This means that the measures $\mu_K = f_K \mathcal{L}^1$, $K \in \mathbb{N}$, weakly converge to $\mu = \delta_0$, the Dirac measure in 0.
But $\delta_0 \perp \mathcal{L}^1$.

Conclusion: Bounded sequences in L^1 need not have weakly converging sub-sequences in L^1.

Approximation and Compactness in BV(A)

We start from the lower semicontinuity

Theorem 1 Let $A \subset \mathbb{R}^n$ be an open set and let $f_K \in BV(A)$, $K \in \mathbb{N}$, be a sequence such that $f_K \xrightarrow[K\to\infty]{} f$ in $L^1_{loc}(A)$. Then

$$\|Df\|(A) \leq \liminf_{K\to\infty} \|Df_K\|(A).$$

Proof. Let $\varphi \in C^1_c(A;\mathbb{R}^n)$ be such that $\|\varphi\|_\infty \leq 1$. Then

$$\int_A f(x)\,\mathrm{div}\,\varphi(x)\,dx = \lim_{K\to\infty}(\inf) \int_A f_K(x)\,\mathrm{div}\,\varphi(x)\,dx$$

$$\leq \liminf_{K\to\infty} \|Df_K\|(A).$$

$\square$

Now we discuss the approximation by smooth functions.

Theorem 2 Let $A \subset \mathbb{R}^n$ be an open set and let $f \in BV(A)$. Then there exists a sequence of functions $f_K \in BV(A) \cap C^\infty(A)$, $K \in \mathbb{N}$, such that:

(1) $f_K \xrightarrow[K\to\infty]{} f$ in $L^1(A)$;

(2) $\|Df_K\|(A) \xrightarrow[K\to\infty]{} \|Df\|(A)$.

Proof. We prove the theorem when $A = \mathbb{R}^n$.
For the general case see [EG] p. 172.

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ function such that $0 \le \eta \le 1$, $\operatorname{spt}(\eta) \subset B_1(0)$ and $\int_{\mathbb{R}^n} \eta \, dx = 1$. For $\varepsilon > 0$ let

We also assume: $\eta(x) = \eta(-x)$

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

Define the functions

$$f_\varepsilon(x) = \eta_\varepsilon * f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y) \, dy.$$

It is well know that $f_\varepsilon \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and that

$$f_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} f \quad \text{in } L^1(\mathbb{R}^n).$$

By lower semicontinuity, it follows that

$$\|Df\|(\mathbb{R}^n) \le \liminf_{\varepsilon \downarrow 0} \|Df_\varepsilon\|(\mathbb{R}^n).$$

Now let $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ be such that $\|\varphi\|_\infty \le 1$.

Consider

$$\int_{\mathbb{R}^n} f_\varepsilon(x) \operatorname{div} \varphi(x) \, dx = \int_{\mathbb{R}^n} \operatorname{div} \varphi(x) \left(\int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y) \, dy \right) dx$$

$$= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \operatorname{div} \varphi(x) \, dx \right) dy$$

$$= - \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} \nabla_x \eta_\varepsilon(x-y) \cdot \varphi(x) \, dx \right) dy$$

$$= \int_{\mathbb{R}^n} f(y) \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial y_i}\left(\eta_\varepsilon(x-y)\right) \varphi_i(x)\, dx\, dy$$

$$= \int_{\mathbb{R}^n} f(y) \underbrace{\left(\sum_{i=1}^n \frac{\partial}{\partial y_i} \underbrace{\int_{\mathbb{R}^n} \overset{\eta_\varepsilon(y-x)}{\overset{\|}{\eta_\varepsilon(x-y)}}\, \varphi_i(x)\, dx}_{\varphi_{i,\varepsilon}(y)} \right)}_{\operatorname{div} \varphi_\varepsilon(y)} dy$$

$$= \int_{\mathbb{R}^n} f(y) \operatorname{div} \varphi_\varepsilon(y)\, dy .$$

Notice that

$$|\varphi_\varepsilon(y)| = \left| \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)\, \varphi(x)\, dx \right| \leq$$

$$\leq \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)\, |\varphi(x)|\, dx \leq 1 \quad \forall y \in \mathbb{R}^n.$$

It follows that

$$\int_{\mathbb{R}^n} f_\varepsilon(x) \operatorname{div} \varphi(x)\, dx = \int_{\mathbb{R}^n} f(y) \operatorname{div} \varphi_\varepsilon(y)\, dy \leq \|Df\|(\mathbb{R}^n)$$

and thus $\|Df_\varepsilon\|(\mathbb{R}^n) \leq \|Df\|(\mathbb{R}^n)$ for all $\varepsilon > 0$.

