

$$
\begin{aligned}
\frac{d}{dt}P_\alpha(E_t)\Big|_{t=0} &= \frac{d}{dt}\int_{\partial E_t\cap\{|x|<\delta,y>0\}} \left[n_1^t(x,y)^2+|x|^{2\alpha}n_2^t(x,y)^2\right]^{1/2} d\mathcal{H}^1\Big|_{t=0}\\
&= \frac{d}{dt}\int_{-\delta}^{\delta}\left[(\varphi'(x)+t\vartheta'(x))^2+|x|^{2\alpha}\right]^{1/2}dx\Big|_{t=0} \qquad (1.2.7)\\
&= \int_{-\delta}^{\delta}\frac{\varphi'(x)\vartheta'(x)}{[\varphi'(x)^2+|x|^{2\alpha}]^{1/2}}dx.
\end{aligned}
$$

We can interchange derivative and integral because

$$
\begin{aligned}
\left|\frac{\partial}{\partial t}\left[(\varphi'(x)+t\vartheta'(x))^2+|x|^{2\alpha}\right]^{1/2}\right| &= \frac{|(\varphi'(x)+t\vartheta'(x))\vartheta'(x)|}{[(\varphi'(x)+t\vartheta'(x))^2+|x|^{2\alpha}]^{1/2}}\\
&\le |\vartheta'(x)|\in L^1(-\delta,\delta).
\end{aligned}
$$

Analogously,

$$
\frac{d}{dt}|E_t|\Big|_{t=0} = \frac{d}{dt}\int_{-\delta}^{\delta}(\varphi(x)+t\vartheta(x))dx\Big|_{t=0} = \int_{-\delta}^{\delta}\vartheta(x)dx = -\int_{-\delta}^{\delta}x\vartheta'(x)dx.
$$

The set E is a solution of Problem (1.2.6), and hence

$$
\frac{P_\alpha(E)^{\frac{Q}{Q-1}}}{|E|}\le\frac{P_\alpha(E_t)^{\frac{Q}{Q-1}}}{|E_t|},\quad |t|<t_0.
$$

Thus

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{P_\alpha(E_t)^{\frac{Q}{Q-1}}}{|E_t|}\Bigg|_{t=0}\\
&= \frac{P_\alpha(E)^{\frac{1}{Q-1}}}{|E|^2}\left(\frac{Q}{Q-1}|E|\int_{-\delta}^{\delta}\frac{\varphi'(x)\vartheta'(x)}{[\varphi'(x)^2+|x|^{2\alpha}]^{1/2}}dx+P_\alpha(E)\int_{-\delta}^{\delta}x\vartheta'(x)dx\right).
\end{aligned} \qquad (1.2.8)
$$

The function $\vartheta\in C_0^1(-\delta,\delta)$ is arbitrary. Therefore it must be

$$
\frac{Q}{Q-1}|E|\frac{\varphi'(x)}{[\varphi'(x)^2+|x|^{2\alpha}]^{1/2}}+P_\alpha(E)x=c,\quad \text{for a.e. } x\in(-\delta,\delta),
$$

for some constant $c\in\mathbb{R}$. The function φ must be even because the set E is x–symmetric. Then φ' is odd and this implies $c=0$. Setting $\lambda=\frac{Q-1}{Q}\frac{P_\alpha(E)}{|E|}$ we find

$$
\varphi'(x)=-\mathrm{sgn}(x)\frac{\lambda|x|^{\alpha+1}}{[1-\lambda^2x^2]^{1/2}}\quad\text{for a.e. } x\in(-\delta,\delta). \qquad (1.2.9)
$$

This equation shows that φ', which a priori is only a locally bounded measurable function, is in fact a continuous function, and the equation is satisfied for all $|x|<1/\lambda$.

Letting $a=\sup\{x>0:|E^x|>0\}$, a regularity argument similar to the one discussed above shows that ∂E is of class C^1 in a neighborhood of $(a,0)$. Then it must be $\varphi(a)=0$, $\varphi'(a)=-\infty$ and $a=1/\lambda$. Hence, for $x\in[0,a]$

$$
\varphi(x)=\int_x^a\frac{t^{\alpha+1}}{a\,(1-(t/a)^2)^{1/2}}\,dt=a^{\alpha+1}\int_{\arcsin(x/a)}^{\pi/2}\sin^{\alpha+1}(t)\,dt.
$$

The parameter $a > 0$ is fixed by means of the volume constraint $|E| = 1$.

If we choose $\lambda = a = 1$ then we find the isoperimetric set E_α in (1.2.2). By (1.2.9) with $\lambda = 1$ and Theorem 1.1.1 we also find

$$P_\alpha(E_\alpha) = 4\int_0^1 \left[\varphi'(x)^2 + |x|^{2\alpha}\right]^{1/2} dx = 4\int_0^1 \frac{|x|^\alpha}{\sqrt{1-x^2}}dx = 2\int_0^\pi \sin^\alpha(t)dt.$$

Moreover $|E_\alpha| = \frac{Q-1}{Q}P_\alpha(E_\alpha)$. Therefore, the isoperimetric constant $c(\alpha)$ is given by

$$c(\alpha) = \frac{|E_\alpha|}{P_\alpha(E_\alpha)^{\frac{Q}{Q-1}}} = \frac{Q-1}{Q}P_\alpha(E_\alpha)^{\frac{1}{1-Q}} = \frac{Q-1}{Q}\left(2\int_0^\pi \sin^\alpha(t)dt\right)^{\frac{1}{1-Q}}.$$

The statement concerning uniqueness follows from Theorem 1.2.1 and from the previous analysis. □

3. Minkowski content and sharp Sobolev inequality

The isoperimetric inequality (1.2.1) can be restated in metric terms. Moreover it implies a sharp Sobolev inequality for the Grushin gradient.

We briefly introduce the definition of the Grushin metric in $\mathbb{R}^2$. General Grushin metrics will be discussed in detail in Section 1 of Chapter 3. Consider the vector fields in the plane $X = \partial_x$ and $Y = |x|^\alpha \partial_y$. A Lipschitz continuous curve $\gamma : [0,1] \to \mathbb{R}^2$ is *admissible* if there exist measurable functions $h = (h_1, h_2) \in L^\infty([0,1]; \mathbb{R}^2)$ such that $\dot\gamma = h_1 X(\gamma) + h_2 Y(\gamma)$ almost everywhere. The length of the curve γ is by definition

$$L_\alpha(\gamma) = \int_0^1 |h(t)|dt.$$

The metric $d_\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ is defined by setting

$$d_\alpha(p,q) = \inf\big\{L_\alpha(\gamma) : \ \gamma \text{ admissible curve such that } \gamma(0) = p \text{ and } \gamma(1) = q\big\}. \tag{1.3.1}$$

Consider a bounded open set $E \subset \mathbb{R}^2$ and define the distance $\text{dist}_\alpha(p; E) = \inf_{q\in E} d_\alpha(p,q)$. The Minkowski content of ∂E in the Grushin plane is defined as

$$\mathfrak{M}_\alpha(\partial E) = \liminf_{\varepsilon \downarrow 0} \frac{|\{p \in \mathbb{R}^2 : 0 < \text{dist}_\alpha(p;E) < \varepsilon\}|}{\varepsilon}. \tag{1.3.2}$$

If E is a bounded open set with boundary of class C^2, then "lim inf" in (1.3.2) can be replaced by "lim" and the identity $\mathfrak{M}_\alpha(\partial E) = P_\alpha(E)$ holds. This can be proved as in [**MSC**] Theorem 5.1.

Let us introduce the following notation for the Grushin gradient of a function $f \in C^1(\mathbb{R}^2)$. We simply write $D_\alpha f(x,y) = (\partial_x f(x,y), |x|^\alpha \partial_y f(x,y))$.

We shall need some general theorems which are proved for Lipschitz vector fields. For this reason we state the next results only for the case $\alpha \geq 1$. The following corollary gives a sharp isoperimetric inequality for Minkowski content.

COROLLARY 1.3.1. *Let $\alpha \geq 1$, $Q = 2 + \alpha$ and let $c(\alpha)$ be the constant in* (1.2.3). *Then, for any bounded open set $E \subset \mathbb{R}^2$ it holds*

$$|E| \leq c(\alpha)\mathfrak{M}_\alpha(\partial E)^{\frac{Q}{Q-1}}. \tag{1.3.3}$$

PROOF. Let $E \subset \mathbb{R}^2$ be a bounded open set and write $\varrho(p) = \mathrm{dist}_\alpha(p; E)$. For any $\varepsilon > 0$ let $E_\varepsilon = \{p \in \mathbb{R}^2 : \varrho(p) < \varepsilon\}$. Without loss of generality we can assume that $|E_\varepsilon \setminus E|$ converges to zero as $\varepsilon \downarrow 0$, otherwise $\mathfrak{M}_\alpha(\partial E) = +\infty$.

By Theorem 3.1 in [**MSC**] we have the Eikonal equation

$$|D_\alpha \varrho(x, y)| = 1 \tag{1.3.4}$$

for almost every $(x, y) \in \mathbb{R}^2 \setminus \bar{E}$. From the coarea formula proved in Theorem 5.2 of [**GN1**] it follows

$$|E_\varepsilon \setminus E| = \int_{E_\varepsilon \setminus E} |D_\alpha \varrho(x, y)| dxdy = \int_0^\varepsilon P_\alpha(E_\tau) d\tau. \tag{1.3.5}$$

Given $\varepsilon > 0$, it cannot be $P_\alpha(E_\tau) > \frac{|E_\varepsilon \setminus E|}{\varepsilon}$ for all $\tau \in (0, \varepsilon)$, otherwise (1.3.5) would be false. Then, for every $\varepsilon > 0$ there exists $\tau(\varepsilon) \in (0, \varepsilon)$ such that

$$P_\alpha(E_{\tau(\varepsilon)}) \leq \frac{|E_\varepsilon \setminus E|}{\varepsilon}.$$

From (1.1.2), by taking the lim inf we find $P_\alpha(E) \leq \mathfrak{M}_\alpha(\partial E)$ and the claim follows from (1.2.1). □

By a straightforward adaptation of the argument in Remark 6.6 of [**FF**], the isoperimetric inequality (1.2.1) implies a sharp Sobolev inequality for the Grushin gradient.

COROLLARY 1.3.2. *Let $\alpha \geq 1$, $Q = 2 + \alpha$ and let $c(\alpha)$ be the constant in* (1.2.3). *Then for any $f \in C_0^\infty(\mathbb{R}^2)$*

$$\left(\int_{\mathbb{R}^2} |f|^{\frac{Q}{Q-1}} dxdy \right)^{\frac{Q-1}{Q}} \leq c(\alpha)^{\frac{Q-1}{Q}} \int_{\mathbb{R}^2} |D_\alpha f| dxdy. \tag{1.3.6}$$

The constant in this inequality is sharp.

PROOF. For any $t > 0$ define $E_t = \{(x, y) \in \mathbb{R}^2 : |f(x, y)| > t\}$ and

$$f_t(x, y) = \begin{cases} t & \text{if } (x, y) \in E_t, \\ |f(x, y)| & \text{if } (x, y) \in \mathbb{R}^2 \setminus E_t. \end{cases}$$

Then, for any $h > 0$, $f_{t+h}(x, y) \leq f_t(x, y) + h\chi_{E_t}(x, y)$ and thus

$$\|f_{t+h}\|_{\frac{Q}{Q-1}} \leq \|f_t + h\chi_{E_t}\|_{\frac{Q}{Q-1}} \leq \|f_t\|_{\frac{Q}{Q-1}} + h|E_t|^{\frac{Q-1}{Q}}.$$

Then

$$\|f\|_{\frac{Q}{Q-1}} = \int_0^{+\infty} \frac{d}{dt} \|f_t\|_{\frac{Q}{Q-1}} dt \leq \int_0^{+\infty} |E_t|^{\frac{Q-1}{Q}} dt \leq c(\alpha)^{\frac{Q-1}{Q}} \int_0^{+\infty} P_\alpha(E_t) dt.$$

By Sard Lemma the sets $\{|f(x,y)| = t\}$ are C^∞ curves for almost every $t > 0$. Then, by the coarea formula and by Theorem 1.1.1

$$\begin{aligned}\int_{\mathbb{R}^2} |D_\alpha f| dxdy &= \int_{\mathbb{R}^2} \Big(\Big(\frac{\partial_x f}{|\nabla f|}\Big)^2 + |x|^{2\alpha} \Big(\frac{\partial_y f}{|\nabla f|}\Big)^2 \Big)^{1/2} |\nabla f| dxdy \\ &= \int_0^{+\infty} \int_{\{|f|=t\}} \big(n_1^2 + |x|^{2\alpha} n_2^2\big)^{1/2} \, d\mathcal{H}^1 \, dt \\ &= \int_0^{+\infty} P_\alpha(A_t) \, dt.\end{aligned}$$

We denoted by $n = (n_1, n_2)$ the unit normal to the level sets $\{|f| = t\}$.

The sharpness of the constant can by proved in the following way. Take a bounded open set $E \subset \mathbb{R}^2$ with boundary of class C^2 and define, as before, $\varrho(p) = \mathrm{dist}_\alpha(p; E)$. For any $\varepsilon > 0$ let

$$f_\varepsilon(p) = \begin{cases} 1 & \text{if } p \in \bar{E}, \\ 1 - \frac{1}{\varepsilon}\varrho(p) & \text{if } 0 < \varrho(p) < \varepsilon \\ 0 & \text{if } \varrho(p) \geq \varepsilon. \end{cases}$$

Apply the Sobolev inequality to f_ε. Letting $\varepsilon \to 0$ and using the Eikonal equation (1.3.4) and the identity $\mathfrak{M}_\alpha(\partial E) = P_\alpha(E)$ we get the isoperimetric inequality (1.2.1). □

4. Grushin and Heisenberg isoperimetric sets

The isoperimetric problem in the Heisenberg group (an interesting still open problem) was the original motivation for our study of the isoperimetric inequality in the Grushin plane. In the next proposition we describe the special interest of the case $\alpha = 1$ and then we discuss some connection and analogy between Grushin and Heisenberg isoperimetric sets.

PROPOSITION 1.4.1. *Let E_1 be the isoperimetric set in* (1.2.2) *for the choice $\alpha = 1$. Then*

$$E_1 = \Big\{ (x,y) \in \mathbb{R}^2 : |y| < \frac{1}{2}\Big(\arccos|x| + |x|\sqrt{1-|x|^2}\Big), \, |x| < 1 \Big\}. \qquad (1.4.1)$$

Moreover, ∂E_1 consists of two geodesics in the metric space $(\mathbb{R}^2, d_1)$, where d_1 is the metric defined in (1.3.1). *These geodesics connect the antipodal points* $(0, \pm\pi/4)$ *of ∂E_1 and are symmetric with respect to the y–axis.*

PROOF. We discuss for a moment the general case $\alpha > 0$. Geodesics in the metric space $(\mathbb{R}^2, d_\alpha)$, i.e. curves with minimal length connecting points, are solution of a particular system of differential equations. Consider the Hamilton function

$$H(x,y,\xi,\eta) = \frac{1}{2}(\xi^2 + |x|^{2\alpha}\eta^2)$$

and the corresponding problem (it is enough to study the case $x \geq 0$)

$$\begin{cases} \dot{x} = \partial_\xi H(x, y, \xi, \eta) = \xi & x(0) = 0 \\ \dot{y} = \partial_\eta H(x, y, \xi, \eta) = x^{2\alpha}\eta & y(0) = y_0 \\ \dot{\xi} = -\partial_x H(x, y, \xi, \eta) = -\alpha x^{2\alpha-1}\eta^2 & \xi(0) = 1 \\ \dot{\eta} = -\partial_y H(x, y, \xi, \eta) = 0 & \eta(0) = -\lambda. \end{cases}$$

Geodesics starting from the point $(0, y_0)$ are to be found (after a reparameterization) among curves $\gamma(t) = (x(t), y(t))$ solving this problem. We refer to [**Be**] for a motivation of this fact. The choice $\xi(0) = 1$ corresponds to arclength parameterization and determines $\dot{x}(0) = 1$. The parameter $\lambda > 0$ controls the direction of the curve. The first, third and fourth equations give $\ddot{x} + \alpha\lambda^2 x^{2\alpha-1} = 0$ and by integration $\dot{x}^2 + \lambda^2 x^{2\alpha} = 1$ and thus $\dot{x} = (1 - \lambda^2 x^{2\alpha})^{1/2}$. Denoting by y' the derivative of y with respect to x we find

$$y'(x) = \frac{dy}{dt}\frac{dt}{dx} = -\frac{\lambda x^{2\alpha}}{(1 - \lambda^2 x^{2\alpha})^{1/2}}.$$

If $\alpha = 1$ this differential equation coincides with the differential equation (1.2.9). Integrating this equation for $y_0 = \pi/4$ and $\lambda = 1$ we find a curve in the quadrant $Q = \{x, y \geq 0\}$ whose support is $\partial E_1 \cap Q$, where E_1 is the isoperimetric set (1.4.1). The union of this curve with its reflection in the $\{y < 0\}$ half space gives a geodesic in $(\mathbb{R}^2, d_1)$ connecting the antipodal points $(0, \pm\pi/4)$ of ∂E_1.

□

The set in $\mathbb{R}^3$ obtained letting rotate E_1 around the y–axis is the conjectured solution of the Heisenberg isoperimetric problem. In $\mathbb{R}^3$ consider the vector fields $X = \partial_x + y\partial_t$ and $Y = \partial_y - x\partial_t$. (Here, the variable t plays the role the variable y did in the Grushin plane). These vector fields are left invariant for the group operation

$$(x, y, t) \cdot (\xi, \eta, \tau) = (x + \xi, y + \eta, t + \tau + \xi y - x\eta).$$

The H–perimeter of a measurable set $E \subset \mathbb{R}^3$ is

$$P_H(E) = \sup_{\varphi \in \mathcal{F}(\mathbb{R}^2)} \int_E (X\varphi_1(x, y, t) + Y\varphi_2(x, y, t))\, dxdydt.$$

If $E \subset \mathbb{R}^3$ has smooth boundary, then its H–perimeter has the following integral representation

$$P_H(E) = \int_{\partial E} \sqrt{(\nu_1 + y\nu_3)^2 + (\nu_2 - x\nu_3)^2}\, d\mathcal{H}^2, \tag{1.4.2}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit Euclidean normal to ∂E. The proof is the same as in Theorem 1.1.1. It can be checked that $P_H(p \cdot E) = P_H(E)$ and $|p \cdot E| = |E|$ for any point $p \in \mathbb{R}^3$. Moreover, $P_H(\delta_\lambda(E)) = \lambda^3 P_H(E)$ and $|\delta_\lambda(E)| = \lambda^4|E|$, where $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$, $\lambda > 0$.

The isoperimetric problem in the Heisenberg group is to find a solution of

$$\min\left\{P_H(E) : E \subset \mathbb{R}^3 \text{ measurable set with } |E| = 1\right\}. \tag{1.4.3}$$

The existence of solutions is proved in [**LR**]. Pansu conjectured in [**P**] that solutions are sets foliated by geodesics for the Heisenberg Carnot–Carathéodory metric, and recently some numerical evidence has been provided supporting this conjecture (see [**LM**]). Moreover, a surface with Heisenberg constant mean curvature must be foliated by geodesics (this was explained to me by S. Pauls) and the boundary of an isoperimetric set, if smooth, has constant mean curvature.

Now, consider the group G of all orthogonal transformations (matrices) $T : \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$T = \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix},$$

where $A \in O(2)$ is a 2×2 orthogonal matrix. It can be checked that $P_H(T(E)) = P_H(E)$ for all $T \in G$. This suggests that sets solving problem (1.4.3) and having barycenter at the origin should satisfy $T(E) = E$ for all $T \in G$.

Definition 1.4.2. An open set $E \subset \mathbb{R}^3$ belongs to the class $\mathcal{A}$ if $E = \{(z,t) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3 : |t| < \varphi(|z|)\}$ for some non negative function $\varphi \in C([0,\varrho]) \cap C^2(0,\varrho)$, $\varrho > 0$, with $\varphi(\varrho) = 0$, $\varphi'(0) = 0$ and $\varphi'(\varrho) = -\infty$.

If solutions are in the class $\mathcal{A}$ then they can be determined explicitly (see Proposition 3.4 in [**Mo**], Theorem 3.3 in [**LM**] and [**DGN2**]). The difficult problem is to show that solutions must have cylindrical symmetry. In the Grushin plane we proved the required symmetry and regularity properties of isoperimetric sets in Theorem 1.2.1. In the Heisenberg three dimensional situation is no longer clear how to "rearrange" sets preserving measure and not increasing H–perimeter.

Proposition 1.4.3. *If the isoperimetric problem* (1.4.3) *has a solution in the class $\mathcal{A}$, then it is a dilation δ_λ of the set*

$$E = \Big\{(z,t) \in \mathbb{C} \times \mathbb{R} : |t| < \frac{1}{2}\big(\arccos|z| + |z|\sqrt{1-|z|^2}\big),\ |z| < 1\Big\}.$$

Moreover, the set E is foliated by a family of Heisenberg geodesics connecting the antipodal points $(0,0,\pm\pi/4)$.

Proof. The statement concerning foliation by geodesics is proved in [**LM**]. We compute the set E. Let $E = \{|t| < \varphi(|z|)\}$ and write $f : D \to [0,+\infty)$, $f(z) = \varphi(|z|)$, $D = \{z \in \mathbb{C} : |z| < \varrho\}$, $\varrho > 0$. We write $z = x + iy$. Denoting by $\nu = (\nu_1,\nu_2,\nu_3)$ the Euclidean outward unit normal to ∂E, from the representation formula (1.4.2) and from the area formula we find

$$\begin{aligned} P_H(E) &= \int_{\partial E} \sqrt{(\nu_1 + y\nu_3)^2 + (\nu_2 - x\nu_3)^2}\, d\mathcal{H}^2 \\ &= 2\int_D \sqrt{(\nu_1 + y\nu_3)^2 + (\nu_2 - x\nu_3)^2}\sqrt{1 + |\nabla f(z)|^2} dxdy, \end{aligned}$$

where in the last integral we have written $\nu = \nu(z, f(z))$ and $\nabla f = (\partial_x f, \partial_y f)$. Using

$$\nu(z, f(z)) = \frac{(-\nabla f(z), 1)}{\sqrt{1 + |\nabla f(z)|^2}},$$

we also get

$$
\begin{aligned}
P_H(E) &= 2\int_D \sqrt{(\partial_x f(z) - y)^2 + (\partial_y f(z) + x)^2}\, dxdy \\
&= 2\int_D \sqrt{|\nabla f(z)|^2 + (x\partial_y f(z) - y\partial_x f(z)) + |z|^2}\, dxdy.
\end{aligned}
$$

Letting $\psi(r) = 2\varphi(\sqrt{r})$, i.e. $f(z) = \frac{1}{2}\psi(|z|^2)$, we have $\partial_x f = x\psi'$ and $\partial_y f = y\psi'$, and using polar coordinates we find

$$
\begin{aligned}
P_H(E) &= 2\int_D \sqrt{|z|^2(\psi'(|z|^2) + 1)}\, dxdy \\
&= 4\pi \int_0^{\varrho} r^2\sqrt{1 + \psi'(r^2)^2}\, dr = 2\pi \int_0^{\varrho^2} \sqrt{r}\sqrt{1 + \psi'(r)^2}\, dr.
\end{aligned}
$$

In the same way

$$
|E| = 2\int_D f(z)\, dxdy = \pi \int_0^{\varrho^2} \psi(r)\, dr.
$$

If E solves problem (1.4.3) then the function ψ minimizes the functional

$$
J(\psi) = 2\pi \int_0^{\sigma} \sqrt{r}\sqrt{1 + \psi'(r)^2}\, dr
$$

among non negative functions satisfying

$$
\psi \in C([0,\sigma]) \cap C^2(0,\sigma), \quad \psi(\sigma) = 0, \quad \psi'(\sigma) = -\infty, \quad \pi \int_0^{\sigma} \psi(r)\, dr = 1, \quad \sigma > 0.
$$

By the Lagrange multiplier theorem for variational problems with integral constraint there exists $\lambda \neq 0$ such that the function ψ solves the Euler–Lagrange equation

$$
\frac{d}{dr}\frac{\partial H(r,\psi,\psi')}{\partial z} = \frac{\partial H(r,\psi,\psi')}{\partial u},
$$

where $H(r,u,z) = 2\pi\sqrt{r}\sqrt{1+z^2} + \pi\lambda u$. This gives the differential equation

$$
\frac{d}{dr}\left(\sqrt{r}\frac{\psi'(r)}{\sqrt{1 + \psi'(r)^2}}\right) = \lambda.
$$

Integrating this equation we obtain

$$
\psi'(r) = -\sqrt{\frac{\lambda^2 r}{1 - \lambda^2 r}}.
$$

The condition $\psi'(\varrho^2) = -\infty$ gives $\lambda^2\varrho^2 = 1$ and using $\psi(\varrho^2) = 0$ we finally find

$$
\varphi(r) = \frac{1}{2}\psi(r^2) = \varrho^2 \int_0^{\arccos(r/\varrho)} \cos^2\vartheta\, d\vartheta = \frac{\varrho^2}{2}\left[\arccos\frac{r}{\varrho} + \frac{r}{\varrho}\sqrt{1 - \left(\frac{r}{\varrho}\right)^2}\,\right].
$$

The parameter ϱ is fixed by the volume constraint $|E| = 1$. ☐

CHAPTER 2

Kelvin transform and critical semilinear equations

1. Introduction

Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $n = m + k$, $\alpha > 0$. We write $z = (x, y) \in \mathbb{R}^n$. The Grushin operator is the subelliptic Laplacian

$$\mathcal{L} = \Delta_x + (\alpha+1)^2 |x|^{2\alpha} \Delta_y, \tag{2.1.1}$$

where

$$\Delta_x = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad \Delta_y = \sum_{i=1}^{k} \frac{\partial^2}{\partial y_i^2}.$$

We can define the Grushin gradient $D_\alpha = (D_x, (\alpha+1)|x|^\alpha D_y)$, where D_x and D_y are the gradients with respect to the variables x and y, respectively. If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ the Grushin divergence of the vector function (f, g) is

$$\mathrm{div}_\alpha(f, g) = \mathrm{div}_x f + (\alpha+1)|x|^\alpha \mathrm{div}_y g,$$

where div_x and div_y are divergences with respect to the x and y variables, respectively. With this notation, $\mathcal{L}u = \mathrm{div}_\alpha D_\alpha u$.

A natural Sobolev space is associated with the gradient D_α. For $u \in C_0^\infty(\mathbb{R}^n)$ define the norm

$$\|u\|_{H^1_\alpha} = \Big(\int_{\mathbb{R}^n} (|u(z)|^2 + |D_\alpha u(z)|^2) dz \Big)^{1/2},$$

and let $H^1_\alpha(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\| \cdot \|_{H^1_\alpha}$.

