Fine Properties of BV functions

Definition Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable. Define for any $x \in \mathbb{R}^n$

$$\mu(x) = \operatorname{ap}\limsup_{y \to x} f(y) = \inf \left\{ t \in \mathbb{R} : \lim_{r \downarrow 0} \frac{\mathcal{L}^n(\{f > t\} \cap B_r(x))}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

$$\lambda(x) = \operatorname{ap}\liminf_{y \to x} f(y) = \sup \left\{ t \in \mathbb{R} : \lim_{r \downarrow 0} \frac{\mathcal{L}^n(\{f < t\} \cap B_r(x))}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

Theorem 1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable. Then

$$\lambda(x) = \mu(x) \in \mathbb{R} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

Moreover, λ and μ are Borel measurable.

Definition Define the "approximate discontinuity set" of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$

$$J = \{ x \in \mathbb{R}^n ; \ \lambda(x) < \mu(x) \}.$$

("Jump set").

By Theorem 1 we have $\mathcal{L}^n(J) = 0$.

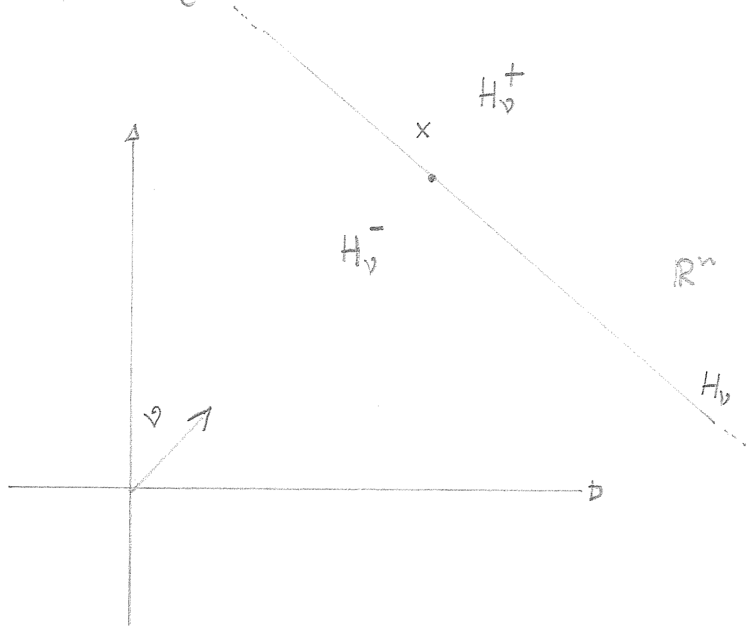
Theorem 2 Let $f \in BV(\mathbb{R}^n)$. Then we have

$$-\infty < \lambda(x) \leq \mu(x) < \infty \quad \text{for } H^{n-1}\text{-a.e. } x \in \mathbb{R}^n$$

Comment: The function $F(x) = \dfrac{\lambda(x) + \mu(x)}{2}$ is finite H^{n-1}-a.e. on $\mathbb{R}^n$, for a function $f \in BV(\mathbb{R}^n)$.

<u>Definition</u> For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ let

$$H_\nu = \{ y \in \mathbb{R}^n ; \; (y-x)\cdot \nu = 0 \},$$
$$H_\nu^+ = \{ y \in \mathbb{R}^n ; \; (y-x)\cdot \nu \geq 0 \},$$
$$H_\nu^- = \{ y \in \mathbb{R}^n ; \; (y-x)\cdot \nu \leq 0 \}.$$



<u>Theorem 3</u> Let $f \in BV(\mathbb{R}^n)$ and let $F(x) = \big(\lambda(x) + \mu(x)\big)/2$.