This implies

$$\limsup_{\varepsilon \downarrow 0} \|Df_\varepsilon\|(\mathbb{R}^n) \leq \|Df\|(\mathbb{R}^n)$$

$\square$

Historical comment

De Giorgi's definition of the $BV(\mathbb{R}^n)$ space.

Let $f \in L^1(\mathbb{R}^n)$ and consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{in } L^1(\mathbb{R}^n). \end{cases}$$

The solution u satisfies $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and

$$\lim_{t \downarrow 0} \| u(x,t) - f(x) \|_{L^1(\mathbb{R}^n)} = 0.$$

In fact, the solution is unique and is

$$u(x,t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

Consider

$$\phi(t) = \int_{\mathbb{R}^n} |\nabla u(x,t)|\, dx, \quad t > 0$$

De Giorgi noticed that $t \mapsto \phi(t)$ is decreasing.

Then the limit

$$\|Df\|(\mathbb{R}^n) := \lim_{t \downarrow 0} \phi(t) \in [0,\infty]$$

does exist.

Definition Let $f \in L^1(\mathbb{R}^n)$. We say that $f \in BV(\mathbb{R}^n)$ if the limit is finite.

This definition is equivalent to the variational definition.

Compactness theorem

Definition We say that a (bounded) open set $A \subset \mathbb{R}^n$ has Lipschitz boundary if for any $x \in \partial A$ there is $r > 0$, there is an isometry $T : \mathbb{R}^n \to \mathbb{R}^n$, there is $B \subset \mathbb{R}^{n-1}$ open and there is $\varphi : B \to \mathbb{R}$ Lipschitz such that

$$T(\partial A \cap B_r(x)) = \{(x', \varphi(x')) \in \mathbb{R}^n : x \in B\}.$$

Theorem Let $A \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $f_k \in BV(A)$, $k \in \mathbb{N}$, be a sequence such that

$$M = \sup_{k \in \mathbb{N}} \|f_k\|_{L^1(A)} + \|Df_k\|(A) < \infty.$$

Then there is $f \in BV(A)$ and a subsequence $\{f_{k_j}\}_{j \in \mathbb{N}}$ such that

$$f_{k_j} \xrightarrow[j \to \infty]{} f \quad \text{in } L^1(A).$$

Example. For $k \geq 1$ let $A_k = \{x \in \mathbb{R}^n : \frac{1}{2k+1} < |x| < \frac{1}{2k}\}$ and let $A = \bigcup_{k=1}^{\infty} A_k$. A is open, bounded.

But at $0 \in \partial A$, ∂A is not (locally) the graph of a Lipschitz function.

We have
$$\mathcal{L}^n(A_k) = \omega_n \left(\left(\frac{1}{2k}\right)^n - \left(\frac{1}{2k+1}\right)^n \right) := c_k \xrightarrow[k \to \infty]{} 0$$

Let $f_k : A \to \mathbb{R}$ be the function

$$f_k(x) = \begin{cases} \frac{1}{c_k} & \text{if } x \in A_k \\ 0 & \text{otherwise} \end{cases}$$

Then we have $\|Df_k\|(A) = 0$ for all $k \in \mathbb{N}$ and $\|f_k\|_{L^1(A)} = 1$. But $(f_k)_{k\in\mathbb{N}}$ has no converging subsequence in $L^1(A)$.

<u>Proof</u> We prove the theorem in the following simplified situation; $A = \mathbb{R}^n$ and there is $R>0$ such that $\operatorname{spt}(f_k) \subset B_R(0)$ for all $k \in \mathbb{N}$.