The Sobolev embedding for functions in $H^1_\alpha(\mathbb{R}^n)$ is proved in [**FGaW**] (see also [**FGuW**] and [**GN1**]). Precisely, there exists a constant $C > 0$ such that for all $u \in H^1_\alpha(\mathbb{R}^n)$

$$\Big(\int_{\mathbb{R}^n} |u(z)|^{\frac{2Q}{Q-2}} dz \Big)^{\frac{Q-2}{2Q}} \leq C \Big(\int_{\mathbb{R}^n} |D_\alpha u(z)|^2 dz \Big)^{1/2}.$$

Here, the number

$$Q = m + (\alpha+1)k \tag{2.1.2}$$

is the "homogeneous dimension" of $\mathbb{R}^n$ for $\mathcal{L}$ and D_α, and $2^* = \frac{2Q}{Q-2}$ is the corresponding Sobolev conjugate exponent.

Non negative extremal functions for the Sobolev inequality are (up to a multiplicative geometric constant) weak solutions of the Euler–Lagrange equation

$$\mathcal{L}u = -u^{2^*-1}. \tag{2.1.3}$$

The exponent $\frac{Q+2}{Q-2} = 2^* - 1$ is the critical exponent for $\mathcal{L}$. We are interested in finding symmetry properties of (and possibly determine all) positive solutions of equation (2.1.3).

Introduce the "norm"

$$\|z\| = \left(|x|^{2(\alpha+1)} + |y|^2\right)^{\frac{1}{2(\alpha+1)}}. \tag{2.1.4}$$

This "norm" is 1–homogeneous for the group of anisotropic dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda > 0$, defined by

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y). \tag{2.1.5}$$

For a suitable constant $c = c(m, k, \alpha) \neq 0$, the function

$$\Gamma(z) = c\|z\|^{2-Q}, \tag{2.1.6}$$

is a fundamental solution with pole at the origin for the operator $\mathcal{L}$.

Proposition 2.1.1. *For all $z \neq 0$ we have $\mathcal{L}\Gamma(z) = 0$.*

The proof of this proposition is in the Appendix at the end of the Chapter.

Proposition 2.1.1 can be improved. The constant $c = c(m, k, \alpha) \neq 0$ in (2.1.6) can be fixed in such a way that

$$\int_{\mathbb{R}^n} \langle D_\alpha \Gamma, D_\alpha \varphi \rangle dx dy = \varphi(0)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. For integers α, integral representations for the fundamental solution of $\mathcal{L}$ with pole at arbitrary points of $\mathbb{R}^n$ have been constructed in [**BGG**]. We do not need these stronger statements, and from now on we choose $c = 1$ in (2.1.6).

2. Inversion and Kelvin transform in Grushin spaces

We define a Kelvin transform for the operator $\mathcal{L}$. To this aim we first introduce an inversion in $\mathbb{R}^n$.

Definition 2.2.1. Define $\mathcal{I} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ by setting

$$\mathcal{I}(z) = \delta_{\|z\|^{-2}}(z), \quad z \neq 0. \tag{2.2.1}$$

Clearly, $\mathcal{I}^2$ is the identity. In the next propositions we prove some basic properties of $\mathcal{I}$. We denote by $J_\mathcal{I}(z) = \det \frac{\partial \mathcal{I}(z)}{\partial z}$ the determinant Jacobian of $\mathcal{I}$ at the point $z \neq 0$.

Lemma 2.2.2. *For all $z \neq 0$ we have $|J_\mathcal{I}(z)| = \Gamma(z)^{\frac{2Q}{Q-2}}$.*

Proof. Let $\Phi(z) = \|z\|$, $S = \{z \in \mathbb{R}^n : \Phi(z) = 1\}$, consider an open set $A \subset S$ and set $\Omega = \{\delta_t(z) : z \in A,\, t > 0\}$. We preliminary show that for $t > 0$

$$\int_{\delta_t(A)} \frac{1}{|\nabla \Phi(z)|} d\mathcal{H}^{n-1}(z) = t^{Q-1} \mu(A), \quad \text{where} \quad \mu(A) = \int_A \frac{1}{|\nabla \Phi(z)|} d\mathcal{H}^{n-1}(z). \tag{2.2.2}$$

Indeed, by the coarea formula

$$\begin{aligned}\int_{\delta_t(A)} \frac{1}{|\nabla\Phi|} d\mathcal{H}^{n-1} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int_{\delta_s(A)} \frac{1}{|\nabla\Phi|} d\mathcal{H}^{n-1}\, ds \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Omega\cap\{t<\|z\|<t+\varepsilon\}} dz = t^Q \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Omega\cap\{1<\|\zeta\|<1+\varepsilon/t\}} d\zeta \\ &= t^Q \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon/t} \int_{\delta_s(A)} \frac{1}{|\nabla\Phi|} d\mathcal{H}^{n-1} ds = t^{Q-1}\mu(A).\end{aligned}$$

We performed the change of variable $z = \delta_t(\zeta)$, which has has determinant Jacobian t^Q.

Now fix a positive number $r > 0$ and for any $\delta > 0$ define the open set $\Omega_\delta = \{\delta_t(z) : z \in A,\ r < t < r+\delta\}$. The inverted set is $\mathcal{I}(\Omega_\delta) = \{\delta_t(z) : z \in A, r < 1/t < r+\delta\}$. By the coarea formula and by (2.2.2)

$$|\Omega_\delta| = \int_r^{r+\delta} \int_{\delta_t(A)} \frac{1}{|\nabla\Phi|} d\mathcal{H}^{n-1} dt = \mu(A) \int_r^{r+\delta} t^{Q-1} dt,$$

and analogously,

$$|\mathcal{I}(\Omega_\delta)| = \mu(A) \int_{1/(r+\delta)}^{1/r} t^{Q-1} dt.$$

If $z \in \mathbb{R}^n$ is a point such that $\|z\| = r > 0$ then

$$|J_{\mathcal{I}}(z)| = \lim_{\delta\to 0} \frac{|\mathcal{I}(\Omega_\delta)|}{|\Omega_\delta|} = \lim_{\delta\to 0} \frac{\displaystyle\int_{1/(r+\delta)}^{1/r} t^{Q-1} dt}{\displaystyle\int_r^{r+\delta} t^{Q-1} dt} = r^{-2Q} = \|z\|^{-2Q}.$$

□

The next theorem and the following corollary describe the conformal nature of $\mathcal{I}$. Let $z = (x, y) \in \mathbb{R}^n$ be a point such that $x \neq 0$ and define the "singular Riemmanian norm" of a vector $\zeta = (\xi, \eta) \in \mathbb{R}^n$ at z as

$$|\zeta|_z = \sqrt{|\xi|^2 + (\alpha+1)^{-2}|x|^{-2\alpha}|\eta|^2}. \tag{2.2.3}$$

THEOREM 2.2.3. *For all $z = (x, y) \in \mathbb{R}^n$ with $x \neq 0$ we have*

$$\lim_{\zeta\to z} \frac{|\mathcal{I}(\zeta) - \mathcal{I}(z)|_{\mathcal{I}(z)}}{|\zeta - z|_z} = |J_{\mathcal{I}}(z)|^{1/Q}. \tag{2.2.4}$$

PROOF. Define $\mathcal{I}_x(z) \in \mathbb{R}^m$ and $\mathcal{I}_y(z) \in \mathbb{R}^k$ by the relation $\mathcal{I}(z) = (\mathcal{I}_x(z), \mathcal{I}_y(z))$, and let $N(z) = |x|^{2(\alpha+1)} + |y|^2$. Then

$$\begin{aligned}|\mathcal{I}(\zeta) - \mathcal{I}(z)|^2_{\mathcal{I}(z)} &= |\mathcal{I}_x(\zeta) - \mathcal{I}_x(z)|^2 + (\alpha+1)^{-2}|\mathcal{I}_x(z)|^{-2\alpha}|\mathcal{I}_y(\zeta) - \mathcal{I}_y(z)|^2 \\ &= \Big|\frac{\xi}{N(\zeta)^{\frac{1}{\alpha+1}}} - \frac{x}{N(z)^{\frac{1}{\alpha+1}}}\Big|^2 + \frac{N(z)^{\frac{2\alpha}{\alpha+1}}}{(\alpha+1)^2|x|^{2\alpha}}\Big|\frac{\eta}{N(\zeta)} - \frac{y}{N(z)}\Big|^2 \\ &= N(z)^{-\frac{2}{\alpha+1}}\bigg\{\Big|\xi\Big(\frac{N(\zeta)}{N(z)}\Big)^{-\frac{1}{\alpha+1}} - x\Big|^2 + \frac{1}{(\alpha+1)^2|x|^{2\alpha}}\Big|\eta\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1} - y\Big|^2\bigg\}.\end{aligned}$$

By a Taylor development of the function $N(\zeta)$ at the point z,

$$\frac{N(\zeta)}{N(z)} = 1 + \frac{1}{N(z)}\big\{2(\alpha+1)|x|^{2\alpha}\langle x, \xi - x\rangle + 2\langle y, \eta - y\rangle\big\} + o(|z-\zeta|), \tag{2.2.5}$$

and therefore

$$\begin{aligned}\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1} &= 1 - \frac{1}{N(z)}\{\cdots\} + o(|z-\zeta|),\\ \Big(\frac{N(\zeta)}{N(z)}\Big)^{-1/(\alpha+1)} &= 1 - \frac{1}{(\alpha+1)N(z)}\{\cdots\} + o(|z-\zeta|),\end{aligned}$$

where the curly bracket is defined as in (2.2.5).

In the following N replaces $N(z)$. Note that $\{\cdots\} = O(|z-\zeta|)$. We get

$$\begin{aligned}|J_{\mathcal{I}}(z)|^{-2/Q}|\mathcal{I}(\zeta) - \mathcal{I}(z)|^2_{\mathcal{I}(z)} &= \Big|\xi - x - \frac{1}{(\alpha+1)N}\{\cdots\}\xi\Big|^2\\ &\quad + \frac{1}{(\alpha+1)^2|x|^{2\alpha}}\Big|\eta - y - \frac{1}{N}\{\cdots\}\eta\Big|^2 + o(|z-\zeta|^2)\\ &= |\zeta - z|_z^2 - \frac{2}{(\alpha+1)N}\langle \xi - x, \xi\rangle\{\cdots\} + \frac{|\xi|^2}{(\alpha+1)^2N^2}\{\cdots\}^2\\ &\quad - \frac{1}{(\alpha+1)^2|x|^{2\alpha}}\frac{2}{N}\langle \eta - y, \eta\rangle\{\cdots\} + \frac{|\eta|^2}{(\alpha+1)^2|x|^{2\alpha}N^2}\{\cdots\}^2\\ &\quad + o(|z-\zeta|^2)\\ &= |\zeta - z|_z^2 + R(z,\zeta).\end{aligned}$$

If $R(z,\zeta) = o(|z-\zeta|^2)$, the proof of the theorem is completed. It is enough to show that the quantity

$$\begin{aligned}\frac{R(z,\zeta)}{\{\cdots\}} &= \frac{2}{N^2(\alpha+1)^2}\Big[-(\alpha+1)N\langle\xi - x, \xi\rangle + |\xi|^2(\alpha+1)|x|^{2\alpha}\langle x, \xi - x\rangle + |\xi|^2\langle y, \eta - y\rangle\\ &\quad - \frac{N}{|x|^{2\alpha}}\langle \eta - y, \eta\rangle + (\alpha+1)|\eta|^2\langle x, \xi - x\rangle + \frac{|\eta|^2}{|x|^{2\alpha}}\langle y, \eta - y\rangle\Big]\\ &= \frac{2}{N^2(\alpha+1)}\langle \xi - x, \xi\rangle\Big(-(|x|^{2(\alpha+1)} + |y|^2) + |x|^{2\alpha}|\xi|^2 + |\eta|^2\Big)\\ &\quad + \frac{2}{N^2(\alpha+1)^2}\langle \eta - y, y\rangle\Big(|\xi|^2 - \frac{|x|^{2(\alpha+1)} + |y|^2}{|x|^{2\alpha}} + \frac{|\eta|^2}{|x|^{2\alpha}}\Big) + o(|z-\zeta|)\end{aligned}$$

is an $o(|z-\zeta|)$. In the last equality we replaced $\langle \xi - x, x\rangle$ with $\langle \xi - x, \xi\rangle$ (and the same we did for η) and we consequently added an $o(|z-\zeta|)$. Now the claim follows from the fact that both the round brackets in the last two lines tend to zero when $\zeta \to z$. ☐

Corollary 2.2.4. *Let $u, v \in C^1(\mathbb{R}^n)$. Then for $z \neq 0$*

$$\langle D_\alpha(u \circ \mathcal{I})(z), D_\alpha(v \circ \mathcal{I})(z)\rangle = |J_{\mathcal{I}}(z)|^{2/Q}\langle D_\alpha u(\mathcal{I}(z)), D_\alpha v(\mathcal{I}(z))\rangle. \tag{2.2.6}$$

PROOF. We preliminary show that if $z = (x, y)$ and $x \neq 0$ then

$$|D_\alpha u(z)| = \limsup_{\zeta \to z} \frac{|u(\zeta) - u(z)|}{|\zeta - z|_z}. \tag{2.2.7}$$

Since u is of class C^1,

$$|u(\zeta) - u(z)| = |\langle Du(z), \zeta - z\rangle + o(|\zeta - z|)| \leq |D_\alpha u(z)||\zeta - z|_z + o(|\zeta - z|),$$

and thus

$$\limsup_{\zeta \to z} \frac{|u(\zeta) - u(z)|}{|\zeta - z|_z} \leq |D_\alpha u(z)|.$$

Choosing $\zeta_i = (\xi_i, \eta_i)$, $i \in \mathbb{N}$, with

$$\xi_i = x + \frac{1}{i} D_x u(z), \quad \eta_i = y + \frac{1}{i}(\alpha + 1)^{-2}|x|^{-2\alpha} D_y u(z)$$

we obtain (2.2.7).

By Theorem 2.2.3 and (2.2.7) we get

$$\begin{aligned} |D_\alpha(u \circ \mathcal{I})(z)| &= \lim_{\zeta \to z} \frac{|\mathcal{I}(\zeta) - \mathcal{I}(z)|_{\mathcal{I}(z)}}{|\zeta - z|_z} \limsup_{\zeta \to z} \frac{|u(\mathcal{I}(\zeta)) - u(\mathcal{I}(z))|}{|\mathcal{I}(\zeta) - \mathcal{I}(z)|_{\mathcal{I}(z)}} \\ &= |J_{\mathcal{I}}(z)|^{1/Q} |D_\alpha u(\mathcal{I}(z))|. \end{aligned}$$

Developing this last identity for the function $u \circ \mathcal{I} + v \circ \mathcal{I}$ we find (2.2.6). □

Now we introduce the Kelvin transform of a function in the Grushin space. The relation of this functional transformation with the geometry of the hyperbolic space will be explained in Section 4.

DEFINITION 2.2.5. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function. The Kelvin transform $u^* : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of u is defined by

$$u^*(z) = \Gamma(z) u(\mathcal{I}(z)), \quad z \neq 0. \tag{2.2.8}$$

We need the following Lemma.

LEMMA 2.2.6. *If $u \in H^1_\alpha(\mathbb{R}^n)$ [1] is a non negative weak solution of $\mathcal{L}u = -u^{2^*-1}$ in $\mathbb{R}^n \setminus \{0\}$, then it is a weak solution in $\mathbb{R}^n$.*

PROOF. For $\varepsilon > 0$ let ψ_ε be the function defined by $\psi_\varepsilon(z) = 0$ for $\|z\| < \varepsilon$, $\psi_\varepsilon(z) = 1$ for $\|z\| > 2\varepsilon$, and $\psi_\varepsilon(z) = \dfrac{\Gamma(z) - \varepsilon^{2-Q}}{(2\varepsilon)^{2-Q} - \varepsilon^{2-Q}}$ for $\varepsilon \leq \|z\| \leq 2\varepsilon$. Since u is a weak solution in $\mathbb{R}^n \setminus \{0\}$, we have for any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (\psi_\varepsilon \langle D_\alpha u, D_\alpha \varphi\rangle + \varphi \langle D_\alpha u, D_\alpha \psi_\varepsilon\rangle) dz = \int_{\mathbb{R}^n} \langle D_\alpha u, D_\alpha(\psi_\varepsilon \varphi)\rangle dz = \int_{\mathbb{R}^n} u^{2^*-1} \psi_\varepsilon \varphi \, dz.$$

[1] Mi sembra che

$$\lim_{\varepsilon \to 0} \varepsilon^{Q-2} \int_{\varepsilon < \|z\| < 2\varepsilon} |D_\alpha u|^2 dz = 0$$

dovrebbe bastare.

By dominated convergence,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u^{2^*-1}\psi_\varepsilon\varphi\,dz = \int_{\mathbb{R}^n} u^{2^*-1}\varphi\,dz, \quad \text{and}$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \psi_\varepsilon\langle D_\alpha u, D_\alpha\varphi\rangle dz = \int_{\mathbb{R}^n}\langle D_\alpha u, D_\alpha\varphi\rangle dz.$$

If we show that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\varphi\langle D_\alpha u, D_\alpha\psi_\varepsilon\rangle dz = 0, \tag{2.2.9}$$

our claim is proved. By Hölder inequality we have

$$\int_{\mathbb{R}^n}|\varphi\langle D_\alpha u, D_\alpha\psi_\varepsilon\rangle|dz \le \frac{\varepsilon^{Q-2}\max|\varphi|}{1-2^{2-Q}}\Big(\int_{\varepsilon<\|z\|<2\varepsilon}|D_\alpha u|^2dz\Big)^{1/2}\Big(\int_{\varepsilon<\|z\|<2\varepsilon}|D_\alpha\Gamma|^2dz\Big)^{1/2},$$

and a dilatation argument shows that

$$\int_{\varepsilon<\|z\|<2\varepsilon}|D_\alpha\Gamma|^2dz = \varepsilon^{2-Q}\int_{1<\|z\|<2}|D_\alpha\Gamma|^2dz.$$

Therefore

$$\int_{\mathbb{R}^n}|\varphi\langle D_\alpha u, D_\alpha\psi_\varepsilon\rangle|dz \le C\Big(\varepsilon^{Q-2}\int_{\varepsilon<\|z\|<2\varepsilon}|D_\alpha u|^2dz\Big)^{1/2},$$

and the last term is infinitesimal as $\varepsilon\to 0$. □

THEOREM 2.2.7. (a) *For any* $u\in H^1_\alpha(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}|u^*(z)|^{\frac{2Q}{Q-2}}dz = \int_{\mathbb{R}^n}|u(z)|^{\frac{2Q}{Q-2}}dz \quad \textit{and} \quad \int_{\mathbb{R}^n}|D_\alpha u^*(z)|^2dz = \int_{\mathbb{R}^n}|D_\alpha u(z)|^2dz.$$

(b) *For any non negative function* $u\in H^1_\alpha(\mathbb{R}^n)$, $\mathcal{L}u = -u^{2^*-1}$ *in weak sense on* $\mathbb{R}^n$ *if and only if* $\mathcal{L}u^* = -(u^*)^{2^*-1}$ *in weak sense on* $\mathbb{R}^n$.

PROOF. We prove statement (b). By Lemma 2.2.6 it suffices to consider test functions $\varphi\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. In this case $\varphi^*\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$. By Lemma 2.2.2 $|J_\mathcal{I}(z)| = \Gamma(z)^{\frac{2Q}{Q-2}}$ and then

$$\int_{\mathbb{R}^n}(u^*)^{2^*-1}\varphi^*dz = \int_{\mathbb{R}^n}u(\mathcal{I}(z))^{2^*-1}\varphi(\mathcal{I}(z))|J_\mathcal{I}(z)|dz = \int_{\mathbb{R}^n}u^{2^*-1}\varphi\,dz.$$

Now let $v = u\circ\mathcal{I}$ and $\psi = \varphi\circ\mathcal{I}$. Using (2.2.6) we find

$$\begin{aligned}\langle D_\alpha u^*, D_\alpha\varphi^*\rangle &= \langle D_\alpha(\Gamma v), D_\alpha(\Gamma\psi))\rangle\\ &= \Gamma^2\langle D_\alpha v, D_\alpha\psi\rangle + v\psi|D_\alpha\Gamma|^2 + \Gamma\langle D_\alpha(\psi v), D_\alpha\Gamma\rangle\\ &= \Gamma^2\langle D_\alpha v, D_\alpha\psi\rangle + \mathrm{div}_\alpha(v\psi\Gamma D_\alpha\Gamma),\end{aligned}$$

because $\mathcal{L}\Gamma = 0$. On the other hand, by (2.2.6)

$$\langle D_\alpha v(z), D_\alpha\psi(z)\rangle = |J_\mathcal{I}(z)|^{2/Q}\langle D_\alpha u(\mathcal{I}(z)), D_\alpha\varphi(\mathcal{I}(z))\rangle,$$

and we get

$$\begin{aligned}\int_{\mathbb{R}^n} \langle D_\alpha u^*, D_\alpha \varphi^* \rangle dz &= \int_{\mathbb{R}^n} \big(\Gamma^2 \langle D_\alpha v, D_\alpha \psi\rangle + \mathrm{div}_\alpha (v\psi \Gamma D_\alpha \Gamma)\big) dz \\ &= \int_{\mathbb{R}^n} \Gamma^2 \langle D_\alpha v, D_\alpha \psi \rangle dz = \int_{\mathbb{R}^n} \langle D_\alpha u(\mathcal{I}(z)), D_\alpha \varphi(\mathcal{I}(z))\rangle |J_{\mathcal{I}}(z)| dz \\ &= \int_{\mathbb{R}^n} \langle D_\alpha u(z), D_\alpha \varphi(z)\rangle dz.\end{aligned}$$

It follows that the statement

$$\int_{\mathbb{R}^n} \langle D_\alpha u, D_\alpha \varphi \rangle dz = \int_{\mathbb{R}^n} u^{2^*-1} \varphi \, dz \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$$

holds for u if and only if it holds for u^*. This ends the proof of (b). Part (a) is proved in the same way.

□

3. Simmetries for semilinear equations

Definition 2.3.1. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function. For $\lambda > 0$, define the functions $\delta_\lambda u$ and u_λ by letting

$$\delta_\lambda u(z) = \lambda^{\frac{Q}{2}-1} u\left(\delta_\lambda(z)\right), \quad u_\lambda(z) = (\delta_\lambda u)^*(z), \quad z \neq 0.$$

[2]

Proposition 2.3.2. *If $\mathcal{L}u = -u^{2^*-1}$ then $\delta_\lambda u$ and u_λ, $\lambda > 0$, solve the same equation.*

Proof. The statement concerning $\delta_\lambda u$ is a simple computation. The statement concerning u_λ is a consequence of Theorem 2.2.7. □

The next theorem is a special case of Bony's Maximum Principle (see Theorem 3.1 in [**Bon**]).

Theorem 2.3.3 (Maximum Principle). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $w \in C^2(\Omega)$ be a function such that $w \geq 0$ and $\mathcal{L}w \leq 0$ in Ω. If there is $z_0 \in \Omega$ such that $w(z_0) = 0$ then $w \equiv 0$ in Ω.*

We also need the following version of Hopf Lemma.

Lemma 2.3.4 (Hopf Lemma). *Let $v \in \mathbb{R}^k$ with $|v| = 1$, $t \in \mathbb{R}$, $\Omega = \{(x, y) \in \mathbb{R}^n : \langle y, v\rangle > t\}$, and $(0, y_0) \in \partial\Omega$. If a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $u > 0$ in Ω, $u(0, y_0) = 0$ and $\mathcal{L}u \leq 0$ in Ω, then $\langle D_y u(0, y_0), v\rangle > 0$.*

Proof. Let $y_1 = y_0 + v$, $z_0 = (0, y_0)$, $z_1 = (0, y_1)$ and $z = (x, y)$. The point $z_1 = z_0 + (0, v)$ belongs to Ω. The function

$$\Lambda(z) = \Gamma(z - z_1) - \Gamma(z_0 - z_1)$$

[2] Piu' preciso

satisfies $\mathcal{L}\Gamma(z) = 0$ for $z \neq z_1$, $\Lambda(z) = 0$ for $\|z - z_1\| = \gamma_1 := \|z_0 - z_1\|$ and $\Lambda(z) = 1$ for $\|z - z_1\| = \gamma_0$, for a suitable $\gamma_0 \in (0, \gamma_1)$.

Define the ring $R = \{z \in \mathbb{R}^n : \gamma_0 < \|z - z_1\| < \gamma_1\} \subset \Omega$. If $\varepsilon > 0$ is small enough, the function $u_\varepsilon(z) = u(z) - \varepsilon\Lambda(z)$ is strictly positive where $\|z - z_1\| = \gamma_1$, because u is strictly positive on this set. Moreover $u_\varepsilon(z) = u(z) \geq 0$ where $\|z - z_1\| = \gamma_0$. Since $u_\varepsilon \geq 0$ on ∂R and

$$\mathcal{L}u_\varepsilon(z) = \mathcal{L}u(z) - \varepsilon\mathcal{L}\Lambda(z) = \mathcal{L}u(z) \leq 0 \quad \text{on } R,$$

by the Maximum Principle it follows that $u \geq \varepsilon\Lambda$ on R. Thus, using $u(0, y_0) = 0$, we find

$$\begin{aligned}
\langle D_y u(0, y_0), v\rangle &= \lim_{t\to 0} \frac{u(0, y_0 + tv)}{t} \geq \varepsilon \lim_{t\to 0} \frac{\Lambda(0, y_0 + tv)}{t} \\
&= \varepsilon \lim_{t\to 0} \frac{1}{t}\Big\{ |y_0 - y_1 + tv|^{\frac{2-Q}{\alpha+1}} - |y_0 - y_1|^{\frac{2-Q}{\alpha+1}} \Big\} \\
&= \varepsilon \lim_{t\to 0} \frac{1}{t}\Big\{ (1 - t)^{\frac{2-Q}{\alpha+1}} - 1 \Big\} = \varepsilon \frac{Q - 2}{\alpha + 1} > 0.
\end{aligned}$$

□

[3]

PROPOSITION 2.3.5. *Let $u \in C^2(\mathbb{R}^n) \cap H^1_\alpha(\mathbb{R}^n)$ be a positive function solving $\mathcal{L}u = -u^{2^*-1}$. Then the function u^* can be continuously extended to a positive function on $\mathbb{R}^n$. In particular,*

$$\lim_{\|z\|\to\infty} \frac{u(z)}{\Gamma(z)} = u^*(0) > 0. \tag{2.3.1}$$

PROOF. By Theorem 2.2.7 $|D_\alpha u^*| \in L^2(\mathbb{R}^n)$ and $\mathcal{L}u^* = -(u^*)^{2^*-1}$ in $\mathbb{R}^n$. Exactly as in Theorem 10.1 in [**GV1**], it can be shown that $u^* \in L^\infty(\mathbb{R}^n)$. Then, the statement follows from Theorem 6.1 in [**CDG2**], the subelliptic version of Serrin's theorem on removability of singularities. □

THEOREM 2.3.6. *Let $u \in C^2(\mathbb{R}^n) \cap H^1_\alpha(\mathbb{R}^n)$ be a positive solution of $\mathcal{L}u = -u^{2^*-1}$. Then there exists $\lambda > 0$ such that $u = u_\lambda$.*

PROOF. By Propositions 2.3.2 and 2.3.5, possibly replacing u with $\delta_\lambda u$ for a suitable $\lambda > 0$, we can assume $u(0) = u^*(0)$. Then we have to prove that $u = u^*$. Let $w_\lambda = u_\lambda - u$ and define

$$\Sigma_\lambda = \{z \in \mathbb{R}^n : \|z\| > \lambda^{1/2}\}, \quad \Omega_\lambda = \{z \in \Sigma_\lambda : w_\lambda(z) < 0\}.$$

Notice that $w_\lambda = 0$ on $\partial\Sigma_\lambda$. We prove the following two statements:

Step 1. There is $R_0 > 0$ such that $\Omega_\lambda = \varnothing$ for all $\lambda > R_0$;
Step 2. We have $\inf\{\lambda \geq 1 : \Omega_\lambda = \varnothing\} = 1$.