Then we have :

(1) $$\lim_{r \downarrow 0} \fint_{B_r(x)} |f(y) - F(x)| \, dy = 0 \quad \text{for } \mathcal{H}^{n-1}\text{- a.e. } x \in \mathbb{R}^n \setminus J ;$$

(2) For $\mathcal{H}^{n-1}$- a.e. $x \in J$ there exists $\nu = \nu(x) \in \mathbb{R}^n$, $|\nu| = 1$, such that

$$\lim_{r \downarrow 0} \fint_{B_r(x) \cap H_\nu^+} |f(y) - \mu(x)| \, dy = 0,$$

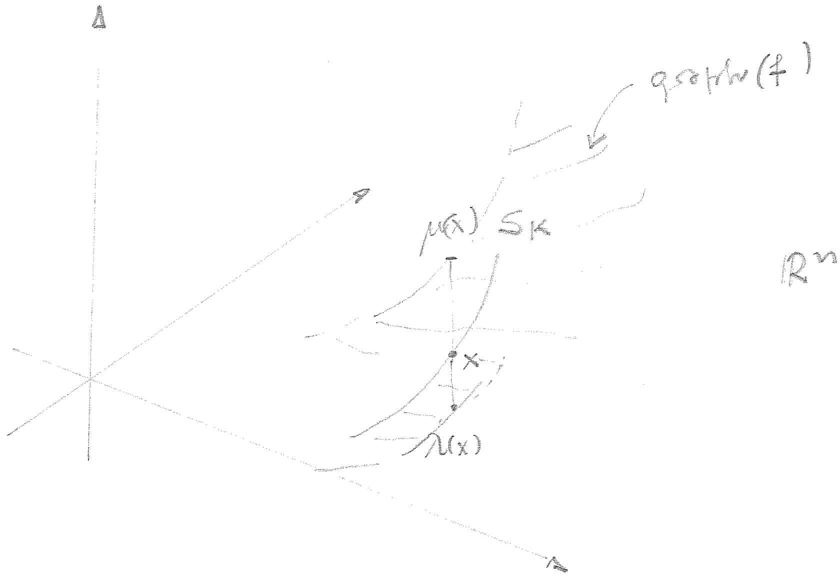
$$\lim_{r \downarrow 0} \fint_{B_r(x) \cap H_\nu^-} |f(y) - \lambda(x)| \, dy = 0.$$

Theorem 4 Let $f \in BV(\mathbb{R}^n)$ and let J be the approximate discontinuity set of f. There exist countably many C^1-hypersurfaces $S_K \subset \mathbb{R}^n$, $K \in \mathbb{N}$, such that

$$\mathcal{H}^{n-1}\left(J \setminus \bigcup_{K=1}^{\infty} S_K\right) = 0 .$$

Comment: J is $\mathcal{H}^{n-1}$-rectifiable.

Picture:



Definition Let $f \in BV(\mathbb{R}^n)$ and let $\mu = \|Df\|$ be the total variation measure of f. We know that

$$\mu = \mu_{ac} + \mu_s \quad \text{with } \mu_{ac} \ll \mathcal{L}^n \text{ and } \mu_s \perp \mathcal{L}^n .$$

We let

$$\mu_j = \mu_s \llcorner J \quad \text{"Jump part of } \mu_s\text{"},$$

$$\mu_c = \mu_s \llcorner (\mathbb{R}^n / J) \quad \text{"Cantor part of } \mu_s\text{"}.$$

<u>Theorem 5</u> Let $f \in BV(\mathbb{R}^n)$ and let $\mu = \|Df\|$ be the total variation measure. Then we have

$$\mu = \underbrace{|\nabla f| \mathcal{L}^n}_{\text{Absolutely continuous Part}} + \underbrace{|\lambda(x) - \mu(x)| \mathcal{H}^{n-1} \llcorner J}_{\text{Jump Part}} + \underbrace{\mu_c}_{\text{Cantor Part}} .$$

Here we have $|\nabla f(x)| = \lim_{r \downarrow 0} \dfrac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))}$, the density of the absolutely continuous part. Moreover

$$\int_J |\lambda(x) - \mu(x)| \, d\mathcal{H}^{n-1} < \infty .$$

<u>Definition</u> We say that $f \in SBV(\mathbb{R}^n)$, special function of bounded variation, if $f \in BV(\mathbb{R}^n)$ and $\mu_c = 0$.