Let $f \in \mathcal{F}$. Then we have $\quad (\|\cdot\| = \|\cdot\|_{L^1(\mathbb{R}^n)})$

$$\|f - f_i\| \leq \underbrace{\|f - f_\varepsilon\|}_{<\,\varepsilon/3} + \underbrace{\|f_\varepsilon - f_{i,\varepsilon}\|}_{\substack{<\,\varepsilon/3 \\ \text{choice of } i}} + \underbrace{\|f_{i,\varepsilon} - f_i\|}_{<\,\varepsilon/3} < \varepsilon.$$

$\square$

<u>Poincaré Inequality</u>

Let $A \subset \mathbb{R}^n$ be a bounded open set. For $f \in L^1_{loc}(\mathbb{R}^n)$ we let

$$f_A = \frac{1}{\mathcal{L}^n(A)} \int_A f(x)\, dx = ⨍_A f(x)\, dx.$$

<u>Theorem</u> Let $A \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary. There exists a constant $C = C(A)$ such that

$$\int_A |f(x) - f_A|\, dx \leq C \int_A |Df(x)|\, dx$$

for all $f \in C^1(A) \cap BV(A)$.

Proof. By contradiction there exists a sequence $(f_k)_{k \in \mathbb{N}}$ such that

$$\int_A |Df_k(x)|\, dx < \frac{1}{k}, \quad \text{but}$$

$$\int_A |f_k(x) - (f_k)_A|\, dx = 1$$

for all $k \in \mathbb{N}$

By compactness, there is $f \in BV(A)$ and there is a sub-sequence $(f_{k_j'})_{j\in\mathbb{N}}$ such that

$$f_{k_j'} \xrightarrow[j\to\infty]{} f \quad \text{in } L^1(A).$$

It follows that $(f_{k_j'})_A \xrightarrow[j\to\infty]{} f_A$ and therefore

$$(*) \qquad \int_A |f(x) - f_A|\,dx = \lim_{j\to\infty} \int_A |f_{k_j'}(x) - (f_{k_j'})_A|\,dx = 1.$$

By lower semicontinuity

$$\|Df\|(A) \leq \liminf_{j\to\infty} \|Df_{k_j'}\|(A) = 0.$$

Now we have

$$\left.\begin{array}{l} \|Df\|(A) = 0 \\ A \text{ connected (open)} \end{array}\right\} \Rightarrow f = \text{constant } \mathcal{L}^n\text{-a.e. on } A.$$

The proof is left as an exercise.

Then we have $f(x) = f_A$ for $\mathcal{L}^n$-a.e. $x \in A$.

This contradicts $(*)$.

$\square$

<u>Comment</u> The general Sobolev-Poincaré inequality is

$$\left(\int_A |f(x) - f_A|^{p^*}dx\right)^{\frac{1}{p^*}} \leq C_{A,p}\left(\int_A |Df(x)|^p\,dx\right)^{\frac{1}{p}}$$

with
- $A \subset \mathbb{R}^n$ bounded with Lipschitz boundary, connected
- $1 \leq p < n$ and $p^* = \frac{pn}{n-p}$
- $f \in W^{1,p}(A)$

Traces and Extensions

Let $A \subset \mathbb{R}^n$ be open and let $f, g \in BV(A)$. The function

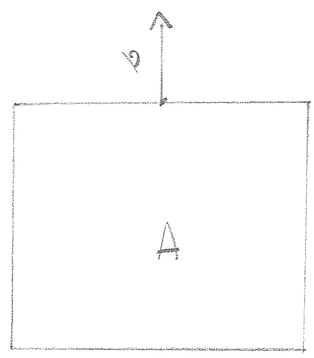
$$d(f,g) = \|f-g\|_{L^1(\mathbb{R}^n)} + \big| \|Df\|(A) - \|Dg\|(A) \big|$$

is a distance on $BV(A)$. The convergence in d is called "strict convergence".

Theorem 1. (Traces) Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. There is a linear and continuous (in the strict convergence) mapping $T : BV(A) \longrightarrow L^1(\partial A; H^{n-1})$ such that

$$\int_A f \operatorname{div} \varphi \, dx = - \int_A \varphi \cdot d[Df] + \int_{\partial A} (\varphi \cdot \nu) \, Tf \, dH^{n-1}$$

for all $f \in BV(A)$ and for all $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$.
Here, ν is the exterior normal to ∂A, that is defined H^{n-1} - a.e. on ∂A.



Definition
The function $Tf : \partial A \longrightarrow [-\infty, \infty]$, $Tf \in L^1(\partial A; H^{n-1})$, is called the trace of $f \in BV(A)$ on ∂A.

<u>Theorem 2</u> $A \subset \mathbb{R}^n$ open bounded, ∂A Lipschitz. Let $f \in BV(A)$.
Then for $\mathcal{H}^{n-1}$-a.e. $x \in \partial A$ we have

$$\lim_{r \downarrow 0} \frac{1}{\mathcal{L}^n(A \cap B_r(x))} \int_{A \cap B_r(x)} | f(y) - Tf(x) | \, dy = 0 .$$

In particular,

$$\lim_{r \downarrow 0} \frac{1}{\mathcal{L}^n(A \cap B_r(x))} \int_{A \cap B_r(x)} f(y) \, dy = Tf(x) .$$

<u>Comment</u> If $f \in BV(A) \cap C(\overline{A})$ then we have

$$Tf(x) = f(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial A .$$

<u>Theorem 3</u> $A \subset \mathbb{R}^n$ open bounded, ∂A Lipschitz. Let $f_1 \in BV(A)$
let $f_2 \in BV(\mathbb{R}^n \setminus \overline{A})$. Define $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \begin{cases} f_1(x) & x \in A \\ f_2(x) & x \in \mathbb{R}^n \setminus A . \end{cases}$$

Then $f \in BV(\mathbb{R}^n)$ and

$$\|Df\|(\mathbb{R}^n) = \|Df_1\|(A) + \|Df_2\|(\mathbb{R}^n \setminus A) + \int_{\partial A} |Tf_1 - Tf_2| \, d\mathcal{H}^{n-1} .$$

For proofs see [EG] pp. 176 - 184.