[3] COntrollare meglio la seguente Proposizione. Chiarire se $u^* \in C^2$.

Proof of Step 1. The function w_λ satisfies

$$\mathcal{L}w_\lambda = \mathcal{L}u_\lambda - \mathcal{L}u = u^{2^*-1} - u_\lambda^{2^*-1} = \frac{Q+2}{Q-2}\bar{u}^{\frac{4}{Q-2}}(u-u_\lambda) = -\psi(z)w_\lambda$$

where $\bar{u} = \bar{u}(z)$ is a number between $u_\lambda(z)$ and $u(z)$ (mean value theorem), and $\psi = \psi(z)$ is defined by the last equality. Let v be a positive function which will be discussed soon. Setting

$$\bar{w}_\lambda = \frac{w_\lambda}{v} = \frac{u_\lambda - u}{v}$$

we have $\mathcal{L}w_\lambda = \bar{w}_\lambda\mathcal{L}v + 2\langle D_\alpha v, D_\alpha \bar{w}_\lambda\rangle + v\mathcal{L}\bar{w}_\lambda$, and therefore $\bar{w}_\lambda$ satisfies

$$\mathcal{L}\bar{w}_\lambda + 2\frac{\langle D_\alpha v, D_\alpha \bar{w}_\lambda\rangle}{v} + \left(\frac{\mathcal{L}v}{v} + \psi\right)\bar{w}_\lambda = 0. \tag{2.3.2}$$

We show that there exist $v > 0$ and $R_0 > 0$ independent from λ such that for all $z \in \Omega_\lambda$ with $\lambda \geq 1$ and $\|z\| > R_0$ we have

$$\frac{\mathcal{L}v(z)}{v(z)} + \psi(z) < 0. \tag{2.3.3}$$

We choose $v = \delta_\varepsilon u$ and we determine an appropriate $\varepsilon > 0$. By Proposition 2.3.2 $\mathcal{L}(\delta_\varepsilon u) = -(\delta_\varepsilon u)^{2^*-1}$, and by Proposition 2.3.5

$$\lim_{\|z\|\to\infty} \frac{\delta_\varepsilon u(z)}{\Gamma(z)} = \lim_{\|z\|\to\infty} \frac{\varepsilon^{\frac{Q}{2}-1}u(\delta_\varepsilon(z))}{\varepsilon^{Q-2}\Gamma(\delta_\varepsilon(z))} = \varepsilon^{1-\frac{Q}{2}}u^*(0),$$

and hence also

$$\lim_{\|z\|\to\infty} \|z\|^4 \left|\frac{\mathcal{L}(\delta_\varepsilon u)(z)}{\delta_\varepsilon u(z)}\right| = \lim_{\|z\|\to\infty} \left(\frac{\delta_\varepsilon u(z)}{\Gamma(z)}\right)^{\frac{4}{Q-2}} = \varepsilon^{-2}u^*(0)^{\frac{4}{Q-2}}. \tag{2.3.4}$$

If $z \in \Omega_\lambda$ then $u_\lambda(z) < u(z)$ and therefore $\bar{u} \leq u(z)$. Thus, by Proposition 2.3.5 there exists a constant $\gamma > 0$ such that

$$\psi(z) \leq 2^* - 1u(z)^{\frac{4}{Q-2}} \leq \gamma\|z\|^{-4}, \quad z \in \Omega_\lambda, \quad \lambda \geq 1.$$

The constant γ does not depend on $\lambda \in [1, +\infty)$. Fixing $\varepsilon > 0$ in such a way that $\varepsilon^{-2}u^*(0)^{\frac{4}{Q-2}} > \gamma$, it follows from (2.3.4) that there exists $R_0 > 1$ independent from $\lambda \in [1, \infty)$ such that for all $z \in \Omega_\lambda$ with $\|z\| > R_0$ the claim (2.3.3) holds.

Now, the following claim also easily follows. There exists $R_0 > 0$ such that if $\lambda \in [1, +\infty)$ and $z \in \Sigma_\lambda$ is a negative minimum point of the function $\bar{w}_\lambda$, then $\|z\| \leq R_0$. The number R_0 is the one fixed above. By contradiction, if z^* with $\|z^*\| > R_0$ is a negative minimum for $\bar{w}_\lambda$, then $\mathcal{L}\bar{w}_\lambda(z^*) \geq 0$. On the other hand, by (2.3.2) and (2.3.3) we have $\mathcal{L}\bar{w}_\lambda(z^*) < 0$.

In order to prove *Step 1* it suffices to show that if $\lambda > R_0$ and $\Omega_\lambda \neq \varnothing$, then w_λ must achieve a negative minimum in Ω_λ. This follows from

$$\begin{aligned}\lim_{\|z\|\to\infty} \bar{w}_\lambda(z) &= \lim_{\|z\|\to\infty} \frac{\Gamma(z)}{v(z)}\left(\frac{u_\lambda(z)}{\Gamma(z)} - \frac{u(z)}{\Gamma(z)}\right) \\ &= \varepsilon^{\frac{Q}{2}-1}\Big(\frac{\lambda^{\frac{Q}{2}-1}u(0) - u^*(0)}{u^*(0)}\Big) = \varepsilon^{\frac{Q}{2}-1}(\lambda^{\frac{Q}{2}-1} - 1) \geq 0\end{aligned} \tag{2.3.5}$$

for $\lambda \in [1, \infty)$. We used the assumption $u(0) = u^*(0)$.

Proof of Step 2. Let

$$\lambda_0 = \inf\{\lambda \geq 1 : \Omega_\lambda = \varnothing\}.$$

By continuity we have $w_{\lambda_0} \geq 0$ and $\mathcal{L}w_{\lambda_0} \leq 0$ in Σ_{λ_0}. By the Maximum Principle Theorem 2.3.3, only two cases can occur: either $w_{\lambda_0} \equiv 0$ on Σ_{λ_0}, or $w_{\lambda_0} > 0$ on Σ_{λ_0}.

In the first case we have $u = u_{\lambda_0}$ on $\|z\| > \lambda_0$, and precisely

$$u(z) = u_{\lambda_0}(z) = \lambda_0^{\frac{Q}{2}-1}\Gamma(z)u\big(\delta_{\lambda_0/\|z\|^2}(z)\big).$$

Dividing this identity by $\Gamma(z)$ and letting $\|z\| \to \infty$ yields $u^*(0) = \lambda_0^{\frac{Q}{2}-1}u(0)$ and thus $\lambda_0 = 1$ because of the choice $u(0) = u^*(0)$.

In the second case we have $w_{\lambda_0} > 0$ on Σ_{λ_0}. Assume by contradiction that $\lambda_0 > 1$. From the definition of λ_0 it follows that there exists a sequence $\lambda_h \in (1, \lambda_0)$ converging to λ_0 such that $\Omega_{\lambda_h} \neq \varnothing$. Every w_{λ_h} has a negative minimum point $z_h \in \Sigma_{\lambda_h}$, by (2.3.5). Moreover, $\|z_h\| \leq R_0$ for all $h \in \mathbb{N}$, and therefore – possibly taking a subsequence – we can assume that $z_h \to z_0 \in \bar{\Sigma}_{\lambda_0}$. Since $w_{\lambda_0} > 0$ on Σ_{λ_0}, it must be $\|z_0\| = \lambda_0^{1/2}$ and $w_{\lambda_0}(z_0) = 0$. Moreover, $\nabla w_{\lambda_h}(z_h) = 0$ for all h, and thus $\nabla w_{\lambda_0}(z_0) = 0$. If $z_0 = (x_0, y_0)$ is such that $x_0 \neq 0$, this contradicts Hopf Lemma for elliptic operators. If $z_0 = (x_0, y_0)$ with $x_0 = 0$ this contradicts Lemma 2.3.4. Therefore it must be $\lambda_0 = 1$.

We have proved that $\lambda_0 = 1$, that is $u^* \geq u$ on $\|z\| > 1$. By the definition of Kelvin transform this implies $u \geq u^*$ on $\|z\| \leq 1$. Repeating the previous argument replacing u and u^* (the assumption $u(0) = u^*(0)$ is symmetric) we get $u^* \leq u$ on $\|z\| > 1$. Thus $u = u^*$ and the theorem is proved.

□

Corollary 2.3.7. *Let $u \in C^2(\mathbb{R}^n) \cap H^1_\alpha(\mathbb{R}^n)$ be a positive solution of $\mathcal{L}u = -u^{2^*-1}$ such that $u = u^*$. Then there exists $y_0 \in \mathbb{R}^k$ such that for all $y \in \mathbb{R}^k$*

$$u(0, y) = u(0, y_0)(1 + |y - y_0|^2)^{\frac{2-Q}{2(\alpha+1)}}. \tag{2.3.6}$$

Proof. The assumption $u = u^*$ reads

$$u(z) = \|z\|^{2-Q} u\left(\delta_{1/\|z\|^2}(z)\right), \tag{2.3.7}$$

where $z = (x, y)$.

For a fixed $b \in \mathbb{R}^k$, define $u_b(z) = u(z + (0, b))$. Clearly, $\mathcal{L}u_b = -u_b^{2^*-1}$ and hence, by Theorem 2.3.6, there exists $\lambda_b > 0$ such that $u_b = (\delta_{\lambda_b} u)^*$, that is

$$u(z + (0, b)) = \left(\|z\|/\lambda_b^{1/2}\right)^{2-Q} u\left(\delta_{\lambda_b/\|z\|^2}(z) + (0, b)\right).$$

Letting $z_b = z - (0, b)$, this identity becomes

$$u(z) = \left(\|z_b\|/\lambda_b^{1/2}\right)^{2-Q} u\left(\delta_{\lambda_b/\|z_b\|^2}(z_b) + (0, b)\right). \tag{2.3.8}$$

Multiplying (2.3.8) by $\|z\|^{Q-2}$ and letting $\|z\| \to \infty$ we find

$$u^*(0) = \lim_{\|z\|\to\infty} \|z\|^{Q-2} u(z) = \lambda_b^{\frac{Q}{2}-1} \lim_{\|z\|\to\infty} \left(\|z_b\|/\|z\|\right)^{2-Q} u\left(\delta_{\lambda_b/\|z_b\|^2}(z_b) + (0,b)\right)$$
$$= \lambda_b^{\frac{Q}{2}-1} u(0,b),$$

and using $u(0) = u^*(0)$ we get

$$\lambda_b^{\frac{Q}{2}-1} = \frac{u(0,0)}{u(0,b)}. \tag{2.3.9}$$

From (2.3.7) and (2.3.8) we also have for $z \in \mathbb{R}^n$

$$\|z\|^{2-Q} u\left(\delta_{1/\|z\|^2}(z)\right) = \left(\|z_b\|/\lambda_b^{1/2}\right)^{2-Q} u\left(\delta_{\lambda_b/\|z_b\|^2}(z_b) + (0,b)\right).$$

Now let $f(y) = u(0,y)$. Setting $x = 0$ in the last identity and using (2.3.9) we obtain

$$|y|^{\frac{2-Q}{\alpha+1}} f\left(\frac{y}{|y|^2}\right) = \frac{f(0)}{f(b)} |y-b|^{\frac{2-Q}{\alpha+1}} f\left(\frac{\lambda_b^{\alpha+1}(y-b)}{|y-b|^2} + b\right),$$

and by a first order Taylor approximation

$$\begin{aligned} |y|^{\frac{2-Q}{\alpha+1}} &\left\{ f(0) + \left\langle \nabla f(0), \frac{y}{|y|^2} \right\rangle + o\left(\frac{1}{|y|}\right) \right\} = \\ &= \frac{f(0)}{f(b)} |y-b|^{\frac{2-Q}{\alpha+1}} \left\{ f(b) + \lambda_b^{\alpha+1} \left\langle \nabla f(b), \frac{y-b}{|y-b|^2} \right\rangle + o\left(\frac{1}{|y-b|}\right) \right\}. \end{aligned} \tag{2.3.10}$$

The function f has a maximum point $y_0 \in \mathbb{R}^k$, because u is infinitesimal at infinity. Without loss of generality, we can assume that $y_0 = 0$ and $\nabla f(0) = 0$. Using again (2.3.9) and rearranging terms in (2.3.10), we get

$$f(0)^{-\frac{2(\alpha+1)}{Q-2}} \left\{ 1 - \left(\frac{|y|}{|y-b|}\right)^{\frac{Q-2}{\alpha+1}} \right\} = \left(\frac{|y|}{|y-b|}\right)^{\frac{Q-2}{\alpha+1}} f(b)^{-\frac{Q+2\alpha}{Q-2}} \left\langle \nabla f(b), \frac{y-b}{|y-b|^2} \right\rangle + o\left(\frac{1}{|y|}\right).$$

We multiply this identity by y_i, $i = 1, ..., k$, and let $y_i \to \infty$. Notice that

$$\lim_{y_i\to+\infty} y_i \left\{ 1 - \left(\frac{|y|}{|y-b|}\right)^{\frac{Q-2}{\alpha+1}} \right\} = -\frac{Q-2}{\alpha+1} b_i,$$

and

$$\lim_{y_i\to+\infty} y_i \left\langle \nabla f(b), \frac{y-b}{|y-b|^2} \right\rangle = \partial_i f(b).$$

Whence,

$$f(0)^{-\frac{2(\alpha+1)}{Q-2}} \nabla(1 + |b|^2) = -\frac{2(\alpha+1)}{Q-2} f(b)^{-\frac{Q+2\alpha}{Q-2}} \nabla f(b) = \nabla(f(b)^{-\frac{2(\alpha+1)}{Q-2}}),$$

and this finally gives for $b \in \mathbb{R}^k$

$$f(b) = f(0)(1 + |b|^2)^{-\frac{Q-2}{2(\alpha+1)}}.$$

This is (2.3.6) with $y_0 = 0$. □

4. Grushin and hyperbolic symmetry

In this section we prove a radial symmetry property of solutions to the equation $\mathcal{L}u = -u^{2^*-1}$. After a suitable functional change of variable, such solutions become radial functions in the hyperbolic space.

DEFINITION 2.4.1. For a given function $u = u(x, y)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$ define the function $U = U(\xi, \eta)$ by

$$U(\xi, \eta) = |\xi|^\beta u\left(|\xi|^{\frac{1}{\alpha+1}}\frac{\xi}{|\xi|}, \eta\right), \qquad \beta = \frac{Q-2}{2(\alpha+1)}. \tag{2.4.1}$$

We write $U = Tu$ and $u = T^{-1}U$.

In order to explain the meaning of the functional transformation T, we choose $m = 1$ and we introduce the hyperbolic space. Let $H = \{\zeta = (\xi, \eta) \in \mathbb{R}\times\mathbb{R}^k : \xi > 0\}$ be the $n = k + 1$ dimensional hyperbolic spaces endowed with the quadratic form $g_H(\zeta) = \xi^{-2}I_n$, where I_n is the identity $n \times n$ matrix. This quadratic form induces the hyperbolic metric d_H on H. The Riemannian hyperbolic Laplacian is

$$\Delta_H = \xi^2\Delta + (1-k)\xi\partial_\xi, \quad \text{where} \quad \Delta = \partial_\xi^2 + \sum_{i=1}^{k}\partial_{\eta_i}^2. \tag{2.4.2}$$

It is sometimes useful to work with the unit ball conformal model for the hyperbolic space. Let $B = \{(x, y) \in \mathbb{R}\times\mathbb{R}^k = \mathbb{R}^n : x^2 + |y|^2 < 1\}$ be the $n = k + 1$ dimensional unit ball endowed with the quadratic form $g_B(x, y) = \frac{4}{(1-(x^2+|y|^2))^2}I_n$, where I_n is the identity $n \times n$ matrix. This quadratic form induces the hyperbolic metric d_B on B. The conformal map $S : B \to H$ defined by

$$S(x, y) = \frac{(1-(x^2+|y|^2), -2y)}{(1+x)^2+|y|^2}, \tag{2.4.3}$$

is an isometry between the hyperbolic ball and the hyperbolic halfspace. Clearly, $S(0) = (1, 0)$ and it can be easily checked that $S^{-1} = S$.

The proof of the following proposition is a computation and it is omitted.

PROPOSITION 2.4.2. *Let $m = 1$. If u is a positive solution to the equation $\mathcal{L}u = -u^{2^*-1}$ in $\{(x, y) \in \mathbb{R}\times\mathbb{R}^k : x > 0\}$, then U is a solution to the equation in H*

$$\Delta_H U + \frac{Q(Q-2)}{4(\alpha+1)^2}U = -\frac{1}{(\alpha+1)^2}U^{2^*-1}, \tag{2.4.4}$$

where Δ_H is the hyperbolic Laplacian (2.4.2).

Equation (2.4.4) is invariant under hyperbolic isometries. Indeed, it can be shown that translations in the variable y of a function $u = u(x, y)$ and Grushin dilations $\delta_\lambda u$ introduced in Definition 2.3.1 correspond to hyperbolic translations of the function U. This observation suggests how to construct the Kelvin transform u^* introduced in (2.2.8).

It is enough to consider the case $m = k = 1$. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $H = \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta > 0\}$ and according to (2.4.3) define $S : D \to H$ by

$$S(z) = \frac{1-z}{1+z}, \quad \text{where } \mathrm{Re}S(z) = \frac{1-|z|^2}{|1+z|^2} \text{ and } \mathrm{Im}S(z) = \frac{-2y}{|1+z|^2}. \tag{2.4.5}$$

The map S is a conformal identification of D with H. Note that S coincides with its inverse and that it maps $z = 1$ to $\zeta = 0$ and $z = -1$ to $\zeta = \infty$. The reflection $I : D \to D$, $I(x + iy) = (-x + iy)$ is a hyperbolic isometry of D and takes -1 to 1.

Proposition 2.4.3. *Fix $m = k = 1$ and for a given function $u = u(x, y)$ in the halfplane $x > 0$ let $v = v(x, y)$ be the function defined by*

$$u \to Tu = U \to SU = U_D \to IU_D = V_D \to SV_D = V \to T^{-1}V = v, \tag{2.4.6}$$

where $SU = U \circ S$, $IU_D = U_D \circ I$, etc. denote compositions. Then $v = u^*$, where u^* is the Kelvin transform defined in (2.2.8).

Proof. Be definition (2.4.1) and (2.4.5) we have for $z = x + iy \in D$

$$U_D(z) = \left(\frac{1-|z|^2}{|1+z|^2}\right)^{\beta} u\left(\left(\frac{1-|z|^2}{|1+z|^2}\right)^{\frac{1}{\alpha+1}}, \frac{-2y}{|1+z|^2}\right),$$

and hence

$$V_D(z) = U_D(I(z)) = \left(\frac{1-|z|^2}{(1-x)^2+y^2}\right)^{\beta} u\left(\left(\frac{1-|z|^2}{(1-x)^2+y^2}\right)^{\frac{1}{\alpha+1}}, \frac{-2y}{(1-x)^2+y^2}\right). \tag{2.4.7}$$

Analogously, for $\zeta = \xi + i\eta \in H$ we have

$$v(\zeta) = \xi^{-\frac{\alpha}{2}} V_D\left(\frac{1-|\xi^{\alpha+1}+i\eta|^2}{|1+\xi^{\alpha+1}+i\eta|^2}, \frac{-2\eta}{|1+\xi^{\alpha+1}+i\eta|^2}\right). \tag{2.4.8}$$

Setting

$$x = \frac{1-(\xi^{2(\alpha+1)}+\eta^2)}{(1+\xi^{\alpha+1})^2+\eta^2}, \qquad y = \frac{-2\eta}{(1+\xi^{\alpha+1})^2+\eta^2},$$

we have

$$1-|z|^2 = \frac{4\xi^{\alpha+1}}{(1+\xi^{\alpha+1})^2+\eta^2}, \quad (1-x)^2+y^2 = \frac{4(\xi^{2(\alpha+1)}+\eta^2)}{(1+\xi^{\alpha+1})^2+\eta^2},$$

and therefore

$$\frac{1-|z|^2}{(1-x)^2+y^2} = \frac{\xi^{\alpha+1}}{\xi^{2(\alpha+1)}+\eta^2}, \qquad \frac{-2y}{(1-x)^2+y^2} = \frac{\eta}{\xi^{2(\alpha+1)}+\eta^2}.$$

Plugging these expressions in (2.4.7), we finally get from (2.4.8)

$$\begin{aligned} v(\zeta) &= \xi^{-\frac{\alpha}{2}} \left(\frac{\xi^{\alpha+1}}{\xi^{2(\alpha+1)}+\eta^2}\right)^{\beta} u\left(\left(\frac{\xi^{\alpha+1}}{\xi^{2(\alpha+1)}+\eta^2}\right)^{\frac{1}{\alpha+1}}, \frac{\eta}{\xi^{2(\alpha+1)}+\eta^2}\right) \\ &= \frac{1}{(\xi^{2(\alpha+1)}+\eta^2)^{\frac{\alpha}{2(\alpha+1)}}} u\left(\frac{\xi}{(\xi^{2(\alpha+1)}+\eta^2)^{\frac{1}{\alpha+1}}}, \frac{\eta}{\xi^{2(\alpha+1)}+\eta^2}\right). \end{aligned}$$

This shows that $v = u^*$, in accordance with the definition of Kelvin transform (2.2.8). □

Proposition 2.4.3 shows that the Kelvin transform in the Grushin plane origins from a reflection in the hyperbolic disk. The construction (2.4.6) not only produces the correct form for the inversion $z \mapsto \delta_{1/\|z\|^2}(z)$, but it also yields the fundamental solution $\Gamma(z) = \|z\|^{2-Q}$ for $\mathcal{L}$ appearing in the definition of u^*.

Now we prove the hyperbolic symmetry theorem. Let $m, k \geq 1$ and for $v \in \mathbb{R}^m$ with $|v| = 1$ consider the "halfspace"

$$H_v = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^k : x = tv,\, t > 0\}.$$

H_v carries a natural structure of $k + 1$ dimensional hyperbolic space. We use the coordinates (t, y) on H_v: by abuse of notation, $(t, y) \in H_v$ stands for $(tv, y) \in H_v$.

THEOREM 2.4.4. *Let $m, k \geq 1$ and $n = m + k$. If $u \in C^2(\mathbb{R}^n) \cap H^1_\alpha(\mathbb{R}^n)$ is a positive solution of $\mathcal{L}u = -u^{2^*-1}$ with $u = u^*$ and $y_0 = 0$ in* (2.3.6)*, then for any $v \in \mathbb{R}^m$ with $|v| = 1$ the function $U = Tu$ restricted to H_v is d_H–radially symmetric about the point $(1, 0) \in H_v$, and precisely it is constant on the k–dimensional spheres*

$$\left\{(t, y) \in H_v : \frac{(1 + t)^2 + |y|^2}{4t} = \frac{r^2}{1 - r^2}\right\}, \quad r \in (0, 1). \tag{2.4.9}$$

PROOF. Let $z = (x, y) \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, $z_b = z - (0, b)$. By Theorem 2.3.6 there exists $\lambda_b > 0$ such that

$$u(z) = \left(\|z_b\|/\lambda_b^{1/2}\right)^{2-Q} u\left(\delta_{\lambda_b/\|z_b\|^2}(z_b) + (0, b)\right).$$

This is (2.3.8) in the proof of Corollary 2.3.7. Moreover, by (2.3.9) λ_b is determined by $u(0)\lambda_b^{1-\frac{Q}{2}} = u(0, b)$, and this, by (2.3.6) with $y_0 = 0$, gives

$$\lambda_b = (1 + |b|^2)^{\frac{1}{\alpha+1}}.$$

Let $\zeta = (\xi, \eta)$, $\zeta_b = \zeta - (0, b)$ and $|\zeta_b| = (|\xi|^2 + |\eta - b|^2)^{1/2}$. By definition (2.4.1) we have

$$U(\zeta) = |\xi|^\beta \left(\frac{|\zeta_b|^{\frac{1}{\alpha+1}}}{\lambda_b^{1/2}}\right)^{2-Q} u\left(\frac{\lambda_b |\xi|^{\frac{1}{\alpha+1}}}{|\zeta_b|^{\frac{2}{\alpha+1}}} \frac{\xi}{|\xi|}, \frac{\lambda_b^{\alpha+1}}{|\zeta_b|^2}(\eta - b) + b\right),$$

and using $u(x, y) = |x|^{1-\frac{Q}{2}} U(|x|^\alpha x, y)$ we finally get

$$U(\zeta) = U\left(\frac{(1 + |b|^2)\xi}{|\zeta_b|^2}, \frac{(1 + |b|^2)(\eta - b)}{|\zeta_b|^2} + b\right), \quad b \in \mathbb{R}^k. \tag{2.4.10}$$

In order to prove the theorem it suffices to choose $m = 1$ and consider the case $\xi > 0$. Let H and B be the hyperbolic halfspace and ball, respectively. The map $I_b : \mathbb{R}^n \setminus \{(0, b)\} \to \mathbb{R}^n \setminus \{(0, b)\}$ given by

$$I_b(\xi, \eta) = \left(\frac{(1 + |b|^2)\xi}{|\zeta_b|^2}, \frac{(1 + |b|^2)(\eta - b)}{|\zeta_b|^2} + b\right)$$

is a spherical inversion with respect to the sphere

$$\Sigma_b = \{(\xi,\eta) \in \mathbb{R}\times\mathbb{R}^k : \xi^2 + |\eta - b|^2 = 1 + |b|^2\}.$$

Clearly, $(1,0) \in \Sigma_b$ for any $b \in \mathbb{R}^k$. Let $\Sigma_b^+ = \Sigma_b \cap \{\xi > 0\}$.