Let $\mathcal{F} = \{ f_k \, ; \, k \in \mathbb{N} \}$. We claim that $\mathcal{F}$ is totally bounded in $L^1(\mathbb{R}^n)$. Because $L^1(\mathbb{R}^n)$ is complete, it follows that $\mathcal{F}$ is precompact in $L^1(\mathbb{R}^n)$ (i.e., $\overline{\mathcal{F}}$ is compact in $L^1(\mathbb{R}^n)$).

Fix $\varepsilon > 0$ and consider $\mathcal{F}_\varepsilon = \{ f_\varepsilon \, ; \, f \in \mathcal{F} \}$ where

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, f(y)\, dy .$$

We have for all $f \in \mathcal{F}$

- $\operatorname{spt} f_\varepsilon \subset B_{R+\varepsilon}(0)$;
- $|f_\varepsilon(x)| \leq \frac{1}{\varepsilon^n} \max_{\mathbb{R}^n} |\eta| \int_{\mathbb{R}^n} |f(y)|\, dy \leq \frac{1}{\varepsilon^n} \max_{\mathbb{R}^n} |\eta|\, M$;
- $|Df_\varepsilon(x)| \leq \frac{1}{\varepsilon^{n+1}} \max_{\mathbb{R}^n} |D\eta| \int_{\mathbb{R}^n} |f(y)|\, dy \leq \frac{1}{\varepsilon^{n+1}} \max_{\mathbb{R}^n} |D\eta|\, M$.

Then $\mathcal{F}_\varepsilon$ is equicontinuous and equibounded.
By Ascoli-Arzelà theorem $\mathcal{F}_\varepsilon$ is totally bounded for the sup-norm. Then $\mathcal{F}_\varepsilon$ is totally bounded for the $L^1(\mathbb{R}^n)$-norm.
(Here we use the fact that functions have uniformly bounded support).

Now we show that $\mathcal{F}_\varepsilon$ is "uniformly close" to $\mathcal{F}$ in $L^1(\mathbb{R}^n)$.
Let $f \in \mathcal{F}$ and assume for a while that $f \in C^1(\mathbb{R}^n)$.

Then

$$f_\varepsilon(x) - f(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y)\, dy - f(x)$$

$$= \int_{\mathbb{R}^n} \eta(z) \left(f(x-\varepsilon z) - f(x) \right) dz$$

$$= \int_{\mathbb{R}^n} \eta(z) \int_0^1 \frac{d}{dt} \left(f(x - t\varepsilon z) \right) dt\, dz$$

$$= \int_{\mathbb{R}^n} \eta(z) \int_0^1 Df(x - t\varepsilon z) \cdot (-\varepsilon z)\, dt\, dz.$$

It follows that

$$\int_{\mathbb{R}^n} |f_\varepsilon(x) - f(x)|\, dx \leq \varepsilon \int_{\mathbb{R}^n} \eta(z) \int_0^1 \int_{\mathbb{R}^n} |Df(x - t\varepsilon z)|\, |z|\, dx\, dt\, dz$$

$$\leq \varepsilon \int_{\mathbb{R}^n} |Df(x)|\, dx = \varepsilon \, \|Df\|(\mathbb{R}^n).$$

By the approximation theorem, this inequality holds for any $f \in BV$.
The conclusion is

$$\sup_{f \in \mathcal{F}} \|f_\varepsilon - f\|_{L^1(\mathbb{R}^n)} \leq \varepsilon M.$$

We prove the main claim. Fix $\delta > 0$ and let $\varepsilon > 0$ be such that $\varepsilon M < \delta/3$.

$\mathcal{F}_\varepsilon$ totally bounded in $L^1(\mathbb{R}^n)$ $\Rightarrow$ $\exists\, f_1, \dots, f_N \in \mathcal{F}$ such that

$$\mathcal{F}_\varepsilon \subset \bigcup_{i=1}^N B_{L^1(\mathbb{R}^n)}(f_{i,\varepsilon}, \delta/3)$$