The conformal map $S : B \to H$ defined in (2.4.3) takes the "plane" $\pi_b = \{(x,y) \in B : x + \langle b, y\rangle = 0\}$ onto the halfsphere Σ_b^+, and $S(0) = (1,0)$. By Theorem 4.3.7 in [**R**] the points $S(\zeta)$ and $S(I_b(\zeta))$ in B are symmetric with respect to the plane π_b for any $\zeta \in H$. Therefore, by (2.4.10), the function $U_B : B \to \mathbb{R}$ defined by $U_B(x,y) = U(S(x,y))$ is symmetric with respect to the plane π_b. Since $b \in \mathbb{R}^k$ is arbitrary, the function U_B is radial about the origin. Now, the claim follows from the fact that S transform the spheres $\{(x,y) \in B : x^2 + |y|^2 = r^2\}$, $r \in (0,1)$, into the spheres (2.4.9). □

Corollary 2.4.5. *Let $m, k \geq 1$ and $n = m + k$. If $u \in C^2(\mathbb{R}^n) \cap H^1_\alpha(\mathbb{R}^n)$ is a positive solution of $\mathcal{L}u = -u^{2^*-1}$ with $u = u^*$ and $y_0 = 0$ in (2.3.6), then the function $v(x) = u(x,0)$, $x \in \mathbb{R}^m$, is a solution of the problem*

$$\begin{cases} \operatorname{div}_x(pD_x v) - qv = -pv^{2^*-1} & |x| < 1 \\ v > 0 & |x| < 1 \\ \dfrac{\partial v}{\partial \nu} + \Big(\dfrac{Q}{2} - 1\Big) v = 0 & |x| = 1, \end{cases} \tag{2.4.11}$$

where $p(x) = (1 - |x|^{2(\alpha+1)})^k$ and $q(x) = k(\alpha+1)(Q-2)(1 - |x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}$.

Proof. Let $x \in \mathbb{R}^m$ be a point such that $0 < |x| < 1$. By Theorem 2.4.4 the function $y \mapsto u(x,y)$ is radial, and therefore $D_y u(x,0) = 0$. Then, for any $i = 1,...,k$

$$\frac{\partial^2 u}{\partial y_i^2}(x,0) = \lim_{\varepsilon\to 0} \frac{2}{\varepsilon^2}\big(u(x,\varepsilon e_i) - u(x,0)\big),$$

where $e_i = (0,...,1,...0) \in \mathbb{R}^k$ with 1 in the i–th coordinate.

Let $U = Tu$ and let $\xi = |x|^\alpha x$. By Theorem 2.4.4, for any $\varepsilon > 0$ there is a unique point $\xi_\varepsilon \in \mathbb{R}^m$ of the form $\xi_\varepsilon = t\xi$ with $t \in (0,1)$ and such that $U(\xi, \varepsilon e_i) = U(\xi_\varepsilon, 0)$. By (2.4.9), ξ_ε is determined by the condition

$$\frac{(1 - |\xi_\varepsilon|)^2}{|\xi_\varepsilon|} = \frac{(1 - |\xi|)^2 + \varepsilon^2}{|\xi|},$$

which gives

$$|\xi_\varepsilon| = \frac{1}{2|\xi|}\Big(1 + |\xi|^2 + \varepsilon^2 - \sqrt{(1 + |\xi|^2 + \varepsilon^2)^2 - 4|\xi|^2}\Big).$$

Letting $\varphi(\varepsilon) = |\xi_{\sqrt{\varepsilon}}|$, we get $\varphi(0) = |\xi|$ and $\varphi'(0) = |\xi|/(|\xi|^2 - 1)$. Using the definition (2.4.1) of U we have

$$u(x, \varepsilon e_i) = \frac{1}{|\xi|^\beta} U(\xi, \varepsilon e_i) = \frac{1}{|\xi|^\beta} U(\xi_\varepsilon, 0) = \left(\frac{|\xi_\varepsilon|}{|\xi|}\right)^\beta u\Big(|\xi_\varepsilon|^{\frac{1}{\alpha+1}} \frac{x}{|x|}, 0\Big).$$

Therefore,

$$\begin{aligned}\frac{\partial^2 u}{\partial y_i^2}(x,0) &= \lim_{\varepsilon\to 0}\frac{2}{\varepsilon}\left(\frac{\varphi(\varepsilon)^\beta}{|\xi|^\beta}u\Big(\varphi(\varepsilon)^{\frac{1}{\alpha+1}}\frac{x}{|x|},0\Big)-u(x,0)\right)\\ &= \frac{2}{|\xi|^\beta}\frac{d}{d\varepsilon}\left(\varphi(\varepsilon)^\beta u\Big(\varphi(\varepsilon)^{\frac{1}{\alpha+1}}\frac{x}{|x|},0\Big)\right)\Big|_{\varepsilon=0}\\ &= -\frac{1}{(\alpha+1)(1-|x|^{2(\alpha+1)})}\big((Q-2)u(x,0)+2\langle D_x u(x,0),x\rangle\big).\end{aligned}$$

The left hand side is a continuous function on $|x|\le 1$, and thus it must be

$$(Q-2)u(x,0)+2\langle D_x u(x,0),x\rangle = 0,\quad \text{for } |x|=1.$$

Moreover,

$$\mathcal{L}u(x,0)=\Delta_x u(x,0)-\frac{k(\alpha+1)|x|^{2\alpha}}{1-|x|^{2(\alpha+1)}}\big((Q-2)u(x,0)+2\langle D_x u(x,0),x\rangle\big).$$

Multiplying the equation $\mathcal{L}u(x,0) = -u(x,0)^{2^*-1}$ by $p(x) = (1-|x|^{2(\alpha+1)})^k$ and letting $q(x)=k(\alpha+1)(Q-2)(1-|x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}$, we finally get

$$\operatorname{div}_x(p(x)D_x u(x,0)) - q(x)u(x,0) = -p(x)u(x,0)^{2^*-1}.$$

□

5. Uniqueness in the case $m = k = 1$

In this section we study the uniqueness of positive solution to the equation (2.1.3) in the case $m=k=1$. Let $u\in C^2(\mathbb{R}^2)\cap H^1_\alpha(\mathbb{R}^2)$ be a positive solution with $u=u^*$ and $y_0=0$ in (2.3.6). Now we have $Q=\alpha+2$ and we also write $2^*=\frac{2Q}{Q-2}=2(\alpha+2)/\alpha$. By abuse of notation write $u(x)=u(x,0)$. Then, by Corollary 2.4.5 the function u solves the problem

$$\begin{cases}(pu')'-qu+pu^{2^*-1}=0, & \text{in } (-1,1)\\ u>0, \quad \text{in } (-1,1)\\ \alpha u(1)+2u'(1)=0\\ \alpha u(-1)-2u'(-1)=0,\end{cases}\tag{2.5.1}$$

where

$$p(x)=(1-|x|^{2(\alpha+1)})\quad\text{and}\quad q(x)=\alpha(\alpha+1)|x|^{2\alpha}.\tag{2.5.2}$$

In a number of steps, we prove the following theorem.

THEOREM 2.5.1. *Problem* (2.5.1) *has at most one solution.*

A first step is the study of the Cauchy problem with data at the point $x=1$. For $\lambda\ge 0$ and $\delta>0$ consider the problem

$$\begin{cases}(pu')'-qu+p|u|^{2^*-2}u=0, & \text{in } (1-\delta,1),\\ u(1)=\lambda,\\ u'(1)=-\lambda\alpha/2.\end{cases}\tag{2.5.3}$$

THEOREM 2.5.2. *There is $\delta > 0$ such that Problem* (2.5.3) *has a unique solution $u_\lambda \in C^1([1-\delta, 1]) \cap C^2([1-\delta, 1))$. Moreover, u_λ and u'_λ continuously depend on λ uniformly on compact intervals.*

The proof of Theorem 2.5.2 is a standard application of the contraction principle and it is given in the Appendix at the end of the Chapter. The proof of Theorem (2.5.1) is based on a variant of the energy method introduced by Kwong and Li in [**KL**]. Let z be the function defined by $z(x) = p(x)^\tau u(x)$, where $\tau \in \mathbb{R}$ is a parameter to be appropriately chosen. The function z solves the equation

$$p^{2-4\tau} z'' + (1-2\tau) p^{1-4\tau} p' z' + Gz + p^{-\tau(2^*+2)+2} z^{2^*-1} = 0,$$

where $G = \tau^2 p^{-4\tau} (p')^2 - \tau p^{1-4\tau} p'' - q p^{1-4\tau}$. The condition ensuring $-\tau(2^*+2)+2 = 0$ is

$$\tau = \frac{1}{2}\frac{Q-2}{Q-1} = \frac{\alpha}{2(\alpha+1)}, \tag{2.5.4}$$

and, using (2.5.2), a computation shows that in this case

$$G(x) = \frac{\alpha^2 |x|^{2\alpha}}{\left(1 - |x|^{2(\alpha+1)}\right)^{\frac{2\alpha}{\alpha+1}}}, \quad |x| < 1. \tag{2.5.5}$$

Clearly, $G' > 0$ on $(0,1)$. After all, the function z solves $p^{2-4\tau} z'' + (1-2\tau)p^{1-4\tau} p' z' + Gz + z^{2^*-1} = 0$, and therefore, introducing the energy

$$E(z) = p^{2-4\tau}(z')^2 + \frac{2}{2^*} z^{2^*} + Gz^2, \tag{2.5.6}$$

we have

$$\frac{d}{dx} E(z(x)) = G'(x) z(x)^2. \tag{2.5.7}$$

We are ready to prove that solutions to Problem (2.5.1) are even functions.

THEOREM 2.5.3. *If $u \in C^2(-1,1) \cap C^1([-1,1])$ solves Problem* (2.5.1) *then $u'(0) = 0$.*

PROOF. Assume by contradiction that $u'(0) < 0$. The function $v(x) = u(-x)$ is a new solution to Problem (2.5.1), because p and q are even functions. Clearly, $v(0) = u(0)$ and $v'(0) > 0$. We claim that $u(x) < v(x)$ for all $x \in (0,1]$.

Let

$$r(x) = \frac{u'(x)}{u(x)} \quad \text{and} \quad R(x) = \frac{v'(x)}{v(x)}. \tag{2.5.8}$$

We have $r(0) < R(0)$. Assume by contradiction that there exists a point $\xi \in (0,1)$ such that $u(\xi) = v(\xi)$, and let ξ be the smallest one. It cannot be $\xi = 1$ because of the uniqueness statement in Theorem 2.5.2. It must be $u'(\xi) > v'(\xi)$ and therefore $r(\xi) > R(\xi)$. Then, by continuity, there exists a point $b \in (0,\xi)$ such that $r(b) = R(b)$. Let b be the smallest one. Then $r(x) < R(x)$ for $x \in (0,b)$, that is $u'/u < v'/v$ on the same interval. This condition is equivalent to $(u/v)' < 0$ on $(0, b)$, and thus the function u/v is strictly decreasing on this interval. Let

$$\eta = \frac{u(b)}{v(b)}. \tag{2.5.9}$$

Since $u(0) = v(0)$, we have $\eta \in (0, 1)$.

Define the functions $z = p^\tau u$ and $\zeta = p^\tau v$, where $\tau > 0$ is the parameter fixed in (2.5.4). Notice that

$$\eta = \frac{z(b)}{\zeta(b)} = \frac{z'(b)}{\zeta'(b)}.$$

The last equality follows from $u'(b)/v'(b) = \eta$, which is implied by $r(b) = R(b)$.

Let $E(z)$ be the energy associated with z as in (2.5.6). Integrating (2.5.7) and using $G(0) = 0$ and $z'(0) = 0$, we get

$$p(b)^{\frac{2}{\alpha+1}}(z'(b))^2 + \frac{2}{2^*}z(b)^{2^*} + G(b)z(b)^2 = \frac{2}{2^*}u(0)^{2^*} + \int_0^b G'(x)z(x)^2 dx. \tag{2.5.10}$$

The same identity holds for ζ. Multiplying it by η^2 we obtain

$$p(b)^{\frac{2}{\alpha+1}}\eta^2(\zeta'(b))^2 + \frac{2}{2^*}\eta^2\zeta(b)^{2^*} + G(b)\eta^2\zeta(b)^2 = \frac{2}{2^*}\eta^2 u(0)^{2^*} + \int_0^b G'(x)\eta^2\zeta(x)^2 dx. \tag{2.5.11}$$

Taking (2.5.9) into account, the difference of (2.5.10) and (2.5.11) yields

$$\frac{2}{2^*}(1 - \eta^{2-2^*})z(b)^{2^*} = \frac{2}{2^*}(1 - \eta^2)u(0)^{2^*} + \int_0^b G'(x)(z(x)^2 - \eta^2\zeta(x)^2)dx.$$

This is a contradiction. Indeed, the right hand side is strictly positive because $G' > 0$ on $(0, 1)$, $z^2 - \eta^2\zeta^2 > 0$ on $(0, b)$ and $\eta \in (0, 1)$. On the other hand $2^* > 2$, and therefore the left hand side is strictly negative. The point ξ cannot exist.

We have proved that $u < v$ on $(0, 1]$. Since u and v are solution to the differential equation in (2.5.1), we have

$$v(pu'' + p'u' - qu + pu^{2^*-1}) = 0,$$
$$u(pv'' + p'v' - qv + pv^{2^*-1}) = 0.$$

Letting $w = uv' - vu'$ and subtracting the equations we obtain

$$(pw)' = puv(u^{2^*-2} - v^{2^*-2}). \tag{2.5.12}$$

Integrating this equations on $(x, 1)$ and using $p(1) = 0$, we get for $x \in (0, 1)$

$$w(x) = \frac{1}{p(x)} \int_x^1 puv(v^{2^*-2} - u^{2^*-2})dt > 0, \tag{2.5.13}$$

and hence $r < R$ on $(0, 1)$. The function u/v is strictly decreasing on $(0, 1)$. Consistently with (2.5.9), let

$$\eta = \frac{u(1)}{v(1)} = \frac{u'(1)}{v'(1)}.$$

The last equality follows from the boundary conditions. Hospital's rule also yields

$$\lim_{x\to 1}\frac{z(x)}{\zeta(x)} = \lim_{x\to 1}\frac{z'(x)}{\zeta'(x)} = \eta.$$

By (2.5.6) and (2.5.7), we have for any $x \in (0,1)$

$$E(z(x)) - \eta^2 E(\zeta(x)) = E(z(0)) - \eta^2 E(\zeta(0)) + \int_0^x G'(t)(z(t)^2 - \eta^2\zeta(t)^2)dt. \quad (2.5.14)$$

We are going to let $x \to 1$ in this identity.

We first show that

$$u(x) - \eta v(x) = O(1-x)^2, \quad \text{for } x \to 1. \quad (2.5.15)$$

The functions u and v are in $C^1([0,1])$ and $v > 0$ on $[0,1]$. Then $u/v \in C^1([0,1])$ and

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = -\frac{w}{v^2}.$$

By the mean value theorem, there exists $\xi \in (x,1)$ such that

$$\eta - \frac{u(x)}{v(x)} = \frac{u(1)}{v(1)} - \frac{u(x)}{v(x)} = \left(\frac{u}{v}\right)'(\xi)(1-x) = -\frac{w(\xi)}{v(\xi)^2}(1-x),$$

From (2.5.13) it follows that $w(\xi) = (1-\xi)(\gamma_1 + o(1))$ for $\xi \to 1$ and for some constant $\gamma_1 \in \mathbb{R}$. Indeed, $p(\xi) = O(1-\xi)$ and the limit

$$\lim_{\xi\to 1}\frac{1}{(1-\xi)^2}\int_\xi^1 puv(v^{2^*-2} - u^{2^*-2})dx$$

exists finite, by Hospital rule. Then $u(x) - \eta v(x) = (1-x)^2(\gamma_2 + o(1))$ for $x \to 1$ and for some new constant $\gamma_2 \in \mathbb{R}$. This proves (2.5.15).

Recalling (2.5.5), we can compute

$$\lim_{x\to 1} G(x)(z(x)^2 - \eta^2\zeta(x)^2) = \alpha^2 \lim_{x\to 1} p(x)^{-\frac{2\alpha}{\alpha+1}+2\tau}(u(x)^2 - \eta^2 v(x)^2) = 0,$$

because $p(x) = O(1-x)$ and $-\frac{2\alpha}{\alpha+1} + \tau + 2 = \frac{\alpha+2}{\alpha+1} > 0$. A similar computation shows that

$$\lim_{x\to 1} p(x)^{2-4\tau}(z'(x)^2 - \eta^2\zeta'(x)^2) = 0.$$

It follows that

$$\lim_{x\to 1} E(z(x)) - \eta^2 E(\zeta(x)) = 0$$

and therefore, using $E(z(0)) = E(\zeta(0)) = \frac{2}{2^*}u(0)^{2^*}$, we obtain from (2.5.14)

$$0 = \frac{2}{2^*}u(0)^{2^*}(1-\eta^2) + \int_0^1 G'(x)(z(x)^2 - \eta^2\zeta(x)^2)dx. \quad (2.5.16)$$

This is a contradiction, because the right hand side is strictly positive. Indeed, $\eta \in (0,1)$ and $z/\zeta = u/v > \eta$ on $(0,1)$. This is not possible.

□

Thanks to Theorem 2.5.3, the uniqueness for Problem (2.5.1) is reduced to the uniqueness for the following Problem:

$$\begin{cases} (pu')' - qu + pu^{2^*-1} = 0, & \text{in } (0,1) \\ u > 0, & \text{in } (0,1) \\ u'(0) = 0 \\ \alpha u(1) + 2u'(1) = 0. \end{cases} \tag{2.5.17}$$

THEOREM 2.5.4. *Two solutions* $u, v \in C^1([0,1]) \cap C^2([0,1))$ *of Problem* (2.5.17) *must intersect at least twice in* $(0,1)$.

PROOF. Let u and v be two solutions of Problem (2.5.17). They must intersect at least once in $(0,1)$. Assume by contradiction that $u < v$ on $(0,1)$. The function $w = uv' - vu'$ satisfies (2.5.12). Integrating this equation over $(0,1)$ with $w(0) = 0$ and $p(1) = 1$ we get

$$\int_0^1 puv(v^{2^*-2} - u^{2^*-2})dx = 0,$$

and this is not possible, because $u < v$ on $(0,1)$.

Now assume by contradiction that u and v intersect only once in $(0,1)$. For example, assume that $u(b) = v(b)$, $u < v$ on $(0,b)$ and $u > v$ on $(b,1)$ for some $b \in (0,1)$. Take a point $x \in (0,1)$. If $x \in (0,b)$, an integration of (2.5.12) over $(0,x)$ yields

$$p(x)w(x) = \int_0^x puv(u^{2^*-2} - v^{2^*-2})dt < 0.$$

If $x \in (b,1)$, an integration of (2.5.12) over $(x,1)$ yields

$$-p(x)w(x) = \int_x^1 puv(u^{2^*-2} - v^{2^*-2})dt > 0.$$

In both cases $w(x) < 0$, or equivalently $r(x) > R(x)$, where r and R are defined as in (2.5.8). Now, the argument following (2.5.8) proves that the function u/v is strictly increasing on $(0,1)$. Let

$$\eta = \frac{u(1)}{v(1)} = \frac{u'(1)}{v'(1)}.$$

Letting $z = p^\tau u$ and $\zeta = p^\tau v$ and arguing as in the last part of the proof of Theorem 2.5.3, we get

$$0 = \frac{2}{2^*}(u(0)^{2^*} - \eta^2 v(0)^{2^*}) + \int_0^1 G'(x)(z^2 - \eta^2\zeta^2)dx.$$

This is a contradiction because the right hand side is strictly negative. Indeed, $u(0) < v(0)$, $\eta > 1$, $G' > 0$ and $z^2 - \eta^2\zeta^2 < 0$ on $(0,1)$. □

Now, Theorem 2.5.1 immediately follows from the following uniqueness theorem. The proof relies upon a shooting argument.

THEOREM 2.5.5. *Problem* (2.5.17) *has at most one solution.*

Proof. Let $u \neq v$ be two solutions to Problem (2.5.17) and assume that $v(1) < u(1)$. Let u_λ be the maximal solution on $[0, 1]$ to the Cauchy Problem (2.5.3) depending on the parameter $\lambda \geq 0$. If $\lambda = v(1)$ then by Theorem 2.5.2 we have $u_\lambda = v$.

Let

$$\lambda^* = \inf\{\lambda \in (0, u(1)) : u \text{ and } u_\lambda \text{ intersect at least twice in } (0, 1)\}.$$

By Theorem 2.5.4, u and v must intersect at least twice on $(0, 1)$, and therefore the above set is nonempty. It must be $\lambda^* > 0$ because u_λ uniformly converges to 0 as $\lambda \to 0$, by Theorem 2.5.2.

The functions u and u_{λ^*} must intersect at least once in $[0, 1)$, because of the continuous dependence of u_λ on λ. For the same reason, u and u_{λ^*} intersect at most twice in $[0, 1)$. There are four cases.

Case 1.a. There is only one intersection point and it is in $(0, 1)$. Then it must be $u_{\lambda^*} \leq u$. This contradicts the uniqueness for the Cauchy problem with data at the intersection point.

Case 1.b. There is only one intersection point and it is $x = 0$. Let

$$r = \frac{u'}{u}, \quad R_\lambda = \frac{u'_\lambda}{u_\lambda}.$$

We claim that $R_{\lambda^*}(0) = 0$, and thus $u'_{\lambda^*}(0) = 0$. Then $u = u_{\lambda^*}$ by the uniqueness for the Cauchy problem with data at the point $x = 0$. This is not possible. We prove the claim. For any $\lambda > \lambda^*$ there are two points $0 < x_1 < x_2 < 1$ such that $u(x_1) = u_\lambda(x_1)$, $u(x_2) = u_\lambda(x_2)$ and $u < u_\lambda$ on (x_1, x_2). Then $r(x_1) < R_\lambda(x_1)$ and $r(x_2) > R_\lambda(x_2)$. By continuity, there exists $\xi_\lambda \in (x_1, x_2)$ such that $r(\xi_\lambda) = R_\lambda(\xi_\lambda)$. Then it must by $\xi_\lambda \to 0$ as $\lambda \to \lambda^*$ decreasing. The functions u_λ and u'_λ depend continuously on λ (uniformly on $[0, 1]$), then

$$R_{\lambda^*}(0) = \lim_{\lambda \downarrow \lambda^*} R_\lambda(\xi_\lambda) = \lim_{\lambda \downarrow \lambda^*} R_u(\xi_\lambda) = 0.$$

Case 2. There are two intersection points and they are 0 and $x^* \in (0, 1)$.

Case 2.a. Assume that $u'_{\lambda^*}(0) < 0$. Then $u'(x^*) = u'_{\lambda^*}(x^*)$. This contradicts the uniqueness for the Cauchy problem with data at x^*.

Case 2.b. We have $u'_{\lambda^*}(0) > 0$. Let

$$\lambda^{**} = \inf\{\lambda \in (0, \lambda^*) : u \text{ and } u_\lambda \text{ intersect only once in } (0, 1)\}.$$

Let $x_\lambda \in (0, 1)$ be the intersection point. If $\lambda \in (\lambda^{**}, \lambda^*)$ then $u'_\lambda(0) > 0$. If for some $\lambda \in (\lambda^{**}, \lambda^*)$ we have $u'_\lambda(0) < 0$ then, by continuity, there exists $\mu \in (\lambda, \lambda^*)$ such that $u'_\mu(0) = 0$. Then u_μ solves Problem (2.5.17) and intersects u only once in $(0, 1)$. By Theorem 2.5.4 this is not possible. Notice that $u_\mu > 0$ on $(0, 1)$, otherwise there would be $\lambda > 0$ such that $u_\lambda \geq 0$ and $u_\lambda = 0$ at some point in $(0, 1)$. This would contradict the uniqueness for the Cauchy problem.

Then for any $\lambda \in (\lambda^{**}, \lambda^*)$ we have $R_\lambda(0) > r(0) = 0$, whereas $R_\lambda(x_\lambda) < r(x_\lambda)$. Thus there exists $\xi_\lambda \in (0, x_\lambda)$ such that $R_\lambda(\xi_\lambda) = r(\xi_\lambda)$. It must be $x_\lambda \to 0$ as

$\lambda \to \lambda^{**}$, whence $u_{\lambda^{**}}(0) = u(0)$ and $u'_{\lambda^{**}}(0) = 0$. But then $u_{\lambda^{**}} = u$, and this is not possible. □

Appendix

Proof of Proposition 2.1.1. Let

$$u(z) = \left(|x|^{2(\alpha+1)} + |y|^2\right)^{\frac{2-Q}{2(\alpha+1)}} = N(z)^\beta,$$

where we set

$$N(z) = |x|^{2(\alpha+1)} + |y|^2, \qquad \beta = \frac{2-Q}{2(\alpha+1)}.$$

The function u is in $C^2(\mathbb{R}^n \setminus \{0\}$ and for $z \neq 0$ we can compute

$$\mathcal{L}u = \mathrm{div}_x D_x u + (\alpha+1)^2 |x|^{2\alpha} \mathrm{div}_y D_y u,$$

where

$$D_x u = 2(\alpha+1)\beta N^{\beta-1}|x|^{2\alpha}x, \quad D_y u = 2\beta N^{\beta-1}y.$$

We easily find

$$\begin{aligned} \mathrm{div}_x D_x u &= 2(\alpha+1)\beta N^{\beta-2}|x|^{2\alpha}\Big\{2(\alpha+1)(\beta-1)|x|^{2(\alpha+1)} + (2\alpha+m)N\Big\}, \\ \mathrm{div}_y D_y u &= 2\beta N^{\beta-2}\Big\{2(\beta-1)|y|^2 + kN\Big\}, \end{aligned}$$

and therefore

$$\Delta_x u + (\alpha+1)^2|x|^{2\alpha}\Delta_y u = 2(\alpha+1)\beta N^{\beta-1}|x|^{2\alpha}\Big\{2(\alpha+1)(\beta-1) + 2\alpha + m + k(\alpha+1)\Big\}.$$

Using $\beta = \frac{2-Q}{2(\alpha+1)}$ and $Q = m + k(\alpha+1)$ it can be checked that

$$2(\alpha+1)(\beta-1) + 2\alpha + m + k(\alpha+1) = 0.$$

This proves that $\mathcal{L}u(z) = 0$ if $z \neq 0$.

□

Proof of Theorem 2.5.2. Existence and uniqueness follow from a standard application of the contraction principle. Consider the integral operator

$$Tu(x) = \lambda + \int_x^1 \frac{1}{p(t)} \int_t^1 \left(q(s)u(s) - p(s)|u(s)|^{2^*-2}u(s)\right) ds\, dt,$$

acting on the complete metric space

$$X = \{u \in C([1-\delta, 1]) : u(1) = \lambda, \|u - \lambda\| \leq M\},$$

for some $\delta > 0$ and $M > 0$. Here, $\|u\| = \max_{x \in [1-\delta,1]} |u(x)|$. Clearly, $u = Tu$ if and only u solves Problem (2.5.3). If $\delta > 0$ is chosen small enough, T takes X into itself. Moreover,

$$|Tu(x) - Tv(x)| \leq L\|u - v\| \int_x^1 \frac{1-t}{p(t)} dt,$$

where L is a Lipschitz constant for $f(x,u) = q(x)u - p(x)|u|^{2^*-2}u$ on $|u| \le \lambda + M$. If $\delta > 0$ is small enough

$$L \int_{1-\delta}^{1} \frac{1-t}{p(t)} dt < 1,$$

and thus T is a contraction. Therefore T has a unique fixed point.

Assume without loss of generality that for $\lambda \in [0, \lambda_0]$ the functions u_λ and u'_λ are defined on $[0,1]$ and are uniformly bounded. Then, for $x \in (0,1)$ we have

$$\begin{aligned} |u_\lambda(x) - u_\mu(x)| &\le |\lambda - \mu| + L \int_x^1 |u_\lambda(t) - u_\mu(t)| \int_x^t \frac{1}{p(s)} ds\, dt \\ &\le |\lambda - \mu| + L_1 \int_x^1 |u_\lambda(t) - u_\mu(t)| |\log(1-t)| dt := \Phi(x). \end{aligned}$$

Here, $L_1 > 0$ is a new uniform constant. Gronwall's Lemma yields

$$\Phi(1) \ge \Phi(x) e^{-L_1(1-x)(1-\log(1-x))}$$

and therefore $|u_\lambda(x) - u_\mu(x)| \le |\lambda - \mu| e^{L_1(1-x)(1-\log(1-x))}$ on $[0,1)$. Analogously,

$$\sup_{x\in[0,1]} |u'_\lambda(x) - u'_\mu(x)| \le C \sup_{x \in [0,1]} |u_\lambda(x) - u_\mu(x)|.$$

□

CHAPTER 3

Regular domains for Grushin metrics

1. Structure of Grushin metrics

In this preliminary section we recall definition and basic properties of the Grushin metric. The structure Theorem 3.1.1 below, which is a special case of the results proved by Franchi and Lanconelli in [**FL**], plays a central role in our study of regular boundaries.

Let $x \in \mathbb{R}^n$ and consider the vector fields

$$X_j = \lambda_j(x) \frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n, \tag{3.1.1}$$

where

$$\lambda_1(x) = 1 \quad \text{and} \quad \lambda_j(x) = \prod_{i=1}^{j-1} |x_i|^{\alpha_i}, \quad j = 2, ..., n. \tag{3.1.2}$$

Assume that the real numbers α_i satisfy

$$\alpha_i = 0 \quad \text{or} \quad \alpha_i \in [1, \infty[. \tag{3.1.3}$$

This condition ensures that the functions λ_j, and thus the vector fields X_j, are locally Lipschitz continuous. If the numbers α_i are integers then the functions λ_j in (3.1.2) could be changed writing $x_i^{\alpha_i}$ instead of $|x_i|^{\alpha_i}$. All results still hold in this smooth case.

The vector fields (3.1.1) induce on $\mathbb{R}^n$ a metric d in the following way (see [**FL**], [**FP**] and [**NSW**]). A Lipschitz continuous curve $\gamma : [0, T] \to \mathbb{R}^n$, $T \geq 0$, is *subunit* if there exists a vector of measurable functions $h = (h_1, ..., h_n) : [0, T] \to \mathbb{R}^n$ such that $\dot{\gamma}(t) = \sum_{j=1}^n h_j(t) X_j(\gamma(t))$ and $|h(t)| \leq 1$ for a.e. $t \in [0, T]$. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ by setting

$$\begin{aligned} d(x, y) = \inf\{T \geq 0 : \; & \text{there exists a subunit curve } \gamma : [0, T] \to \mathbb{R}^n \\ & \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned} \tag{3.1.4}$$

The definition of the metric d still makes sense for any system of vector fields $X_1, ..., X_m$, even with $m < n$, provided $d(x, y)$ is finite for all x, y. This happens, for instance if the vector fields are smooth and satisfy Hörmander condition (see [**NSW**]). We denote by $B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$ the balls in $\mathbb{R}^n$ defined by the metric d.

For all $j = 1, ..., n$ define inductively the functions $F_j : \mathbb{R}^n \times [0, +\infty) \to [0, +\infty)$ by

$$\begin{aligned} & F_1(x, r) = r, \qquad F_2(x, r) = r\lambda_2(|x_1| + F_1(x, r)), \quad \dots \\ & F_j(x, r) = r\lambda_j\big(|x_1| + r, |x_2| + F_2(x, r), \dots, |x_{j-1}| + F_{j-1}(x, r)\big). \end{aligned} \tag{3.1.5}$$

An inspection of the explicit form (3.1.2) of the functions λ_j shows that

$$F_{j+1}(x,r) = F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j}, \quad j = 1,\dots,n-1. \tag{3.1.6}$$

Note that $F_j(x,r)$ actually depends only on $x_1,\dots,x_{j-1}$. Moreover, $r \mapsto F_j(x,r)$ is increasing and it satisfies the following doubling property

$$F_j(x,2r) \le CF_j(x,r), \quad x\in\mathbb{R}^n,\, 0<r<\infty \tag{3.1.7}$$

for all $j = 1,...,n$. By a direct computation, the following estimates can also be established

$$F_j(x,r+s) \le C(F_j(x,r) + F_j(x,s)), \quad 0<r,s<\infty, \tag{3.1.8}$$

$$F_j(x + F_k(x,r)e_k, s) \le CF_j(x,s), \quad x\in\mathbb{R}^n,\, 0<r\le s<+\infty, \tag{3.1.9}$$

$$\begin{aligned} F_j(x,\varrho r) &\le \varrho F_j(x,r), \qquad \varrho\le 1, r>0,\\ (1+\eta)F_j(x,r) &\le F_j\big(x,(1+\eta)r\big), \qquad \eta\ge 0. \end{aligned} \tag{3.1.10}$$

For all $j = 1,...,n$, define inductively the real numbers d_j by

$$d_1 = 1, \quad d_2 = 1+\alpha_1, \quad \dots \quad , d_j = 1 + \sum_{i=1}^{j-1} d_i\alpha_i = (1+\alpha_1)\cdot\ldots\cdot(1+\alpha_{j-1}). \tag{3.1.11}$$

We say that d_j is the *degree of the variable* x_j. Note that $F_j(0,r) = r^{d_j}$.

The structure of the balls $B(x,r)$ can be described by means of the boxes

$$\mathrm{Box}(x,r) := \{x+h : |h_j| < F_j(x,r),\ j=1,...,n\}. \tag{3.1.12}$$

For any fixed $x\in\mathbb{R}^n$ the function $F_j(x,\cdot)$ is strictly increasing and maps $]0,+\infty[$ onto itself. We denote its inverse by $G_j(x,\cdot) = F_j(x,\cdot)^{-1}$. The following structure theorem is proved in [**FL**].

THEOREM 3.1.1. *There exists a constant $C>0$ such that:*

$$\begin{aligned} &\mathrm{Box}(x,C^{-1}r) \subset B(x,r) \subset \mathrm{Box}(x,Cr), \qquad x\in\mathbb{R}^n, r\in\,]0,+\infty[\\ &C^{-1}d(x,y) \le \sum_{j=1}^n G_j(x,|y_j-x_j|) \le Cd(x,y), \qquad x,y\in\mathbb{R}^n. \end{aligned} \tag{3.1.13}$$

Denote in the following by c_ϱ any positive constant depending on $\varrho>0$ such that $c_\varrho \to 0$, as $\varrho \downarrow 0$. The proof of the following Lemma is in the Appendix at the end of the Chapter.

LEMMA 3.1.2. $\mathrm{Box}(y,r) \subset \mathrm{Box}\big(x,(1+c_\varrho)r\big)$ *for all x,y,r satisfying $d(x,y)\le\varrho r$.*

2. Domains with admissible boundary

We introduce the notion of admissible boundary for a domain $\Omega \subset \mathbb{R}^n$ of class C^1. A point $x \in \partial\Omega$ is said to be *characteristic* if all the vector fields $X_1, ..., X_n$ are tangent to $\partial\Omega$ at x. We give a definition of regular surface requiring a uniform control on the flatness behavior of the surface near characteristic points. The study of a domain near the characteristic set of its boundary will be referred to as the "characteristic case".

If $j = 1, ..., n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we write $\hat{x}_j = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \equiv (x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$. Given a point $x \in \partial\Omega$, we write locally $\partial\Omega$ as a graph of the form $x_j = \varphi(\hat{x}_j)$ for some $j = 1, \dots, n$. We first discuss the case $j = n$.

Introduce the $(n-1)$–dimensional box

$$\text{Box}_n(\hat{x}_n, r) = \{\hat{x}_n + \hat{h}_n \in \mathbb{R}^{n-1} : |h_i| < F_i(\hat{x}_n, r),\ i = 1, \dots, n-1\}, \tag{3.2.1}$$

and let

$$\Lambda_n(\hat{x}_n, r) = \sup_{\hat{y}_n \in \text{Box}_n(\hat{x}_n, r)} |\lambda_n(\hat{y}_n) - \lambda_n(\hat{x}_n)|. \tag{3.2.2}$$

The following proposition collects some properties of the functions Λ_n. The proof is in the Appendix at the end of the Chapter.

PROPOSITION 3.2.1. *Assume that at least one of the numbers α_j, $j = 1, \dots, n$, is strictly positive. Then there exists a constant $\eta > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $r > 0$ and $a \in]0, 1]$*

$$\Lambda_n(\hat{x}_n, ar) \le h(a)\Lambda_n(\hat{x}_n, r), \quad \textit{where} \quad h(a) = \frac{a}{a + \eta(1-a)}. \tag{3.2.3}$$

Moreover, $\Lambda_n(\hat{x}_n, r)) \ge r^{d_n - 1}$, $\Lambda_n(\hat{x}_n, r) \le (C/r)F_n(x, r)$ and $\Lambda_n(x, 2r) \le C\Lambda_n(x, r)$ for some constant $C > 0$, and for all $r > 0$ and $\hat{x}_n \in \mathbb{R}^{n-1}$.

In order to introduce the notion of "admissible surface" we first give the definition for a graph of the form $x_n = \varphi(\hat{x}_n)$. This is the most degenerate case and contains all the difficulties of the problem. Then we will show that a graph of the form $x_j = \varphi(\hat{x}_j)$ with $j \ne n$ can be studied reducing to the previous case. Finally, in Definition 3.2.6 we introduce the notion of domain with admissible boundary.

If $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$ is a function, recall the standard notation

$$\text{osc}(f, A) := \sup_{x,y \in A} |f(x) - f(y)|.$$

DEFINITION 3.2.2. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_n = \varphi(\hat{x}_n)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$

$$\sum_{i \ne n} \text{osc}(X_i\varphi, \text{Box}_n(\hat{x}_n, r)) \le C\Big(r \sum_{i \ne n} |X_i\varphi(\hat{x}_n)|^{\frac{d_n - 2}{d_n - 1}} + \Lambda_n(\hat{x}_n, r)\Big). \tag{3.2.4}$$

The meaning of condition (3.2.4) can be explained as follows. The oscillation of the derivatives of the function φ along the vector fields $X_1, ..., X_{n-1}$ is bounded by a sum of two terms. The first term vanishes on the characteristic set, while the second term $\Lambda_n(\hat{x}_n, r)$ gives an amount of oscillation admitted also at characteristic points. This term is connected with the vertical size of metric balls in the n–coordinate. The right balance between the two terms is given by the power $\frac{d_n-2}{d_n-1}$. Note that, if at least one of the numbers α_i is strictly positive, then $d_n \geq 2$ and the exponent $\frac{d_n-2}{d_n-1}$ is nonnegative.

Actually, we are interested in surfaces which are the boundary of bounded sets. Definition 3.2.2 can be stated also for a bounded graph $x_n = \varphi(\hat{x}_n)$, letting $\hat{x}_n$ belong to a bounded open set of $\mathbb{R}^{n-1}$.

Proposition 3.2.3. *Let $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfy (3.2.4). Then there exists $C > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$*

$$\operatorname{osc}(\varphi, \operatorname{Box}_n(\hat{x}_n, r)) \leq C\Big(r \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + r\Lambda_n(\hat{x}_n, r)\Big). \tag{3.2.5}$$

Proof. Fix $\hat{x}_n, \hat{y}_n \in \mathbb{R}^{n-1}$ and let $\delta = d(\hat{x}_n, \hat{y}_n)$. Then there is a subunit curve $\gamma : [0, \delta] \to \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\}$ such that $\gamma(0) = \hat{x}_n$ and $\gamma(\delta) = \hat{y}_n$. Then we have

$$|\varphi(\hat{x}_n) - \varphi(\hat{y}_n)| \leq \int_0^\delta \sum_{i \neq n} |X_i\varphi(\gamma(t))| dt \leq \delta \sup_{\operatorname{Box}_n(\hat{x}_n,\delta)} \sum_{i \neq n} |X_i\varphi|. \tag{3.2.6}$$

By (3.2.4)

$$\begin{aligned}\sup_{\operatorname{Box}_n(\hat{x}_n,\delta)} \sum_{i \neq n} |X_i\varphi| &\leq \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \sum_{i \neq n} \operatorname{osc}(X_i\varphi, \operatorname{Box}_n(\hat{x}_n, \delta)) \\ &\lesssim \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \delta \sum_{i \neq n} |X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} + \Lambda_n(\hat{x}_n, \delta) \\ &\lesssim \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \Lambda_n(\hat{x}_n, \delta).\end{aligned}$$

We used Hölder inequality $\delta|X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} \lesssim \delta^{d_n-1} + |X_i\varphi(\hat{x}_n)|$ and the inequality $\delta^{d_n-1} \leq \Lambda_n(\hat{x}_n, \delta)$ proved in Proposition 3.2.1. Now, (3.2.5) follows from the doubling property $\Lambda_n(\hat{x}_n, 2r) \leq C\Lambda_n(\hat{x}_n, r)$ proved in Proposition 3.2.1.

□

Next we introduce admissible surfaces of the form $\{x_j = \varphi(\hat{x}_j)\}$, $j \neq n$. We would like to give a definition similar to Definition 3.2.2. The set $\operatorname{Box}_n(\hat{x}_n, r)$ is the intersection of $\operatorname{Box}(x, r)$ with the plane $\{y \in \mathbb{R}^n : y_n = x_n\}$. When $j \neq n$, the intersection of $\operatorname{Box}(x, r)$ with the hyperplane $\{y \in \mathbb{R}^n : y_j = x_j\}$ depends on x_j. Thus (3.2.1) can not be trivially generalized. But, roughly speaking, the vector fields $X_{j+1}, \ldots, X_n$ are "more degenerate" than X_j, and this suggests that the dependence of the function $\varphi(\hat{x}_j)$ on $x_{j+1}, \ldots, x_n$ needs a less careful control than the dependence on $x_1, \ldots, x_{j-1}$. In order to make this remark rigorous, define new functions and vector fields

$$\widetilde{\lambda}_i(x) = \begin{cases} \lambda_i(x) & \text{if } i \leq j, \\ \lambda_j(x) & \text{if } i \geq j. \end{cases} \qquad \text{and } \widetilde{X}_i = \widetilde{\lambda}_i \partial_i, \quad i = 1, \ldots, n. \tag{3.2.7}$$

The variable x_j can now be viewed as the $n-$th variable with respect to the new vector fields. All previous results hold for these vector fields. The functions $\widetilde{F}_i(x,r)$ are defined exactly as in (3.1.5). Set $\widetilde{\mathrm{Box}}(x,r) = \{x+h : |h_i| < \widetilde{F}_i(x,r),\ i=1,...,n\}$ and denote by $\widetilde{d}$ the metric constructed as in (3.1.4) using subunit curves with respect to the vector fields $\widetilde{X}_j$. Let $\widetilde{B}(x,r)$ be the corresponding balls. In the following proposition we list some easy relations between the distances d and $\widetilde{d}$.

Proposition 3.2.4. *For any $C_1 > 0$ there is $C_2 > 0$ such that:*
(i) *if $|x|, |y|, r < C_1$ then $B(x,r) \subset \widetilde{B}(x, C_2 r)$ and $\widetilde{d}(x,y) \le C_2 d(x,y)$;*
(ii) *writing $x' = (x_1, ..., x_j)$ and $x'' = (x_{j+1}, ..., x_n)$, we have $d((x',x''),(y',x'')) \simeq \widetilde{d}((x',x''),(y',x''))$.*

Proof. We have $\widetilde{F}_i(\hat{x}_j, r) = \widetilde{F}_i(\hat{x}_j, r)$, if $i \le j$, while for $i > j$ it is $\widetilde{F}_i(x,r) = F_j(x,r)$. Then, if $i > j$,

$$F_i(x,r) = F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j} \cdots \big(|x_{i-1}| + F_{i-1}(x,r)\big)^{\alpha_{i-1}}$$
$$\le C F_j(x,r) \le F_j(x, Cr) = \widetilde{F}_i(x, Cr),$$

as soon as $|x|, r \le C$. Then $\mathrm{Box}(x,r) \subset \widetilde{\mathrm{Box}}(x, Cr)$. Thus (i) follows by Theorem 3.1.1.

In order to see (ii) recall that the function $\widetilde{G}_i(x,\cdot)$ is the inverse of $\widetilde{F}_i(x,\cdot)$. Moreover, if $i \le j$ then $\widetilde{F}_i(x,r) = F_i(x,r)$. Thus Theorem 3.1.1 gives

$$d\big((x',x''),(y',x'')\big) \simeq \sum_{i=1}^{j} G_i(x', |x_i - y_i|) = \sum_{i=1}^{j} \widetilde{G}_i(x', |x_i - y_i|) \simeq d\big((x',x''),(y',x'')\big)$$

This concludes the proof of (ii). □

The sections of the boxes $\widetilde{\mathrm{Box}}(x,r)$ with the planes $\{y \in \mathbb{R}^n : y_j = x_j\}$ do not depend on x_j. Thus we can set

$$\widetilde{\mathrm{Box}}_j(\hat{x}_j, r) = \{\hat{x}_j + \hat{h}_j : |h_i| < \widetilde{F}_i(\hat{x}_j, r),\ i \neq j\} \tag{3.2.8}$$

$$\text{and} \quad \widetilde{\Lambda}_j(\hat{x}_j) = \sup_{\hat{y}_j \in \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)} |\widetilde{\lambda}_j(\hat{y}_j) - \widetilde{\lambda}_j(\hat{x}_j)|. \tag{3.2.9}$$

The function $\widetilde{\Lambda}_j$ enjoys the properties of Proposition 3.2.1 (replace the subscript n with j).

We are ready to give the general definition of admissible surface and of domain with admissible boundary.

Definition 3.2.5. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_j = \varphi(\hat{x}_j)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_j \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$

$$\sum_{i \neq j} \mathrm{osc}(\widetilde{X}_i \varphi, \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)) \le C\Big(r \sum_{i \neq j} |\widetilde{X}_i \varphi(\hat{x}_j)|^{\frac{d_j - 2}{d_j - 1}} + \widetilde{\Lambda}_j(\hat{x}_j, r)\Big). \tag{3.2.10}$$

DEFINITION 3.2.6 (Domain with admissible boundary). A connected bounded open set $\Omega \subset \mathbb{R}^n$ is said to be *with admissible boundary* if it is of class C^1 and for all $x \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of x such that $\partial\Omega \cap \mathcal{U}$ is an admissible surface according to Definitions 3.2.2 and 3.2.5.

3. An example in $\mathbb{R}^3$

We give examples of admissible surfaces and of bounded domains with admissible boundary in $\mathbb{R}^3$. Consider the functions $\lambda_1 \equiv 1$, $\lambda_2 = |x_1|^{\alpha_1}$, $\lambda_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}$ and the corresponding vector fields

$$X_1 = \partial_1, \quad X_2 = |x_1|^{\alpha_1}\partial_2, \quad X_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}\partial_3. \tag{3.3.1}$$

We consider the case $\alpha_i \geq 1$, $i = 1, 2$. The degrees of the variables x_1, x_2 and x_3 are respectively $d_1 = 1$, $d_2 = 1 + \alpha_1$, $d_3 = (1 + \alpha_1)(1 + \alpha_2)$.

We begin with the study of admissible surfaces of the form $\{x_3 = \varphi(x_1, x_2)\}$. We write $x = (x_1, x_2)$ and $|X\varphi| = |X_1\varphi| + |X_2\varphi|$. If $\varphi \in C^1(\mathbb{R}^2)$ condition (3.2.4) reads

$$\sum_{i=1,2} \operatorname{osc}(X_i\varphi, \operatorname{Box}_3(x, r)) \lesssim r|X\varphi(x)|^{\frac{d_3-2}{d_3-1}} + \Lambda_3(x, r), \tag{3.3.2}$$

where $\operatorname{Box}_3(x, r) = \{(x_1 + u_1F_1(x, r), x_2 + u_2F_2(x_1, r)) : |u_1|, |u_2| \leq 1\}$ and $\Lambda_3(x, r) = \sup_{\operatorname{Box}_3(x,r)} |\lambda_3 - \lambda_3(x)|$. Here $F_1(x, r) = r$ and $F_2(x, r) = r(|x_1| + r)^{\alpha_1}$.

Using the relation ($\alpha \geq 1$)

$$(t + r)^\alpha - t^\alpha \simeq \alpha r(t + r)^{\alpha-1}, \quad t \geq 0, \quad r \geq 0, \tag{3.3.3}$$

we can write explicitly (see also (3.A.6) in the Appendix)

$$\begin{aligned} \Lambda_3(x) &\gtrsim r\big(|x_1| + r\big)^{\alpha_1-1}\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2} + |x_1|^{\alpha_1}F_2(x_1, r)\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2-1} \\ &\gtrsim r\big(|x_1| + r\big)^{\alpha_1-1}\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2}. \end{aligned} \tag{3.3.4}$$

THEOREM 3.3.1. *Let $N(x) = |x_1|^{2d_2} + x_2^2$ and assume that $\varphi(x) = g(N(x))$, where $g \in C^2(0, +\infty)$ is a function such that for some constant $C > 0$*

$$0 \leq g'(t) \leq Ct^{\frac{d_3}{2d_2}-1} \equiv Ct^{\frac{\alpha_2-1}{2}}, \quad |g''(t)| \leq C\frac{g'(t)}{t}, \quad g'(2t) \leq Cg'(t), \quad t > 0. \tag{3.3.5}$$

Then the surface $\{x_3 = \varphi(x_1, x_2)\}$ is admissible.

PROOF. We check (3.3.2). Without loss of generality we assume $x_1, x_2 > 0$. A short computation gives

$$\begin{aligned} |X_1\varphi(x)| &\simeq x_1^{2\alpha_1+1}g'(N(x)) = x_1^{\alpha_1}\{x_1^{d_2}g'(N(x))\} := x_1^{\alpha_1}h_1(x), \\ |X_2\varphi(x)| &\simeq x_1^{\alpha_1}\{x_2g'(N(x))\} := x_1^{\alpha_1}h_2(x). \end{aligned} \tag{3.3.6}$$

Note that $|h(x)| = |(h_1(x), h_2(x))| = N(x)^{1/2}g'(x)$. Then $|X\varphi(x)| \simeq |x_1|^{\alpha_1}N(x)^{1/2}g'(N(x))$. Moreover

$$\begin{aligned}\operatorname{osc}(X_i\varphi, \mathrm{Box}_3(x,r)) &\lesssim |(x_1+r)^{\alpha_1}h_i(x+F(x,r)) - x_1^{\alpha_1}h_i(x)| \\ &\leq ((x_1+r)^{\alpha_1} - x_1^{\alpha_1})h_i(x) + (x_1+r)^{\alpha_1}(h_i(x+F(x,r)) - h_i(x)) \\ &\lesssim \alpha_1 r(x_1+r)^{\alpha_1-1}h_i(x) + (x_1+r)^{\alpha_1}(h_i(x+F(x,r)) - h_i(x)),\end{aligned}$$

where we used (3.3.3). Writing $h = (h_1, h_2)$ we find the estimate from above for the oscillation

$$\sum_{i=1,2}\operatorname{osc}(X_i\varphi, \mathrm{Box}_3(x,r)) \lesssim r(x_1+r)^{\alpha_1-1}|h(x)| + (x_1+r)^{\alpha_1}|h(x+F(x,r)) - h(x)|. \tag{3.3.7}$$

We already know that $|h(x)| \simeq N(x)^{1/2}g'(N(x))$. In order to estimate the last term in the right hand side we use the following inequality (as in (3.2.6))

$$|h_i(x+F(x,r)) - h_i(x)| \lesssim r \sup_{y\in\mathrm{Box}_3(x,r),\, k=1,2}|X_k h_i(y)|.$$

A computation of second derivatives and condition $g''(t) \leq Cg'(t)/t$ give

$$\begin{aligned}&X_1h_1(x) \simeq x_1^{\alpha_1}\{g'(N(x)) + x_1^{2(\alpha_1+1)}g''(N(x))\} \simeq x_1^{\alpha_1}g'(N(x)),\\ &X_2h_1(x) \simeq x_2x_1^{2\alpha_1+1}g''(N(x)) \lesssim \frac{x_1^{\alpha_1+1}x_2}{N(x)}x_1^{\alpha_1}g'(N(x)) \lesssim x_1^{\alpha_1}g'(N(x)),\\ &X_1h_2(x) = x_2x_1^{2\alpha_1+1}g''(N(x)) \lesssim x_1^{\alpha_1}g'(N(x)),\\ &X_2h_2(x) = x_1^{\alpha_1}\{g'(N(x)) + x_2^2g''(N(x))\} \simeq x_1^{\alpha_1}g'(N(x)).\end{aligned}$$

Hence we find

$$|h(x+F(x,r)) - h(x)| \lesssim r(x_1+r)^{\alpha_1}g'\big(N(x+F(x,r))\big).$$

Coming back to (3.3.7), we see that condition (3.3.2) is guaranteed by

$$\begin{aligned}&r(x_1+r)^{\alpha_1-1}N(x)^{1/2}g'(N(x)) + r(x_1+r)^{2\alpha_1}g'(N(x+F(x,r)))\\ &\lesssim r\big\{x_1^{\alpha_1}N(x)^{1/2}g'(N(x))\big\}^{\frac{d_3-2}{d_3-1}} + r\big(x_1+r\big)^{\alpha_1-1}\big(x_2+F_2(x_1,r)\big)^{\alpha_2},\end{aligned} \tag{3.3.8}$$

where the first term in the right hand side is provided by (3.3.6) and the second one comes from (3.3.4).

Now two cases need to be distinguished: (A) $x_2 \geq x_1^{\alpha_1+1}$; (B) $x_2 < x_1^{\alpha_1+1}$.

Study of Case (A). We ignore the contribution of the first term in the right hand side of (3.3.8) and we consider the second one only. Thus (3.3.8) is implied by

$$N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim (x_2+F_2(x_1,r))^{\alpha_2}. \tag{3.3.9}$$

Notice that in Case (A) $N(x) \simeq x_2^2$.

We distinguish the following two subcases: (A1) $x_2 \leq r^{\alpha_1+1}$; (A2) $x_2 > r^{\alpha_1+1}$.

Case (A1). We majorize the left hand side of (3.3.9) using $x_1 \leq r$ and $x_2 \leq r^{\alpha_1+1}$, and we set $x = 0$ in the right hand side obtaining the stronger condition

$r^{\alpha_1+1}g'(r^{2(\alpha_1+1)}) \lesssim r^{\alpha_2(\alpha_1+1)}$, which can be rewritten as $g'(r^{2d_2}) \lesssim r^{d_3-2d_2}$. This last inequality is satisfied by assumption (3.3.5).

Case (A2). Condition (3.3.9) is implied by

$$N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim x_2^{\alpha_2}.$$

We can use $x_1^{\alpha_1+1}, r^{\alpha_1+1} \le x_2$ and $N(x) \simeq N(x+F(x,r)) \simeq x_2^2$. This gives $x_2 g'(x_2^2) \lesssim x_2^{\alpha_2}$, i.e. $g'(x_2^2) \lesssim x_2^{d_3/d_2-2}$. The latter inequality holds by assumption.

Study of Case (B). Here we have $N(x) \simeq x_1^{2(\alpha_1+1)}$. Two subcases must be distinguished: (B1) $x_1 \le r$; (B2) $x_1 > r$.

Case (B1). In this case we ignore the contribution of the first term in the right hand side of (3.3.8) and consider the second term only. Condition (3.3.8) is guaranteed by

$$N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim (x_2+F_2(x_1,r))^{\alpha_2}. \quad (3.3.10)$$

Set $x = 0$ in the right hand side of (3.3.10) and use $x_1 \le r$ and $x_2 < r^{\alpha_1+1}$. We find the stronger inequality $r^{\alpha_1+1}g'(r^{2(\alpha_1+1)}) \le r^{\alpha_2(\alpha_1+1)}$, i.e. $g'(r^{2d_2}) \le r^{d_3-2d_2}$.

Case (B2). We use here the contribution of the first term in the right hand side of (3.3.8). Then we get the stronger inequality

$$(x_1+r)^{\alpha_1-1}N(x)^{1/2}g'(N(x)) + (x_1+r)^{2\alpha_1}g'(N(x+F(x,r))) \lesssim \left\{x_1^{\alpha_1}N(x)^{1/2}g'(N(x))\right\}^{\frac{d_3-2}{d_3-1}}.$$

Since $r \le x_1$ and $x_2 \le x_1^{\alpha_1+1}$, we finally find the stronger condition $x_1^{2\alpha_1}g'(x_1^{2d_2}) \lesssim \{x_1^{\alpha_1+d_2}g'(x_1^{2d_2})\}^{\frac{d_3-2}{d_3-1}}$, i.e. $g'(x_1^{2d_2}) \lesssim x_1^{d_3-2d_2}$. □

Finally, we give an example in $\mathbb{R}^3$ of bounded open set with admissible boundary.

THEOREM 3.3.2. *The open set* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : \big(|x_1|^{2(\alpha_1+1)} + x_2^2\big)^{1+\alpha_2} + x_3^2 < 1\}$ *has admissible boundary.*

PROOF. Let $\varepsilon \in (0,1)$ be fixed. The surface $\partial\Omega \cap \{|x_3| > \varepsilon\}$ can be studied by means of Theorem 3.3.1. Indeed, the lower cap can be written in the form

$$x_3 = -\big(1 - N(x_1,x_2)^{1+\alpha_2}\big)^{1/2} = g(N(x_1,x_2))$$

where $N(x_1,x_2) = |x_1|^{2(\alpha_1+1)} + x_2^2$, and, for any fixed $t_0 < 1$, it is easy to see that the function $g(t) = -\big(1-t^{1+\alpha_2}\big)^{1/2}$ satisfies conditions (3.3.5) for $t \in (0,t_0)$.

The surface $\partial\Omega \cap \{|x_3| < \varepsilon\}$ is noncharacteristic, and hence admissible, away from a neighborhood of its intersection with the plane $x_1 = 0$. To complete the proof of the theorem it is enough to show that $\partial\Omega$ is admissible in a neighborhood of $(0,1,0)$. Here, $\partial\Omega$ can be parameterized as follows

$$x_2 = \Big(\big(1-x_3^2\big)^{\frac{1}{1+\alpha_2}} - x_1^{2(\alpha_1+1)}\Big)^{1/2} := \varphi(x_1,x_3).$$

We check that the function φ satisfies condition (3.2.10). To this aim, as suggested by (3.2.7), we consider the vector fields $\widetilde{X}_1 = \partial_1$, $\widetilde{X}_2 = |x_1|^{\alpha_1}\partial_2$, $\widetilde{X}_3 = |x_1|^{\alpha_1}\partial_3$. We have to check

$$\sum_{i=1,3} \operatorname{osc}(\widetilde{X}_i\varphi, \widetilde{\mathrm{Box}}_2(\hat{x}_2, r)) \lesssim r(|\widetilde{X}_1\varphi(\hat{x}_2)| + |\widetilde{X}_3\varphi(\hat{x}_2)|)^{\frac{d_2-2}{d_2-1}} + \widetilde{\Lambda}_2(\hat{x}_2, r), \tag{3.3.11}$$

where $d_2 = 1 + \alpha_1$ and

$$\widetilde{\mathrm{Box}}_2(\hat{x}_2, r) = \{\hat{x}_2 + \hat{h}_2 : |h_1| < \widetilde{F}_1(\hat{x}_2, r) = r,\ |h_3| < \widetilde{F}_3(\hat{x}_2, r) = r(|x_1| + r)^{\alpha_1}\}.$$

Write $x = (x_1, x_3)$, $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_3)$. An easy computation yields

$$|\widetilde{X}_1\varphi(x)| = h_1(x)|x_1|^{2\alpha_1+1} \quad \text{and} \quad |\widetilde{X}_2\varphi(x)| = h_2(x)|x_3|\,|x_1|^{\alpha_1},$$

where h_1 and h_2 are positive Lipschitz continuous functions in a neighborhood of the origin (we do not need their explicit form here). Assume without loss of generality $x_1, x_3 \geq 0$.

Next we estimate the left hand side of (3.3.11):

$$\begin{aligned} \operatorname{osc}(\widetilde{X}_1\varphi, \widetilde{\mathrm{Box}}_2(\hat{x}_2, r)) &\lesssim \left|h_1(x + \widetilde{F}(x,r))(x_1 + r)^{2\alpha_1+1} - h_1(x)x_1^{2\alpha_1+1}\right| \\ &\lesssim \left|h_1(x + \widetilde{F}(x,r)) - h_1(x)\right| x_1^{2\alpha_1+1} \\ &\qquad + h_1(x + \widetilde{F}(x,r))\left|(x_1 + r)^{2\alpha_1+1} - x_1^{2\alpha_1+1}\right| \\ &\lesssim r x_1^{2\alpha_1+1} + r(x_1 + r)^{2\alpha_1} \lesssim r(x_1 + r)^{2\alpha_1}. \end{aligned} \tag{3.3.12}$$

We used the Lipschitz continuity of h_1 and the estimate $|\widetilde{F}(x,r)| \lesssim r$. Moreover

$$\begin{aligned} \operatorname{osc}(\widetilde{X}_3\varphi, \widetilde{\mathrm{Box}}(\hat{x}_2, r)) &\lesssim \left|h_2(x + \widetilde{F}(x,r))(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - h_2(x)x_1^{\alpha_1}x_3\right| \\ &\lesssim \left|h_2(x + \widetilde{F}(x,r)) - h_2(x)\right| x_1^{\alpha_1}x_3 \\ &\qquad + h_2(x + \widetilde{F}(x,r))\left|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\right| \\ &\lesssim r x_1^{\alpha_1}x_3 + \left|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\right|. \end{aligned} \tag{3.3.13}$$

The last term can be evaluated as follows

$$\begin{aligned} &(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3 \\ &\qquad \lesssim (x_1 + r)^{\alpha_1} - x_1^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) + x_1^{\alpha_1}\big((x_3 + \widetilde{F}_3(x_1, r)) - x_3\big) \\ &\qquad \lesssim r(x_1 + r)^{\alpha_1-1} + \widetilde{F}_3(x_1, r)x_1^{\alpha_1} \lesssim r(x_1 + r)^{\alpha_1-1}. \end{aligned} \tag{3.3.14}$$

Taking into account (3.3.12), (3.3.13), (3.3.14) and the equivalence $\widetilde{\Lambda}_2(\hat{x}_2, r) \simeq r(x_1 + r)^{\alpha_1-1}$, we conclude that condition (3.3.11) is implied by $(x_1 + r)^{2\alpha_1} + x_1^{\alpha_1}x_3 + (x_1 + r)^{\alpha_1-1} \lesssim (x_1 + r)^{\alpha_1-1}$, which is trivially satisfied. □

4. John domains for Grushin metrics

We show that admissible domains introduced in Section 2 are John domains in the Grushin metric space $(\mathbb{R}^n, d)$.

DEFINITION 3.4.1. A bounded open set $\Omega \subset (\mathbb{R}^n, d)$ is a *John domain* if there exist $x_0 \in \Omega$ and $\sigma > 0$ such that for all $x \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\operatorname{dist}\big(\gamma(t), \partial\Omega\big) \geq \sigma \operatorname{diam}\big(\gamma|_{[0,t]}\big). \tag{3.4.1}$$

A curve satisfying (3.4.1) will be called a *John curve*, x_0 will be called the *center* and σ the *John constant* of Ω.

In general metric spaces the definition of John domain is given with length$(\gamma|_{[0,t]})$ replacing diam$(\gamma|_{[0,t]})$ in (3.4.1). Anyway, by a general result due to Martio and Sarvas (see [**MS**, Theorem 2.7]), such definitions are in fact equivalent in doubling metric spaces with geodesics. The metric space $(\mathbb{R}^n, d)$ is doubling endowed with Lebesgue measure and moreover it is geodesics. We do not address this latter question here. In our proofs we shall always work with John curves γ satisfying diam$(\gamma|_{[0,t]}) \simeq d(\gamma(t), \gamma(0))$.

We need the following proposition. The proof is easy and can be found in [**MM2**].

PROPOSITION 3.4.2. *Let $\Omega \subset (\mathbb{R}^n, d)$ be a bounded open set and for any $r > 0$ define $\Omega_r = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > r\}$. Assume that there exist $r > 0$ and $\sigma > 0$ such that Ω_r is arcwise connected and such that for any $x \in \Omega$ there is a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) \in \Omega_r$ and*

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \sigma \operatorname{diam}(\gamma|_{[0,t]}) \tag{3.4.2}$$

for all $t \in [0,1]$. Then Ω is a John domain.

The main result in this section is the following:

THEOREM 3.4.3. *If $\Omega \subset \mathbb{R}^n$ is a domain with admissible boundary according to Definition 3.2.6, then it is a John domain in the metric space $(\mathbb{R}^n, d)$.*

PROOF. We use Proposition 3.4.2. Given $\bar{x} \in \partial\Omega$ we show that there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ and $\sigma > 0$ such that for all $x \in \Omega \cap \mathcal{U}$ there exists a curve γ starting from x and satisfying (3.4.2). The claim follows choosing by compactness a finite covering of $\partial\Omega$.

Fix $\bar{x} \in \Omega$ and write locally $\partial\Omega$ as a graph of the form $x_j = \varphi(\hat{x}_j)$ for some $j = 1, \dots, n$, where φ is a C^1 function. We begin with the basic case $j = n$. Let $\varphi \in C^1(\mathbb{R}^{n-1})$ be a function satisfying the admissibility condition (3.2.4) and assume for the sake of simplicity that $\Omega = \{x_n > \varphi(\hat{x}_n)\}$.

We have to construct a John curve starting from a point $x = \hat{x}_n + x_n e_n \in \Omega$. To this aim two different situations need to be distinguished:

$$\begin{aligned} &\max_{i<n} |X_i \varphi(\hat{x}_n)| \leq \lambda_n(\hat{x}_n) \qquad \text{(Case 1)},\\ &\max_{i<n} |X_i \varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \qquad \text{(Case 2)}. \end{aligned} \tag{3.4.3}$$

In Case 1, the characteristic case, we construct a John curve starting from x of the form $x + te_n$, $t \geq 0$. In Case 2 the path must be split into two pieces. The first one starts from x in the coordinate direction e_k, where $k < n$ is such that the derivative $|X_k\varphi(\hat{x}_n)|$ is "maximal" among all the $|X_i\varphi(\hat{x}_n)|$, $i = 1, \dots, n-1$, and moves in this direction for a time $\bar{t} = \bar{t}(x)$ which must be established in a careful way (compare (3.4.7)). The second part of the path will be of the form $\gamma(\bar{t}) + (t - \bar{t})e_n$.

First of all we introduce the following notation

$$\nu_i = \nu_i(\hat{x}_n) = -\partial_i\varphi(\hat{x}_n), \quad N_i = \frac{\nu_i}{|\nu_i|}, \quad \text{if } \nu_i \neq 0, \quad i \neq n, \quad w(\hat{x}_n) = \sum_{i\neq n} |X_i\varphi(\hat{x}_n)|.$$

Case 1. Define

$$\gamma(t) = x + te_n = \hat{x}_n + (x_n + t)e_n \quad \text{and} \quad \delta \equiv \delta(t) = G_n(\hat{x}_n, t) \simeq d(\gamma(t), \gamma(0)). \quad (3.4.4)$$

Consider for small $\sigma > 0$

$$\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ \big(x_n + t + u_n F_n(x, \sigma\delta)\big)e_n + \hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta) : |u_i| \leq 1, \, i = 1, ..., n \Big\}.$$

We used $F_i(x + te_n, \delta) = F_i(x, \delta)$.

We claim that there exists $\sigma > 0$ independent from x such that $\mathrm{Box}(\gamma(t), \sigma\delta) \subset \Omega$, i.e. such that the following condition holds:

$$x_n + t + u_n F_n(x, \sigma\delta) > \varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big), \quad \delta > 0, \, |u_i| \leq 1. \quad (3.4.5)$$

This is the John condition (3.4.2). Since $x \in \Omega$ then $x_n - \varphi(\hat{x}_n) \geq 0$. Take the worst case $u_n = -1$ in (3.4.5). Moreover, $F_n(x, \sigma\delta) \leq F_n(x, \delta) = F_n(x, G_n(x, t)) = t$. Thus, condition (3.4.5) is easily seen to be implied by

$$\big|\varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| \leq (1 - \sigma)t, \quad \delta > 0, \, |\hat{u}_n| \leq 1.$$

Using the control (3.2.5) for the oscillation of φ, Case 1 and Proposition 3.2.1 we may estimate the left hand side as follows

$$\begin{aligned}\big|\varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| &\leq \mathrm{osc}(\varphi, \mathrm{Box}_n(\hat{x}_n, \sigma\delta)) \\ &\lesssim \sigma\delta w(x) + \sigma\delta\Lambda_n(\hat{x}_n, \sigma\delta)) \\ &\lesssim \sigma\delta\lambda_n(\hat{x}_n) + \sigma F_n(\hat{x}_n, \delta) \simeq \sigma F_n(\hat{x}_n, \delta).\end{aligned}$$

In the last equivalence we used the trivial estimate $\delta\lambda_n(\hat{x}_n) \leq F_n(x, \delta)$. Thus, (3.4.5) is implied by $\sigma F_n(x, \delta) \leq \sigma_0 t$, where σ_0 is a small but absolute constant. Since $t = F_n(x, \delta)$, this inequality holds as soon as $\sigma \leq \sigma_0$.

Case 2. Assume that x satisfies Case 2 in (3.4.3). Take any $k = 1, \dots, n-1$ such that

$$|X_k\varphi(\hat{x}_n)| \geq \frac{1}{2} \max_{i<n} |X_i\varphi(\hat{x}_n)| > \frac{1}{4}\lambda_n(\hat{x}_n). \quad (3.4.6)$$

(The factors $\frac{1}{2}$ and $\frac{1}{4}$ will become relevant in the next section, where we prove that admissible domains are non–tangentially accessible).

Fix $\varepsilon_0 > 0$ and let $\bar{\delta} = \bar{\delta}(x)$ be the solution of the following equation

$$\Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0|X_k\varphi(\hat{x}_n)|. \quad (3.4.7)$$

(The function $\Lambda_n(x, \cdot)$ is a homeomorphism of $[0, \infty[$ onto itself). The number ε_0 will be fixed and will become an absolute constant in (3.4.16). Finally, set $\bar{t} = F_k(x, \bar{\delta})$.

Define the first piece of the John curve letting for $t \in [0, \bar{t}]$

$$\gamma(t) = x + tN_k e_k, \qquad \text{and} \quad \delta = \delta(t) = G_k(x,t) \simeq d(\gamma(0), \gamma(t)).$$

For $\sigma > 0$ consider the $\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ \big(tN_k e_k + \sum_{i=1}^n \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i : |u_i| \le 1,\ i = 1, ..., n \Big\}$. We claim that there exist $\varepsilon_0, \sigma > 0$ independent of x such that the following John condition holds

$$\mathrm{Box}(\gamma(t), \sigma\delta(t)) \subset \Omega, \quad t \in [0, \bar{t}]. \tag{3.4.8}$$

Points of the box belong to Ω as soon as for all u, $|u_i| \le 1$, $i = 1, \dots, n$, we have

$$x_n + u_n F_n(x + tN_k e_k, \sigma\delta) > \varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i\Big).$$

Take the worst case $u_n = -1$ and use $x_n > \varphi(\hat{x}_n)$. The inequality above is implied by

$$\varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i\Big) - \varphi(\hat{x}_n) + F_n(x + tN_k e_k, \sigma\delta) \le 0,$$

which can be rewritten as

$$\mathrm{I} + \mathrm{II} + F_n(x + tN_k e_k, \sigma\delta) \le 0, \tag{3.4.9}$$

where we set

$$\begin{aligned} \mathrm{I} &= \varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i\Big) - \varphi(tN_k e_k + \hat{x}_n), \\ \mathrm{II} &= \varphi(tN_k e_k + \hat{x}_n) - \varphi(\hat{x}_n). \end{aligned}$$

We claim that ε_0 in (3.4.7) can be fixed independently from x in such a way that

$$\mathrm{II} \le -\frac{1}{2} t |\nu_k| \quad \text{ for all } t \in [0, \bar{t}]. \tag{3.4.10}$$

Indeed, by the mean value theorem there exists $\vartheta \in [0, 1]$ such that

$$\begin{aligned} \varphi(\hat{x}_n + tN_k e_k) - \varphi(\hat{x}_n) &= \partial_k \varphi(\hat{x}_n + \vartheta t N_k e_k) t N_k \\ &= \partial_k \varphi(\hat{x}_n) t N_k + \{\partial_k \varphi(\hat{x}_n + \vartheta t N_k e_k) - \partial_k \varphi(\hat{x}_n)\} t N_k \\ &= -|\nu_k| t + \{\partial_k \varphi(\hat{x}_n + \vartheta t N_k e_k) - \partial_k \varphi(\hat{x}_n)\} t N_k. \end{aligned}$$

Notice that Case 2 in (3.4.3) ensures $\nu_k \ne 0$ and $\lambda_k(\hat{x}_n) \ne 0$. The curly brackets can be estimated by (3.2.4) as follows (notice that λ_k does not depend on x_k and $t = F_k(x, \delta)$)

$$\begin{aligned} \big|\{\partial_k \varphi(\hat{x}_n + \vartheta t N_k e_k) - \partial_k \varphi(\hat{x}_n)\}\big| &= \frac{1}{\lambda_k(\hat{x}_n)} \big| X_k \varphi(\hat{x}_n + \vartheta t N_k e_k) - X_k \varphi(\hat{x}_n) \big| \\ &\le \frac{1}{\lambda_k(\hat{x}_n)} \operatorname{osc}(X_k \varphi, \mathrm{Box}_n(\hat{x}_n, \delta)) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)} \big(\delta w(\hat{x}_n)^{(d_n - 2)/(d_n - 1)} + \Lambda_n(\hat{x}_n, \delta)\big) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)} \big(\delta |X_k \varphi(\hat{x}_n)|^{(d_n - 2)/(d_n - 1)} + \Lambda_n(\hat{x}_n, \delta)\big). \end{aligned}$$

In the last inequality we used (3.4.6). Now, (3.4.10) is guaranteed by

$$C_0\big(\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)}+\Lambda_n(\hat{x}_n,\delta)\big)\leq\frac{1}{2}|\nu_k|\lambda_k(\hat{x}_n)=\frac{1}{2}|X_k\varphi(\hat{x}_n)|,\qquad(3.4.11)$$

where C_0 is a big but absolute constant. By Hölder inequality and Proposition 3.2.1

$$C_0\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)}\leq\frac{1}{4}|X_k\varphi(\hat{x}_n)|+C_1\delta^{d_n-1}\leq\frac{1}{4}|X_k\varphi(\hat{x}_n)|+C_2\Lambda_n(\hat{x}_n,\delta),$$

where C_2 is a new big absolute constant. Using $\Lambda_n(\hat{x}_n,\delta)\leq\Lambda_n(\hat{x}_n,\bar{\delta})=\varepsilon_0|X_k\varphi(\hat{x}_n)|$ (this is (3.4.7)) we see that (3.4.11) is guaranteed by a choice of $\varepsilon_0>0$ such that $4(C_0+C_2)\varepsilon_0\leq 1$.

Now, by the estimate on II, inequality (3.4.9) is implied by

$$\mathrm{I}+F_n(x+tN_ke_k,\sigma\delta)\leq\frac{1}{2}t|\nu_k|,\quad t\in[0,\bar{t}].\qquad(3.4.12)$$

We claim that this inequality holds as soon as $\sigma>0$ is small enough independently from x.

First of all by (3.2.5) we find

$$\begin{aligned}\mathrm{I}&\leq\Big|\varphi\Big(tN_ke_k+\sum_{i\neq n}\big(x_i+u_iF_i(x+tN_ke_k,\sigma\delta)\big)e_i\Big)-\varphi(tN_ke_k+\hat{x}_n)\Big|\\&\leq\operatorname{osc}\big(\varphi,\operatorname{Box}_n(\hat{x}_n+tN_ke_k,\sigma\delta)\big)\\&\lesssim\sigma\delta w(\hat{x}_n+tN_ke_k)+\sigma\delta\Lambda_n(\hat{x}_n+tN_ke_k,\sigma\delta),\end{aligned}$$

and by (3.2.4)

$$\begin{aligned}w(\hat{x}_n+tN_ke_k)&=\sum_{i\neq n}|X_i\varphi(\hat{x}_n+tN_ke_k)|\\&\leq\sum_{i\neq n}|X_i\varphi(\hat{x}_n)|+\sum_{i\neq n}|X_i\varphi(\hat{x}_n+tN_ke_k)-X_i\varphi(\hat{x}_n)|\\&\leq w(\hat{x}_n)+\sum_{i\neq n}\operatorname{osc}(X_i\varphi,\operatorname{Box}_n(\hat{x}_n,\delta))\\&\lesssim w(\hat{x}_n)+\delta w(\hat{x}_n)^{(d_n-2)/(d_n-1)}+\Lambda_n(\hat{x}_n,\delta)\\&\lesssim w(\hat{x}_n)+\Lambda_n(\hat{x}_n,\delta).\end{aligned}\qquad(3.4.13)$$

We used once again Hölder inequality and Proposition 3.2.1. Since $\Lambda_n(\hat{x}_n+tN_ke_k,\sigma\delta)\lesssim\Lambda_n(\hat{x}_n,\delta)$, using $w(\hat{x}_n)\leq|X_k\varphi(\hat{x}_n)|$ we finally get

$$\mathrm{I}\lesssim\sigma\delta(|X_k\varphi(\hat{x}_n)|+\Lambda_n(\hat{x}_n,\delta)).\qquad(3.4.14)$$

Now we show that the second term in the left hand side of (3.4.12) satisfies the same estimate (we will need (3.1.9)):

$$\begin{aligned}F_n(x+tN_ke_k,\sigma\delta)&\leq\sigma F_n(x+tN_ke_k,\delta)\lesssim\sigma F_n(x,\delta)\\&=\sigma\delta\lambda_n(\hat{x}_n+\hat{F}_n(x,\delta))\leq\sigma\delta(\Lambda_n(\hat{x}_n,\delta)+\lambda_n(\hat{x}_n))\\&\lesssim\sigma\delta(\Lambda_n(\hat{x}_n,\delta)+|X_k\varphi(\hat{x}_n)|).\end{aligned}\qquad(3.4.15)$$

We used (3.4.6).

Taking into account (3.4.14) and (3.4.15), and recalling that $\Lambda_n(\hat{x}_n, \delta) \leq \Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0 |X_k\varphi(\hat{x}_n)|$, we see that condition (3.4.12) holds as soon as $\sigma\delta|X_k\varphi(\hat{x}_n)| \lesssim t|\nu_k|$, i.e. $C_0\sigma\delta|X_k\varphi(\hat{x}_n)| \leq t|\nu_k|$, i.e. $C_0\sigma\delta\lambda_k(\hat{x}_n) \leq t$ for all $t > 0$. Here $C_0 > 0$ is a big but absolute constant. This inequality holds if $\sigma > 0$ is chosen in such a way that $\sigma C_0 \leq 1$, because $\delta\lambda_k(\hat{x}_n) \leq F_k(\hat{x}_n, \delta) = t$. This proves (3.4.12), and hence claim (3.4.8), as well.

So far we have defined a John curve γ starting from a point $x \in \Omega$ satisfying Case 2 in (3.4.3) for a time $t \in [0, \bar{t}]$, where

$$\bar{t} = F_k(x, \bar{\delta}) \quad \text{and } \bar{\delta} \text{ solves} \quad \Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0|X\varphi(\hat{x}_n)|. \tag{3.4.16}$$

The constant $\varepsilon_0 > 0$ is from now on fixed. Now we define γ for times $t \geq \bar{t}$. Let

$$\gamma(t) = x + \bar{t}N_k e_k + (t - \bar{t})e_n, \qquad t \geq \bar{t}. \tag{3.4.17}$$

We shall write $s = t - \bar{t}$. Set

$$\delta = \delta(t) = \bar{\delta} + G_n(x, t - \bar{t}) = \bar{\delta} + G_n(x, s) \simeq d(\gamma(0), \gamma(t)). \tag{3.4.18}$$

For $\sigma > 0$ consider the box

$$\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ s e_n + \bar{t}N_k e_k + \sum_{i=1}^{n} (x_i + u_i F_i(x + \bar{t}N_k e_k, \sigma\delta))e_i : |u_i| \leq 1 \Big\}.$$

Since $\varphi(\hat{x}_n) \leq x_n$, taking the worst case $u_n = -1$, the John condition $\mathrm{Box}(\gamma(t), \sigma\delta) \subset \Omega$ is implied by

$$\mathrm{J} + \mathrm{JJ} + F_n(x + \bar{t}N_k e_k, \sigma\delta) \leq s, \tag{3.4.19}$$

where we set

$$\begin{aligned} \mathrm{J} &= \varphi\Big(\bar{t}N_k e_k + \sum_{i\neq n} (x_i + u_i F_i(x + \bar{t}N_k e_k, \sigma\delta))e_i\Big) - \varphi(\bar{t}N_k e_k + \hat{x}_n), \\ \mathrm{JJ} &= \varphi(\bar{t}N_k e_k + \hat{x}_n) - \varphi(\hat{x}_n). \end{aligned}$$

By (3.4.10) with $t = \bar{t}$, we have $\mathrm{JJ} \leq -\frac{1}{2}|\nu_k|\bar{t}$. Hence, (3.4.19) is guaranteed by

$$\mathrm{J} + F_n(x + \bar{t}N_k e_k, \sigma\delta) \leq s + \frac{1}{2}|\nu_k|\bar{t}. \tag{3.4.20}$$

We begin with the estimate of J. By (3.2.5)

$$\begin{aligned} \mathrm{J} &\leq \mathrm{osc}\,\big(\varphi, \mathrm{Box}_n(\hat{x}_n + \bar{t}N_k e_k, \sigma\delta)\big) \\ &\lesssim \sigma\delta\big(w(\hat{x}_n + \bar{t}e_k N_k) + \Lambda_n(\hat{x}_n + \bar{t}e_k N_k, \sigma\delta)\big), \end{aligned}$$

and by (3.4.13), (3.4.6) and Proposition 3.2.1 (use $\bar{\delta} \leq \delta$)

$$\delta w(\hat{x}_n + \bar{t}e_k N_k) \lesssim \delta w(\hat{x}_n) + \delta\Lambda_n(\hat{x}_n, \bar{\delta}) \lesssim \delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \delta).$$

On the other hand, by Proposition 3.2.1, (3.1.9), (3.4.18) and (3.1.8)

$$\begin{aligned} \delta\Lambda_n(\hat{x}_n + \bar{t}e_k N_k, \delta) &\lesssim F_n(\hat{x}_n + \bar{t}e_k N_k, \delta) \\ &\lesssim F_n(\hat{x}_n, \delta) = F_n(\hat{x}_n, \bar{\delta} + G_n(\hat{x}_n, s)) \\ &\lesssim F_n(\hat{x}_n, \bar{\delta}) + F_n(\hat{x}_n, G_n(\hat{x}_n, s)) = F_n(\hat{x}_n, \bar{\delta}) + s. \end{aligned}$$

Thus (3.4.20) is ensured by the inequality

$$\sigma(\delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n,\bar{\delta}) + s) \lesssim s + \frac{1}{2}|\nu_k|\bar{t},$$

which reduces to (recall that $\delta - \bar{\delta} = G_n(x,\delta)$, by (3.4.18))

$$\sigma C_0\big(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + G_n(\hat{x}_n,s)\lambda_k(\hat{x}_n) + F_n(\hat{x}_n,\bar{\delta})\big) \le s + |\nu_k|\bar{t} \tag{3.4.21}$$

for some absolute big constant $C_0 > 0$. If in (3.4.21) we put $s = 0$ we get

$$\sigma C_0(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n,\bar{\delta}))) \le |\nu_k|\bar{t},$$

which holds for σ small enough (we have already proved it when we proved (3.4.12), see also (3.4.14), (3.4.15) and (3.4.7)).

To complete the estimate (3.4.21), it will be enough to show that

$$\sigma G_n(\hat{x}_n,s)\lambda_k(\hat{x}_n)|\nu_k| \le s + |\nu_k|\bar{t}, \quad \text{for all } s > 0, \tag{3.4.22}$$

as soon as $\sigma > 0$ is small enough independently of x. Now, (3.4.22) is equivalent to

$$G_n(\hat{x}_n,s) \le \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|} \quad \Leftrightarrow \quad s \le F_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big).$$

Notice that the function $f_n(\hat{x}_n,r) = \frac{1}{r}F_n(\hat{x}_n,r)$ is increasing in the variable r. From

$$s + |\nu_k|\bar{t} \ge |\nu_k|\bar{t} = |\nu_k|F_k(\hat{x}_n,\bar{\delta}) \ge |\nu_k|\bar{\delta}\lambda_k(\hat{x}_n) \ge \sigma|\nu_k|\bar{\delta}\lambda_k(\hat{x}_n)$$

it follows

$$f_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big) \ge f_n(x,\bar{\delta}) \gtrsim \Lambda_n(\hat{x}_n,\bar{\delta}),$$

by Proposition 3.2.1. Finally, recalling (3.4.7) we find out that (3.4.22) is implied by

$$s \le \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k(\hat{x}_n)|\nu_k|}\Lambda_n(\hat{x}_n,\bar{\delta}) = \frac{\varepsilon_0}{\sigma}(s + |\nu_k|\bar{t}),$$

which holds for all $s > 0$ as soon as $\sigma \le \varepsilon_0$. Ultimately, this proves (3.4.19) and ends the discussion of Case 2 and of the parameterization $x_n = \varphi(\hat{x}_n)$.

Now assume that $\bar{x} \in \partial\Omega$ is a point such that for a neighborhood $\mathcal{U}$ of $\bar{x}$ the piece of boundary $\partial\Omega \cap \mathcal{U}$ is a surface of type $\{x_j = \varphi(\hat{x}_j)\}$ for some $j \ne n$ and for some function φ of class C^1 which satisfies the admissibility condition (3.2.10). We explain how to construct a John curve starting from points $x \in \mathcal{U} \cap \{x_j > \varphi(\hat{x}_j)\}$.

The functions $\widetilde{\lambda}_i$ and the vector fields $\widetilde{X}_i$ are defined in (3.2.7). By $\widetilde{d}$ we denote the metric induced on $\mathbb{R}^n$ by the vector fields $\widetilde{X}_i$. The boxes $\widetilde{\mathrm{Box}}_j(\hat{x}_j,r)$ and the function $\widetilde{\Lambda}_j$ have been defined in (3.2.8) and (3.2.9), respectively. Without loss of generality, we can assume $\mathcal{U} \subset \{|x_i| < 1 : i = 1,...,n\}$ and $|\partial_i\varphi(\hat{x}_j)| \le 1$, $i > j$. Then $|\widetilde{X}_i\varphi(\hat{x}_j)| \le \widetilde{\lambda}_j(\hat{x}_j)$, for all $i > j$. Thus the distinction of cases (3.4.3) simply is

$$\max_{i<j}|\widetilde{X}_i\varphi(\hat{x}_j)| \le \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 1)},$$

$$\max_{i<j}|\widetilde{X}_i\varphi(\hat{x}_j)| > \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 2)}.$$

In Case 1 we define a curve γ moving directly in the direction e_j, analogously to (3.4.4). In Case 2 we first define $\gamma(t) = x + tN_k$, where $k = 1, ..., j-1$ is any index such that

$$|\widetilde{X}_k\varphi(\hat{x}_j)| \geq \frac{1}{2}\max_{i<j}|\widetilde{X}_i\varphi(\hat{x}_j)| > \frac{1}{4}\widetilde{\lambda}_j(\hat{x}_j),$$

and $t \in [0, \bar{t}]$, where now $\bar{t} = F_k(\hat{x}_j, \bar{\delta})$ and $\bar{\delta}$ solves $\widetilde{\Lambda}_j(\hat{x}_j, \bar{\delta}) = \varepsilon_0|\widetilde{X}_k\varphi(\hat{x}_j)|$ instead of (3.4.7). Then we let γ move in the direction e_j, analogously to (3.4.17).

The curve γ so defined satisfies, for some $\sigma > 0$ independent of x, the John condition with respect to the metric $\widetilde{d}$, i.e. $\widetilde{B}(\gamma(t), \sigma\widetilde{\mathrm{diam}}(\gamma|_{[0,t]})) \subset \Omega$, $t \in [0,1]$, where $\widetilde{B}$ denote balls in the metric $\widetilde{d}$. The proof of this is exactly the same as for the case $j = n$. By Proposition 3.2.4 (ii) it follows that $\widetilde{\mathrm{diam}}(\gamma|_{[0,t]}) \simeq \mathrm{diam}(\gamma|_{[0,t]})$ and by (i) it also follows that $B(\gamma(t), \sigma\mathrm{diam}(\gamma|_{[0,t]})) \subset \Omega$. This remark ends the proof of the theorem. □

5. Non–tangentially accessible domains

We continue the analysis of domains with admissible boundary. We prove that they satisfy a condition stronger than the John condition. In fact, admissible domains are non–tangentially accessible. We begin with the definition of uniform domain.

Definition 3.5.1. An open set $\Omega \subset (\mathbb{R}^n, d)$ is a *uniform domain* if there exists $\varepsilon > 0$ such that for every $x, y \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,

$$\mathrm{diam}(\gamma) \leq \frac{1}{\varepsilon} d(x,y), \tag{3.5.1}$$

and for all $t \in [0,1]$

$$\mathrm{dist}(\gamma(t), \partial\Omega) \geq \varepsilon \min\{\mathrm{diam}(\gamma|_{[0,t]}), \mathrm{diam}(\gamma|_{[t,1]})\}. \tag{3.5.2}$$

For bounded domains, the uniform property is equivalent to the (ε, δ)–property introduced in [**Jo**] in the Euclidean case and in [**GN2**] for Carnot–Carathéodory spaces. This property requires that (3.5.1) and (3.5.2) hold only for pairs of points $x, y \in \Omega$ such that $d(x,y) \leq \delta$, where δ is a positive number.

Usually, in the definition of uniform domain the curves γ are required to be rectifiable and the diameter in (3.5.2) is replaced by the length (see, for instance, [**V**]). Anyway, in doubling metric spaces with geodesics this stronger definition is equivalent to the weaker one we are giving here (this is proved in [**MS**, Theorem 2.7]).

Next, we introduce the corkscrew condition.

Definition 3.5.2. An open set $\Omega \subset (\mathbb{R}^n, d)$ satisfies the *interior* (*exterior*) *corkscrew condition* if there exist $r_0 > 0$ and $\varepsilon > 0$ such that for all $r \in (0, r_0)$ and $x \in \partial\Omega$ the set $B(x,r) \cap \Omega$ (the set $B(x,r) \cap (\mathbb{R}^n \setminus \overline{\Omega})$) contains a ball of radius εr. An open set Ω satisfies the *corkscrew condition* if it satisfies both the interior and the exterior corkscrew condition.

If both Ω and $\mathbb{R}^n \setminus \bar{\Omega}$ are John domains then Ω satisfies the corkscrew condition.

The notion of non–tangentially accessible domain was introduced in the Euclidean case by Jerison and Kenig in [**JK**], and then generalized to the setting of metric spaces associated with vector fields in [**CG**].

Let $\Omega \subset (\mathbb{R}^n, d)$ be an open set and $\alpha \geq 1$. A sequence of balls $B_0, B_1, ..., B_k \subset \Omega$ is an *α–Harnack chain* in Ω if $B_i \cap B_{i-1} \neq \emptyset$ for all $i = 1, ..., k$, and $\alpha^{-1}\text{dist}(B_i, \partial\Omega) \leq r(B_i) \leq \alpha\text{dist}(B_i, \partial\Omega)$, where $\text{dist}(B_i, \partial\Omega) = \inf_{x\in B_i, y\in\partial\Omega} d(x, y)$ and $r(B_i)$ is the radius of B_i.

DEFINITION 3.5.3. A bounded open set Ω is a *non–tangentially accessible domain* in the metric space $(\mathbb{R}^n, d)$ if the following conditions hold:

(i) there exists $\alpha \geq 1$ such that for all $\eta > 0$ and for all $x, y \in \Omega$ such that $\text{dist}(x, \partial\Omega) \geq \eta$, $\text{dist}(y, \partial\Omega) \geq \eta$ and $d(x, y) \leq C\eta$ for some $C > 0$, there exists an α–Harnack chain $B_0, B_1, ..., B_k \subset \Omega$ such that $x \in B_0$, $y \in B_k$ and k depends on C but not on η;

(ii) Ω satisfies the corkscrew condition.

REMARK 3.5.4. If Ω is a uniform domain according to Definition 3.5.1, then condition (i) in Definition 3.5.3 is fulfilled (see [**CT**, Proposition 4.2]).

The following Lemma gives a useful sufficient condition for an open set to be uniform. The proof is in the Appendix at the end of the Chapter.

LEMMA 3.5.5. *Let $\Omega \subset (\mathbb{R}^n, d)$ be an open set. Assume that there exist constants $\sigma, C_3, C_2 > 0$ such that for all $x, y \in \Omega$ there are John curves $\gamma_x : [0, t_x] \to \Omega$ and $\gamma_y : [0, t_y] \to \Omega$ of parameter σ, with $\gamma_x(0) = x$ and $\gamma_y(0) = y$ and such that*

$$\text{diam}(\gamma_x) \geq C_3 d(x, y), \tag{3.5.3}$$

$$d(\gamma_x(t_x), \gamma_y(t_y)) \leq \frac{\sigma}{2} C_3 d(x, y), \tag{3.5.4}$$

and

$$\max\left\{\text{diam}(\gamma_x), \text{diam}(\gamma_y)\right\} \leq C_2 d(x, y). \tag{3.5.5}$$

Then Ω is a uniform domain.

We recall some results established in the proof of Theorem 3.4.3, and in particular we recall how to construct a John curve starting from a point in a domain with admissible boundary. Consider an open set of the form $\Omega = \{x_n > \varphi(\hat{x}_n)\}$, take a point $x = \hat{x}_n + x_n e_n \in \Omega$ and introduce the following notation

$$\nu_i = \nu_i(\hat{x}_n) = -\partial_i\varphi(\hat{x}_n) \quad \text{and} \quad N_i = \frac{\nu_i}{|\nu_i|}, \quad \text{if } \nu_i \neq 0, \quad i \neq n. \tag{3.5.6}$$

In order to construct a John curve $\gamma_x : [0, 1] \to \Omega$ starting from x, two different situations need to be distinguished:

$$\begin{aligned} &\max_{i<n} |X_i\varphi(\hat{x}_n)| \leq \lambda_n(\hat{x}_n) \qquad \text{(Case 1)}, \\ &\max_{i<n} |X_i\varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \qquad \text{(Case 2)}. \end{aligned} \tag{3.5.7}$$

In Case 1, the characteristic case, define the curve

$$\gamma_x(t) = x + te_n = \hat{x}_n + (x_n + t)e_n, \quad t \geq 0. \tag{3.5.8}$$

In Case 2, the curve γ_x is defined in two steps. First of all, take any $k = 1, \dots, n-1$ such that $|X_k\varphi(\hat{x}_n)|$ is "maximal" in the following sense (this choice is not unique)

$$|X_k\varphi(\hat{x}_n)| \geq \frac{1}{2}\max_{i<n}|X_i\varphi(\hat{x}_n)| > \frac{1}{4}\lambda_n(\hat{x}_n), \tag{3.5.9}$$

and let $\delta_k(x)$ be the solution of the following equation in the variable δ

$$\Lambda_n(\hat{x}_n, \delta) = \varepsilon_0|X_k\varphi(\hat{x}_n)|. \tag{3.5.10}$$

The solution is unique because $\Lambda_n(\hat{x}_n, \cdot)$ is strictly increasing. Here, $\varepsilon_0 > 0$ is a suitable constant which depends on the surface and whose choice is discussed in the proof of Theorem 3.4.3. Finally, define the positive time $t(x) = t_k(x)$ by

$$t_k(x) = F_k(x, \delta_k(x)). \tag{3.5.11}$$

The first piece of γ_x is defined for $t \in [0, t_k(x)]$ by letting

$$\gamma_x(t) = x + tN_ke_k. \tag{3.5.12}$$

Here, $N_k = N_k(x)$ depends on x. The number $\delta_k(x)$ essentially represents the diameter of the first piece of the path. The second piece is

$$\gamma(t) = x + t_k(x)N_ke_k + (t - t_k(x))e_n, \qquad t \geq t_k(x). \tag{3.5.13}$$

In Theorem 3.4.3 we have proved the following. Assume that $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfies (3.2.4) and let $\Omega = \{x_n > \varphi(\hat{x}_n)\}$. Then there exists a constant $\sigma > 0$ such that: if $x \in \Omega$ and Case 1 holds, then the curve γ_x defined as in (3.5.8) is a John curve of parameter σ; if $x \in \Omega$ and Case 2 holds, then for any k such that (3.5.9) holds, the curve γ_x defined in (3.5.12)–(3.5.13) is a John curve of parameter σ.

Now we start the core of our discussion. For any $x \in \Omega$ for which Case 2 in (3.5.7) holds, fix a $k = k(x) \in \{1, \dots, n-1\}$ such that $|X_k\varphi(\hat{x}_n)| = \max_{i<n}|X_i\varphi(\hat{x}_n)|$. Introduce now the parameter $\Delta(x)$ (equivalent to the diameter of the first piece of the path γ_x starting from x) as follows:

$$\Delta(x) = \begin{cases} 0 & \text{if } x \text{ satisfies (3.5.7), Case 1,} \\ \delta_k(x) & \text{if } x \text{ satisfies (3.5.7), Case 2,} \end{cases}$$

where, if x satisfies Case 2, $\delta_k(x)$ is given by (3.5.10).

Let $\varrho > 0$ be a constant that will be fixed later. Given a pair of points x and $y \in \Omega$, we distinguish two cases. The first case is

$$d(x, y) > \varrho \max\{\Delta(x), \Delta(y)\} \qquad \text{(Case A)}.$$

If Case A does not hold, assuming for instance $\Delta(x) \geq \Delta(y)$, it should be $d(x, y) \leq \varrho\Delta(x)$. Moreover, if $k = k(x)$ is the number selected above, we can write $\Delta(x) =$

$\delta_k(x)$. Then the second case is

$$\begin{cases} |X_k\varphi(\hat{x}_n)| = \max_{i\neq n} |X_i\varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \\ d(x,y) \le \varrho\delta_k(x). \end{cases} \tag{Case B}$$

Case B is the more delicate one. The problem here is that if the points x and y are very near and we want to connect them by a curve with total diameter comparable with $d(x,y)$, we have to use only the first piece of the paths γ_x and γ_y starting from x and y. The following Lemma provides the suitable tools to prove that if y is near x (in other words, if we are in Case B and ϱ is small) then we can choose a John curve γ_y from y which starts in the *same direction* of the curve γ_x starting from x. This Lemma gives the correct bound on the oscillation of the horizontal derivatives $X_i\varphi$ near characteristic points. The properties established in this lemma are crucial.

LEMMA 3.5.6. *Let $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfy* (3.2.4)*. There are a constant $\varrho_0 > 0$ and a function $\varrho \mapsto c_\varrho$ from $(0,\varrho_0)$ to $\mathbb{R}^+$, with $\lim_{\varrho\downarrow 0} c_\varrho = 0$ and such that, if Case B holds for a pair of points $x, y \in \{x_n > \varphi(\hat{x}_n)\}$ and for a number $k = 1,\dots,n-1$, then we have*

$$|X_i\varphi(\hat{x}_n) - X_i\varphi(\hat{y}_n)| \le c_\varrho |X_k\varphi(\hat{x}_n)| \quad \forall\, i = 1,\dots,n-1, \tag{3.5.14}$$

$$|X_k\varphi(\hat{y}_n)| \ge (1-c_\varrho)\lambda_n(\hat{y}_n), \tag{3.5.15}$$

and, denoting by $\delta_k(y)$ the solution of (3.5.10) *with $\hat{y}_n$ replacing $\hat{x}_n$,*

$$\delta_k(y) \ge \frac{1}{2}\delta_k(x). \tag{3.5.16}$$

PROOF. Fix $k \in \{1,\dots,n-1\}$ such that $|X_k\varphi(\hat{x}_n)| = \max_{i=1,\dots,n-1} |X_i\varphi(\hat{x}_n)|$. Then (3.2.4) gives

$$\begin{aligned} |X_i\varphi(\hat{x}_n) - X_i\varphi(\hat{y}_n)| &\le \operatorname{osc}\big(X_i\varphi, \operatorname{Box}_n(\hat{x}_n, d(x,y))\big) \\ &\le C\Big(d(x,y)|X_k\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} + \Lambda_n\big(\hat{x}_n, d(x,y)\big)\Big) \\ &\le C\Big(\varrho\delta_k(x)|X_k\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} + C\varrho\Lambda_n\big(\hat{x}_n, \delta_k(x)\big)\Big), \end{aligned}$$

by Case B and Proposition 3.2.1. Now, in order to estimate the right hand side note that by (3.5.10) $\Lambda_n(\hat{x}_n, \delta_k(x)) = \varepsilon_0 |X_k\varphi(\hat{x}_n)|$. Moreover, by Proposition 3.2.1

$$\delta_k(x) \le \Lambda_n\big(\hat{x}_n, \delta_k(x)\big)^{1/(d_n-1)} = \big(\varepsilon_0|X_k\varphi(\hat{x}_n)|\big)^{1/(d_n-1)}.$$

Then (3.5.14) is proved. Letting $i = k$ in (3.5.14) we get

$$|X_k\varphi(\hat{y}_n)| \ge (1-c_\varrho)|X_k\varphi(\hat{x}_n)|. \tag{3.5.17}$$

We are now ready to prove (3.5.15). By the definition of Λ_n we have

$$\begin{aligned} \lambda_n(\hat{y}_n) &\le \lambda_n(\hat{x}_n) + \Lambda_n\big(\hat{x}_n, d(x,y)\big) \le \lambda_n(\hat{x}_n) + \Lambda_n\big(\hat{x}_n, \varrho\delta_k(x)\big) \\ &\le \lambda_n(\hat{x}_n) + c_\varrho\varepsilon_0|X_k\varphi(\hat{x}_n)| \le (1+c_\varrho)|X_k\varphi(\hat{y}_n)|, \end{aligned}$$

where we used Case B to estimate $\lambda_n(\hat{x}_n)$ and (3.5.17). Then (3.5.15) is proved.

We prove (3.5.16). By (3.5.17) and by the definition of δ_k, we have

$$\begin{aligned}\Lambda_n\big(\hat y_n, \delta_k(y)\big) &= \varepsilon_0|X_k\varphi(\hat y_n)| \geq \varepsilon_0(1-c_\varrho)|X_k\varphi(\hat x_n)| \\ &\geq (1-c_\varrho)\Lambda_n\big(\hat x_n, \delta_k(x)\big).\end{aligned} \tag{3.5.18}$$

Assume by contradiction that $\delta_k(y) < \frac{1}{2}\delta_k(x)$. Then, we have

$$\mathrm{Box}_n(\hat y_n, \delta_k(y)) \subset \mathrm{Box}_n\Big(\hat y_n, \frac{1}{2}\delta_k(x)\Big) \subset \mathrm{Box}_n\Big(\hat x_n, \frac{1}{2}(1+c_\varrho)\delta_k(x)\Big),$$

by Lemma 3.1.2 (recall that $d(x,y) \leq \varrho\delta_k(x)$, by Case B). Then

$$\begin{aligned}\Lambda_n\big(\hat y_n, \delta_k(y)\big) &= \sup_{\hat z_n \in \mathrm{Box}_n(\hat y_n, \delta_k(y))} |\lambda_n(\hat z_n) - \lambda_n(\hat y_n)| \\ &\leq \Lambda_n\Big(\hat x_n, \frac{1}{2}(1+c_\varrho)\delta_k(x)\Big) + |\lambda_n(\hat x_n) - \lambda_n(\hat y_n)| \\ &\leq \Lambda_n\Big(\hat x_n, \frac{1}{2}(1+c_\varrho)\delta_k(x)\Big) + \Lambda_n(\hat x_n, \varrho\delta_k(x)) \\ &\leq \Big(h\Big(\frac{1}{2}(1+c_\varrho)\Big) + h(\varrho)\Big)\Lambda_n(\hat x_n, \delta_k(x)),\end{aligned}$$

where h is the function introduced in Proposition 3.2.1. By the properties of h, we immediately see that the last chain of inequalities is in contradiction with (3.5.18), if ϱ is small enough. This finishes the proof of Lemma 3.5.6. □

Using Lemma 3.5.6 we can prove that domains with admissible boundary are non–tangentially accessible.

Theorem 3.5.7. *If $\Omega \subset \mathbb{R}^n$ is an admissible domain then it is a non–tangentially accessible domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. We show that Ω is a uniform domain in the sense of Definition 3.5.1, and this will prove condition (i) in Definition 3.5.3. Condition (ii) is a direct consequence of Theorem 3.4.3.

It will be enough to consider the case $\Omega = \{x_n > \varphi(\hat x_n)\}$ where $\varphi \in C^1(\mathbb{R}^{n-1})$ is a function satisfying (3.2.4). We start the discussion with Case B. Let $x, y \in \Omega$ and $k \in \{1, \dots, n-1\}$ be as in Case B for some $\varrho > 0$. The estimates provided by Lemma 3.5.6 and a choice of ϱ small enough easily imply

$$|X_k\varphi(\hat y_n)| \geq \frac{1}{2}|X_i\varphi(\hat y_n)|, \quad \text{for all } i \neq n, \tag{3.5.19}$$

$$|X_k\varphi(\hat y_n)| > \frac{1}{2}\lambda_n(\hat y_n). \tag{3.5.20}$$

By Theorem 3.4.3 and (3.5.9) there are two John curves γ_x and γ_y of parameter $\sigma > 0$, starting respectively from x and y, which are of the form (compare (3.5.12))

$$\gamma_x(t) = x + tN_ke_k, \quad t \leq t_k(x) \qquad \text{and} \qquad \gamma_y(t) = y + tN_ke_k, \quad t \leq t_k(y). \tag{3.5.21}$$

The numbers $t_k(x)$ and $t_k(y)$ are respectively defined by $t_k(x) = F_k(x, \delta_k(x))$ and $t_k(y) = F_k(y, \delta_k(y))$, where $\delta_k(x)$ and $\delta_k(y)$ are solutions of equation (3.5.10) written in x and y, respectively. Moreover, note that γ_x and γ_y are parallel. This is a consequence of the fact that (3.5.19) and (3.5.20) give (3.5.9) with y instead of x. In addition, $X_k\varphi(\hat{x}_n)$ and $X_k\varphi(\hat{y}_n)$ must have the same sign by (3.5.14) and thus $N_k(x) = N_k(y)$ (recall (3.5.6)). We denoted both by N_k.

We claim that if $\varrho > 0$ is small enough, there exist constants $C_2, C_3 > 0$ (independent of x and y) and times $t_x \le t_k(x)$ and $t_y \le t_k(y)$ such that the curves γ_x and γ_y satisfy assumptions (3.5.3)–(3.5.5) of Lemma 3.5.5. This will show that Ω is a uniform domain.

Define the numbers

$$\delta^* = \frac{1}{2\varrho} d(x,y) \quad \text{and} \quad t^* = F_k(x, \delta^*). \tag{3.5.22}$$

Since we are in Case B, we trivially have $\delta^* \le \frac{\delta_k(x)}{2}$ and by (3.5.16) $\delta^* \le \delta_k(y)$. It follows that $t^* \le t_k(x), t_k(y)$. We would like to apply Lemma 3.5.5 for the times $t_x = t_y = t^*$. This would require the estimate (3.5.4), i.e. $d(\gamma_x(t^*), \gamma_y(t^*)) \le \frac{\sigma C_3}{2} d(x,y)$. Unfortunately, it may happen that $\gamma_x(t^*)$ belongs (or is very near) to the plane $\{x_k = 0\}$. In this case the size of the boxes may become too small (this can be seen letting $x_k = 0$ in (3.1.5)) and estimate (3.5.4) does not seem to hold. To overcome this problem we operate as follows.

Consider the projection of x onto the k'th coordinate plane $x_k = 0$ and denote it by $\pi(x) = \sum_{i \ne k} x_i e_i$. We distinguish the following two cases:

$$d\big(x + t^* N_k e_k, \pi(x)\big) \ge \frac{1}{4} d\big(x, \pi(x)\big), \quad \text{and} \tag{3.5.23}$$

$$d\big(x + t^* N_k e_k, \pi(x)\big) < \frac{1}{4} d\big(x, \pi(x)\big). \tag{3.5.24}$$

We first study case (3.5.23). Case (3.5.24) can be reduced to the first one (this is discussed after equation (3.5.30)). Choose $t_x = t_y = t^*$ and let $\gamma_x : [0, t^*] \to \Omega$ and $\gamma_y : [0, t^*] \to \Omega$ be as in (3.5.21). We first check (3.5.3), which is easier. By Theorem 3.1.1

$$\operatorname{diam}(\gamma_x) \ge C_0 \delta^* = C_0 \frac{d(x,y)}{2\varrho}, \tag{3.5.25}$$

where $C_0 < 1$ is an absolute constant. Then (3.5.3) holds with

$$C_3 = \frac{C_0}{2\varrho}. \tag{3.5.26}$$

Now we have to check (3.5.4), which is

$$d(\gamma_x(t^*), \gamma_y(t^*)) = d(x + t^* N_k e_k, y + t^* N_k e_k) \le \frac{\sigma C_0}{4\varrho} d(x,y). \tag{3.5.27}$$

We claim that there exists a constant $C_4 > 0$ independent of ϱ, x, y such that

$$d(\gamma_x(t^*), \gamma_y(t^*)) \le C_4 d(x,y), \tag{3.5.28}$$

whenever x satisfies (3.5.23). Then (3.5.27) follows choosing ϱ small enough to ensure $C_4 \le \frac{\sigma C_0}{4\varrho}$.

To prove (3.5.28), first of all notice that by Theorem 3.1.1 condition (3.5.23) implies $G_k(\pi(x), |x_k + t^* N_k e_k|) \geq C G_k(\pi(x), |x_k|)$ and thus

$$|x_k + t^* N_k e_k| \geq F_k(\pi(x), C G_k(\pi(x), |x_k|)) \geq C F_k(\pi(x), G_k(\pi(x), |x_k|)) = C|x_k|,$$

for some absolute (small) constant C. This estimate together with the explicit form (3.1.2) and (3.1.5) of the vector fields also implies

$$F_i(x + t^* N_k e_k, s) \geq \varepsilon_1 F_i(x, s), \quad \forall s > 0, i = 1, \dots, n, \tag{3.5.29}$$

where $\varepsilon_1 > 0$ is a new absolute small constant. Then

$$|y_i - x_i| = F_i(x, G_i(x, |y_i - x_i|)) \leq F_i(x, C d(x, y)) \leq \varepsilon_1^{-1} F_i(x + t^* N_k e_k, C d(x, y)).$$

This is equivalent to saying that $y + t^* N_k e_k \in \mathrm{Box}\big(x + t^* N_k e_k, C d(x, y)\big)$, which gives (3.5.28) (by Theorem 3.1.1) provided C_4 is large enough. Note that all such estimates do not depend on ϱ. This proves the claim (3.5.27).

We have proved hypotheses (3.5.3) and (3.5.4) of Lemma 3.5.5 under condition (3.5.23). We discuss later condition (3.5.5).

Now we study case (3.5.24). We shall show that it can be essentially reduced to case (3.5.23). By continuity there is $t^{**} < t^*$ such that

$$d\big(x + t^{**} N_k e_k, \pi(x)\big) = \frac{1}{4} d\big(x, \pi(x)\big). \tag{3.5.30}$$

In this case we choose $t_x = t_y = t^{**}$ and we define define δ^{**} by $t^{**} = F_k(x, \delta^{**})$.

Now we are using shorter paths. We have to make sure that their diameter is large enough to ensure that (3.5.3) continue to hold. In order to check (3.5.3), notice that the triangle inequality and (3.5.24) give

$$d(x, \pi(x)) \geq d(x, \gamma_x(t^*)) - d(\pi(x), \gamma_x(t^*)) > d(x, \gamma_x(t^*)) - \frac{1}{4} d(x, \pi(x)),$$

which yields $d(x, \pi(x)) \geq \frac{4}{5} d(x, \gamma_x(t^*))$. Thus, by (3.5.30)

$$\begin{aligned} \mathrm{diam}(\gamma_x|_{[0,t^{**}]}) &\geq d(x, \pi(x)) - d(\gamma_x(t^{**}), \pi(x)) = \frac{3}{4} d(x, \pi(x)) \\ &\geq \frac{3}{5} d(x, \gamma_x(t^*)) \geq \frac{3}{5} C_3 d(x, y), \end{aligned}$$

where C_3 is given by (3.5.26). In other words, changing δ^* with δ^{**} does not give any problem in checking (3.5.3). We just have to modify slightly the constant C_3 in (3.5.26).

Moreover, since (3.5.30) holds, we can prove (3.5.28) and ultimately (3.5.27) with t^{**} instead of t^*. This shows that (3.5.4) holds in case (3.5.24), as well.

In order to finish the proof of the theorem in Case B, we have to check condition (3.5.5). We check the upper bound for t^*, which is greater than t^{**}. The estimate $\mathrm{diam}(\gamma_x|_{[0,t^*]}) \lesssim C \frac{d(x,y)}{\varrho}$ follows from the definition of δ^*. It remains to estimate the diameter of γ_y. Since by Theorem 3.1.1 $\mathrm{diam}(\gamma_y|_{[0,t^*]}) \simeq G_k(y, t^*)$, the proof is concluded as soon as we show that

$$G_k(y, t^*) \leq 2 G_k(x, t^*).$$

Since $t^* = F_k(x, \delta^*)$, the claim is equivalent to

$$G_k\big(y, F_k(x, \delta^*)\big) \le 2\delta^* \quad \Leftrightarrow \quad F_k(x, \delta^*) \le F_k(y, 2\delta^*),$$

which holds (also with $1 + c_\varrho$ instead of 2) in force of (3.A.3) (in the statement of Lemma 3.1.2 x and y can be interchanged). The proof of Case B is concluded.

Case A is the easy part. Denote by $\widetilde{x}$ e $\widetilde{y}$ the endpoints of the paths γ_x e γ_y at the end of their first piece, i.e. $\widetilde{x} = x + t_{k(x)}(x)N_{k(x)}e_{k(x)}$ and $\widetilde{y} = y + t_{k(y)}(y)N_{k(y)}e_{k(y)}$. Here $k(x)$ may be different from $k(y)$. This does not matter because the points are not too near. It could also be $\widetilde{x} = x$ or $\widetilde{y} = y$ if one or both of the points belong to Case 1 in (3.5.7). At any rate we have

$$d(x, \widetilde{x}) \le \Delta(x) \le \frac{1}{\varrho} d(x, y).$$

The same estimate holds for $d(y, \tilde{y})$ (we are assuming $\Delta(x) \ge \Delta(y)$). Here ϱ is small but has been fixed in the proof of Case B. We have the paths

$$\gamma_x(s) = \widetilde{x} + se_n \quad \text{and} \quad \gamma_y(s) = \widetilde{y} + se_n,$$

with $s \ge 0$. The proof of Case A can be concluded noting that by invariance with respect to translations along the $n-$th direction we have, independently of s,

$$d(\widetilde{x} + se_n, \widetilde{y} + se_n) = d(\widetilde{x}, \widetilde{y}) \le d(\widetilde{x}, x) + d(x, y) + d(\widetilde{y}, y) \le \Big(\frac{1}{\varrho} + 1 + \frac{1}{\varrho}\Big) d(x, y).$$

□

Appendix

Proof of Lemma 3.1.2. By definition, $z \in \mathrm{Box}(y, r)$ if and only if $|z_j - y_j| \le F_j(y, r)$ for all $j = 1, \dots, n$. We need to prove

$$|z_j - x_j| \le F_j\big(x, (1 + c_\varrho)r\big), \qquad j = 1, \dots, n. \tag{3.A.1}$$

The assumptions of the lemma, Theorem 3.1.1 and the first inequality in (3.1.10) give

$$\begin{aligned} |z_j - x_j| &\le |z_j - y_j| + |y_j - x_j| \\ &\le F_j(y, r) + F_j(x, Cd(x, y)) \le F_j(y, r) + c_\varrho F_j(x, r). \end{aligned} \tag{3.A.2}$$

We claim that

$$F_k(y, r) \le F_k\big(x, (1 + c_\varrho)r\big) \quad \text{for all } k = 1, \dots, n. \tag{3.A.3}$$

If the claim is proved, then inserting (3.A.3) in (3.A.2) we conclude

$$|z_j - x_j| \le F_j\big(x, (1 + c_\varrho)r\big) + c_\varrho F_j(x, r) \le (1 + c_\varrho) F_j\big(x, (1 + c_\varrho)r\big) \le F_j\big(x, (1 + c_\varrho)^2 r\big),$$

by (3.1.10) (in our notations $(1 + c_\varrho)^2 = 1 + c_\varrho$). Then the lemma is proved.

In order to show (3.A.3) we use induction on k. The statement is trivial for $k = 1$. If (3.A.3) holds for some k then by (3.1.6)

$$\begin{aligned} F_{k+1}(y, r) &= F_k(y, r)\big(|y_k| + F_k(y, r)\big)^{\alpha_k} \\ &\le F_k\big(x, (1 + c_\varrho)r\big)\Big(|x_k| + |y_k - x_k| + F_k\big(x, (1 + c_\varrho)r\big)\Big)^{\alpha_k}. \end{aligned} \tag{3.A.4}$$

Recall that, by Theorem 3.1.1, $|y_k - x_k| \le F_k(x, Cd(x,y)) \le c_\varrho F_k(x,r)$, and

$$c_\varrho F_k(x,r) + F_k\big(x,(1+c_\varrho)r\big) \le (1+c_\varrho)F_k\big(x,(1+c_\varrho)r\big) \le F_k\big(x,(1+c_\varrho)^2 r\big),$$

by (3.1.10). Inserting the last inequality into the second line of (3.A.4) we immediately conclude the proof of (3.A.3). □

Proof of Proposition 3.2.1. Let $\hat{x}_n \in \mathbb{R}^{n-1}$ and assume without loss of generality $x_i \ge 0$, $i = 1, ..., n-1$. Take $a < 1$. The "reverse doubling" property (3.2.3) is equivalent to

$$\frac{\Lambda_n(x,R) - \Lambda_n(x,aR)}{\Lambda_n(x,aR)} \ge \frac{\eta(1-a)}{a}.$$

It is easy to realize that $\Lambda_n(x,t) = \prod_{j=1}^{n-1} \big(x_j + F_j(x,t)\big)^{\alpha_j} - \prod_{j=1}^{n-1} x_j^{\alpha_j}$. Then

$$\frac{\Lambda_n(x,R) - \Lambda_n(x,aR)}{\Lambda_n(x,aR)} = \frac{\prod_{j=1}^{n-1}\big(x_j + F_j(x,R)\big)^{\alpha_j} - \prod_{j=1}^{n-1}\big(x_j + F_j(x,aR)\big)^{\alpha_j}}{\prod_{j=1}^{n-1}\big(x_j + F_j(x,aR)\big)^{\alpha_j} - \prod_{j=1}^{n-1}\big(x_j\big)^{\alpha_j}} =: \frac{N}{D}.$$

To write N recall that given nonnegative numbers $m_j \le M_j$, $j = 1, \dots p$, the difference of their products can be written as follows

$$M_1 M_2 \cdots M_p - m_1 m_2 \cdots m_p = \sum_{k=1}^{p} (M_k - m_k) \prod_{i=1}^{k-1} M_i \prod_{i=k+1}^{p} m_i \tag{3.A.5}$$

Then

$$\begin{aligned} N = {} & \sum_{k=1}^{n-1} \Big\{ \big(x_k + F_k(x,R)\big)^{\alpha_k} - \big(x_k + F_k(x,aR)\big)^{\alpha_k} \Big\} \\ & \cdot \prod_{i=1}^{k-1} \big(x_i + F_i(x,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \\ \gtrsim {} & \sum_{k=1}^{n-1} \alpha_k \big(F_k(x,R) - F_k(x,aR)\big)\big(x_k + F_k(x,R)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}. \end{aligned}$$

Now note that, letting $F_k(x,t) = t f_k(x,t)$ (see (3.1.5)) we get

$$\begin{aligned} F_k(x,R) - F_k(x,aR) &= R f_k(x,R) - aR f_k(x,aR) \\ &\ge R(1-a) f_k(x,aR) = \frac{1-a}{a} F_k(x,aR). \end{aligned}$$

Then

$$N \gtrsim \frac{1-a}{a} \sum_{k=1}^{n-1} \alpha_k F_k(x,aR)\big(x_k + F_k(x,aR)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i} \simeq \frac{1-a}{a} D,$$

again by (3.A.5) and (3.3.3). Thus $N/D \ge \sigma \frac{1-a}{a}$. Therefore the proof of (3.2.3) is concluded.

To prove the remaining statements use again (3.A.5) and (3.3.3) to write

$$\Lambda_n(x,R) \simeq \sum_{k=1}^{n-1} \alpha_k F_k(x,R)\big(x_k + F_k(x,R)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}. \tag{3.A.6}$$

Now, for any $k = 1, ..., n-1$

$$\alpha_k F_k(x,R)(x_k + F_k(x,R))^{\alpha_k - 1} \prod_{i=1}^{k-1} (x_i + F_i(x,R))^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i} \leq$$

$$\leq \alpha_k \prod_{i=1}^{n-1} (x_i + F_i(x,R))^{\alpha_i} = \frac{1}{R} F_n(x,R).$$

Then the second statement follows. Incidentally, note that the explicit estimate of $\Lambda_n(x,r)$ given in (3.A.6), together with (3.1.7), shows the doubling property $\Lambda_n(x,2r) \leq C\Lambda_n(x,r)$.

Finally, in order to prove that $\Lambda_n(\hat{x}_n, R)) \geq R^{d_n - 1}$ it is enough to estimate from below the right hand side of (3.A.6) using $x_j + F_j(x,R) \geq F_j(x,R) \geq F_j(0,R)$, $j = 1, \ldots, n$

$$\Lambda_n(x,R) \gtrsim \sum_{k=1}^{n-1} \big(F_k(0,R)\big)^{\alpha_k} \prod_{i=1}^{k-1} \big(F_i(0,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} \big(F_i(0,R)\big)^{\alpha_i} = CR^{d_n - 1}.$$

This ends the proof. □

Proof of Lemma 3.5.5. There exists a continuous curve $\tilde{\gamma}$ joining the point $\gamma_x(t_x)$ to the point $\gamma_y(t_y)$ and satisfying the condition $\mathrm{diam}(\tilde{\gamma}) \leq d(\gamma_x(t_x), \gamma_y(t_y))$. Consider the sum path $\gamma = -\gamma_y + \tilde{\gamma} + \gamma_x$, where $-\gamma_y$ stands for a reverse parameterization. We first show condition (3.5.1):

$$\mathrm{diam}(\gamma) \leq \mathrm{diam}(\gamma_x) + \mathrm{diam}(\tilde{\gamma}) + \mathrm{diam}(\gamma_y)$$

$$\leq C_2 d(x,y) + \frac{\sigma}{2} C_3 d(x,y) + C_2 d(x,y) \leq \Big(\frac{\sigma}{2} C_3 + 2C_2\Big) d(x,y).$$

Now we check (3.5.2). Take a point $\gamma_x(t)$ with $t \leq t_x$. Since γ_x is a John curve of parameter σ we have

$$\mathrm{dist}(\gamma_x(t), \partial\Omega) \geq \sigma \mathrm{diam}(\gamma_x|_{[0,t]}) \geq \sigma \min\{\mathrm{diam}(\gamma_x|_{[0,t]}), \mathrm{diam}\big(-\gamma_y + \tilde{\gamma} + \gamma_x|_{[t,t_x]}\big)\}.$$

The same argument works for a point $\gamma_y(t)$, $t \leq t_y$. Finally, given a point $w \in \tilde{\gamma}$, by the triangle inequality, (3.5.3) and (3.5.4) we get

$$\mathrm{dist}(w, \partial\Omega) \geq \mathrm{dist}(\gamma_x(t_x), \partial\Omega) - d(w, \gamma_x(t_x)) \geq \sigma \mathrm{diam}(\gamma_x) - \frac{\sigma}{2} C_3 d(x,y)$$

$$\geq \sigma \mathrm{diam}(\gamma_x) - \frac{\sigma}{2} \mathrm{diam}(\gamma_x) = \frac{\sigma}{2} \mathrm{diam}(\gamma_x).$$

In order to provide a lower bound for the last term it is enough to note that the hypotheses of the lemma ensure that $\mathrm{diam}(\gamma_x) \simeq \mathrm{diam}(\gamma)$ through constants depending on σ, C_3 and C_2. □

Bibliography

[A] L. Ambrosio, Fine properties of sets of finite perimeter in doubling metric measure spaces, Calculus of variations, nonsmooth analysis and related topics, Set-Valued Anal. **10** (2002), no. 2-3, 111–128.

[AT] L. Ambrosio, P. Tilli, Selected topics on "Analysis in Metric Spaces", Scuola Normale Superiore, Pisa 2000.

[ASC] L. Ambrosio, F. Serra Cassano (Editors), Lecture Notes on Analysis in Metric Spaces, Scuola Normale Superiore, Pisa 2000.

[Au] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geometry **11** (1976), 573–598.

[BM] Z. Balogh, R. Monti, Accessible domains in the Heisenberg group, Proc. Amer. Math. Soc. (to appear).

[BGG] R. Beals, P. Greiner, B. Gaveau, Green's functions for some highly degenerate elliptic operators, J. Funct. Anal. **165** (1999), 407–429.

[B] W. Beckner, On the Grushin operator and hyperbolic symmetry, Proc. Amer. Math. Soc. **129** (2001), no. 4, 1233–1246

[Be] A. Bellaïche, The tangent space in sub–Riemannian geometry, in Sub–Riemannian geometry, 1–78, Progr. Math., **144**, Birkhäuser, Basel, 1996.

[BR] A. Bellaïche, J. Riesler (Editors), Sub–Riemannian Geometry, Progress in Mathematics, **144**, Birkhäuser, Basel, 1996.

[Bes] O. V. Besov, Integral representations of functions in a domain with the flexible horn condition, and imbedding theorems (Russian), Dokl. Akad. Nauk. SSSR **273** (1983), 1294–1297.

[Bo] B. Bojarski, Remarks on Sobolev imbedding inequalities, in Proc. of the Conference on Complex Analysis (Joensu 1987), 52–68, Lecture Notes in Math. 1351, Springer 1988.

[Bon] J. M. Bony, Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier (Grenoble) **19** (1969) fasc. 1, 277–304.

[BK] S. Buckley, P. Koskela, Sobolev–Poincaré implies John, Math. Res. Lett. **2** (1995), 577–593.

[BKL] S. Buckley, P. Koskela, G. Lu, Boman equals John, XVIth Rolf Nevanlinna Colloquium (Joensuu, 1995), 91–99, de Gruyter, Berlin, 1996.

[CGS] L. A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.

[CDG1] L. Capogna, D. Danielli, N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality, Comm. Anal. Geom., **12** (1994), 203–215.

[CDG2] L. Capogna, D. Danielli, N. Garofalo, Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations, Amer. J. Math. **118** (1996), 1153–1196.

[CG] L. Capogna, N. Garofalo, Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics, J. Fourier Anal. Appl. **4** (1998), 4-5, 403–432.

[CGN] L. Capogna, N. Garofalo, D. M. Nhieu, Examples of uniform and NTA domains in Carnot groups, Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 103–121, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[CGP] L. Capogna, N. Garofalo, S. W. Pauls, On a class of NTA domains in the Heisenberg groups, preprint.

[CT] L. Capogna, P. Tang, Uniform domains and quasiconformal mappings on the Heisenberg group, Manuscripta Math. **86** (1995), no. 3, 267–281.

[CL] W. X. Chen, C. Liongming, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. **63** (1991), no. 3, 615–622.

[C] J.-L. Clerc, Kelvin transform and multi-harmonic polynomials, Acta Math. **185** (2000), no. 1, 81–99.

[CDKR] M. Cowling, A. H. Dooley, A. Korányi, F. Ricci, H-type groups and Iwasawa decompositions, Adv. Math. **87** (1991), 1–41.

[DS] G. David, S. Semmes, Fractured Fractals and Broken Dreams. Self-Similar Geometry through Metric and Measure, Oxford University Press, 1997.

[DG] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** 1958, 33–44.

[DGN1] D. Danielli, N. Garofalo, D. M. Nhieu, Non–doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot–Carathéodory spaces, preprint.

[DGN2] D. Danielli, N. Garofalo, D. M. Nhieu, Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups, preprint.

[FF] H. Federer, W. H. Fleming, Normal and integral currents, Ann. of Math. (2) **72** 1960 458–520.

[FP] C. Fefferman, D. H. Phong, Subelliptic eigenvalue problems. Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 590–606, Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983.

[FeF] F. Ferrari, B. Franchi, Geometry of the boundary and doubling property of the harmonic measure for Grushin type operators, Rend. Sem. Univ. e Politec. Torino (to appear).

[FGaW] B. Franchi, S. Gallot, R. L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics, Math. Ann. **300** (1994), 557–571.

[FGuW] B. Franchi, C. E. Gutiérrez, R. L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators, Comm. Partial Differential Equations **19** (1994), no. 3-4, 523–604.

[FL] B. Franchi, E. Lanconelli, Une métrique associée à une classe d'opérateurs elliptiques dégénérés, Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 105–114.

[GNN] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[GN1] N. Garofalo, D. M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[GN2] N. Garofalo, D. M. Nhieu, Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot Carathéodory spaces, Jour. Anal. Math. **74** (1998), 67–97.

[GV1] N. Garofalo, D. Vassilev, Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups, Math. Ann. **318** (2000), no. 3, 453–516.

[GV2] N. Garofalo, D. Vassilev, Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type, Duke Math. J. **106** (2001), no. 3, 411–448.

[G1] A. V. Greshnov, Extension of differentiable functions beyond the boundary of the domain on Carnot Groups, Siberian Advances in Mathematics **7** (1997), 20–62.

[G2] A. V. Greshnov, Uniform and NTA-domains on Carnot groups, (Russian) Sibirsk. Mat. Zh. **42** (2001), no. 5, 1018–1035, ii.

[G] M. Gromov, Carnot–Carathéodory spaces seen from within, Sub-Riemannian geometry, 79–323, Progr. Math., **144**, Birkhäuser 1996.

[HK1] P. Hajłasz, P. Koskela, Isoperimetric inequalities and embedding theorems in irregular domains, J. London Math. Soc. **58** (1998), 425–450.

[HK2] P. Hajłasz, P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. **688**, 2000.

[HH] W. Hansen, H. Hueber, The Dirichlet problem for sublaplacians on nilpotent groups – Geometric criteria for regularity, Math. Ann. **276** (1987), 537–547.

[H] J. Heinonen, Lectures on analysis on metric spaces, Springer, New York, 2001.

[J] D. Jerison, The Poincaré inequality for vector fields satisfying Hörmander's condition, Duke Math. J. **53** (1986), 503–523.

[JK] D. Jerison, C. E. Kenig, Boundary behavior of harmonic functions in non–tangentially accessible domains, Adv. Math. **46** (1982), 80–147.

[JL1] D. Jerison, J. M. Lee, The Yamabe problem on CR manifolds, J. Differential Geom. **25** (1987), 167–197.

[JL2] D. Jerison, J. M. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. Amer. Math. Soc. **1** (1988), 1–13.

[Jo] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. **147** (1981), 71–88.

[Joh] F. John, Rotation and strain, Comm. Pure Appl. Math. **14** (1961), 391–413.

[K] A. Korányi, Kelvin transforms and harmonic polynomials on the Heisenberg group, J. Funct. Anal. **49** (1982), 177–185.

[KOT] P. Koskela, J. Onninen, J. T. Tyson, Quasihyperbolic boundary conditions and capacity: Poincaré domains, Math. Ann. (to appear).

[KL] M. K. Kwong, Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, Trans. Amer. Math. Soc. **333** (1992), no. 1, 339–363.

[LM] G. Leonardi, S. Masnou, On the isoperimetric problem in the Heisenberg group, preprint 2002.

[LR] G. Leonardi, S. Rigot, Isoperimetric sets on Carnot groups, Houston J. Math. **29** (2003), no. 3, 609–637

[L] G. Lu, The sharp Poincaré inequality for free vector fields: an endpoint result, Rev. Mat. Iberoamericana **10** (1994), no. 2, 453–466.

[M] O. Martio, John domains, bilipschitz balls and Poincaré inequality, Rev. Roumaine Math. Pures Appl. **33** (1988), 107–112.

[MS] O. Martio, J. Sarvas, Injectivity theorems in plane and space, Ann. Acad. Sci. Fenn., Ser. A I Math. **4** (1978/1979) 383–401.

[Ma] V. G. Maz'ya, Classes of domains and imbedding theorems for function spaces, Dokl. Acad. Nauk. SSSR **133**, 527–530.

[MP] V. G. Maz'ya, S. V. Poborchi, Differentiable functions on bad domains, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[Mo] R. Monti, Brunn–Minkowski and isoperimetric inequality in the Heisenberg group, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 99–109.

[MM1] R. Monti, D. Morbidelli, Trace theorems for vector fields, Math. Z. **239** (2002), 747–776.

[MM2] R. Monti, D. Morbidelli, Regular domains in homogeneous groups, Trans. Amer. Math. Soc. (to appear).

[MM3] R. Monti, D. Morbidelli, Isoperimetric inequality in the Grushin plane, J. Geom. Anal. (to appear).

[MM4] R. Monti, D. Morbidelli, John domains for the control distance of diagonal vector fields, Jour. Anal. Math. (to appear).

[MM5] R. Monti, D. Morbidelli, Boundary regularity for vector fields: metric techniques and applications, submitted.

[MSC] R. Monti, F. Serra Cassano, Surface measures in Carnot-Carathéodory spaces, Calc. Var. Partial Differential Equations **13** (2001), no. 3, 339–376.

[NSW] A. Nagel, E. M. Stein, S. Wainger, Balls and metrics defined by vector fields I: Basic properties, Acta Math. **155** (1985), 103–147.

[P] P. Pansu, An isoperimetric inequality on the Heisenberg group. Conference on differential geometry on homogeneous spaces (Turin, 1983), Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 159–174 (1984)

[R] J. Ratcliffe, Foundations of hyperbolic manifolds, Springer, New York, 1994.

[Re] Yu. G. Reshetnyak, Integral representations of differentiable functions in domains with nonsmooth boundary (Russian), Sib. Mat. Sb. **21** (1980), 108–116.

[SS] W. Smith, D. Stegenga, Hölder domains and Poincaré domains, Trans Amer. Math. Soc. **319** (1990), 67–100.

[T] G. Talenti, The standard isoperimetric theorem, Handbook of convex geometry, Vol. A, B, 73–123, North-Holland, Amsterdam, 1993.

[V] J. Väisälä, Uniform domains, Tohoku Math. J. **40** (1988), 101–118.

[VSC] N. Th. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and geometry on groups, Cambridge Tracts in Mathematics, **100**, Cambridge University Press, Cambridge, 1992.

[VG] S. K. Vodop'yanov, A. V. Greshnov, On the continuation of functions of bounded mean oscillation from domains in a space of homogeneous type with intrinsic metric, Siberian Math. J. **36** (1995), no. 5, 873–901.