Sets of finite perimeter in $\mathbb{R}^n$

Definition We say that a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ has locally finite perimeter (is a Caccioppoli set) if $\chi_E \in BV_{loc}(\mathbb{R}^n)$, i.e.,

$$\|\partial E\|(A) := \sup \left\{ \int_E \operatorname{div}\varphi \, dx \; ; \; \varphi \in C_c^1(A;\mathbb{R}^n), \; \|\varphi\|_\infty \le 1 \right\} < \infty$$

for any open set $A \subset\subset \mathbb{R}^n$.

We say that E has finite perimeter if $\|\partial E\|(\mathbb{R}^n) < \infty$.

Notation
- We let $P(E;A) = \|\partial E\|(A)$ and $P(E) = P(E;\mathbb{R}^n)$.
- Let μ and $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}$, $|\sigma| = 1$, be the measure and vector given by Riesz Theorem. We have

$$\int_E \operatorname{div}\varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \sigma \, d\mu \qquad \forall \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

Here $E \subset \mathbb{R}^n$ has finite perimeter in $\mathbb{R}^n$.

We call

$\|\partial E\| = \mu$ the perimeter measure of E

$\nu_E = -\sigma$ the exterior (measure theoretic) unit normal to ∂E.

Examples

(1) $E \subset \mathbb{R}^n$ bounded open set with smooth (C^∞, C^1, Lipschitz) boundary. Then, by the divergence theorem

$$\int_E \operatorname{div}\varphi \, dx = \int_{\partial E} \varphi \cdot \underset{\substack{\uparrow \\ \text{outer} \\ \text{normal}}}{\nu} \, dH^{n-1}$$

It follows that

$$\sup_{\|\varphi\|_\infty \leq 1} \int_E \operatorname{div} \varphi \, dx = \mathcal{H}^{n-1}(\partial E) < \infty .$$

Then E has finite perimeter in $\mathbb{R}^n$ and $\|\partial E\|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial E)$.

(2) Let $\mathbb{Q}^n = \{ q_i \in \mathbb{Q}^n : i \in \mathbb{N} \}$ be an enumeration of $\mathbb{Q}^n$.
Fix radii $r_i > 0$ to be chosen and let

$$E_K = \bigcup_{i=1}^{K} \underbrace{B_{r_i}(q_i)}_{\text{open ball}} \quad \text{open set in } \mathbb{R}^n$$

$$E = \bigcup_{i=1}^{\infty} B_{r_i}(q_i) \quad \text{open set in } \mathbb{R}^n.$$

Then

$$\mathcal{L}^n(E) \leq \sum_{i=1}^{\infty} \mathcal{L}^n(B_{r_i}(q_i))$$

$$\leq \alpha(n) \sum_{i=1}^{\infty} r_i^n$$

Fix $r_i > 0$ such that $\alpha(n) \sum_{i=1}^{\infty} r_i^n < \varepsilon$, $\varepsilon > 0$.

We can apply the divergence theorem to each E_K.

$$\int_{E_K} \operatorname{div} \varphi \, dx = \int_{\partial E_K} \varphi \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial E_K) \leq$$

$\|\varphi\|_\infty \leq 1$ — ν exterior normal to ∂A_K

$$\leq \sum_{i=1}^{K} \mathcal{H}^{n-1}(\partial B_{r_i}(q_i)) = n\alpha(n) \sum_{i=1}^{K} r_i^{n-1}$$

It follows that

$$\|\partial E_K\|(\mathbb{R}^n) \leq n\alpha(n) \sum_{i=1}^{K} r_i^{n-1}$$
$$\leq n\alpha(n) \sum_{i=1}^{\infty} r_i^{n-1}$$

We fix $r_i > 0$ such that $n\alpha(n) \sum_{i=1}^{\infty} r_i^{n-1} < \varepsilon$.

Because

$$\chi_{E_K} \xrightarrow[K\to\infty]{} \chi_E \quad \text{in } L^1(\mathbb{R}^n),$$

by lower semicontinuity:

$$\|\partial E\|(\mathbb{R}^n) \leq \liminf_{K\to\infty} \|\partial E_K\|(\mathbb{R}^n) \leq \varepsilon.$$

Conclusion:

For any $\varepsilon > 0$ there is an open set $E \subset \mathbb{R}^n$ such that

- $\mathcal{L}^n(E) \leq \varepsilon$
- $\|\partial E\|(\mathbb{R}^n) \leq \varepsilon$
- $\overline{E} = \mathbb{R}^n$
- $\mathcal{L}^n(\partial E) = \mathcal{L}^n(\overline{E}\setminus E) = \mathcal{L}^n(\mathbb{R}^n \setminus E) = \infty.$

Remarks

(1) E and $\mathbb{R}^n \setminus E$ have the same perimeter.

In fact, for any $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \operatorname{div} \varphi(x)\, dx = 0 \implies \int_{\mathbb{R}^n \setminus E} \operatorname{div} \varphi\, dx = -\int_E \operatorname{div} \varphi\, dx$$

(2) The measure $\|\partial E\|$ is concentrated on the topological boundary ∂E. Proof:

$$\|\partial E\| (\operatorname{int}(E)) = \sup \left\{ \int_{\operatorname{int}(E)} \operatorname{div} \varphi\, dx \;:\; \begin{array}{l} \varphi \in C^1_c(\operatorname{int}(E); \mathbb{R}^n) \\ \|\varphi\|_\infty \le 1 \end{array} \right\} = 0$$

and

$\|\partial E\| (\operatorname{ext}(E)) = 0$, similar.

<u>Example 1</u> Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Fix $m \in \mathbb{R}$ such that $0 < m < \mathcal{L}^n(A)$. Introduce the family of sets

$$\mathcal{A} = \{ E \subset A ;\ E\ \mathcal{L}^n\text{-measurable},\ \mathcal{L}^n(E) = m \}.$$

Let $P : \mathcal{A} \to [0, \infty]$ be the functional

$$P(E) = \|\partial E\|(A).$$

Fix on $\mathcal{A}$ the topology induced by the metric

$$\begin{aligned} d(E,F) = \int_{\mathbb{R}^n} |\chi_E - \chi_F|\, dx &= \mathcal{L}^n(E \setminus F) + \mathcal{L}^n(F \setminus E) \\ &= \mathcal{L}^n(E \setminus F \cup F \setminus E) \\ &= \mathcal{L}^n(E \Delta F). \end{aligned}$$

This is the L^1-topology of characteristic functions.

We claim that the minimum problem

$$\min\{ P(E) : E \in \mathcal{A} \}$$

has a solution. Notice that $\mathcal{A} \neq \emptyset$ and there is $E \in \mathcal{A}$ with $P(E) < \infty$.

Consider a minimizing sequence $E_k \in \mathcal{A}$, $k \in \mathbb{N}$:

$$\lim_{k \to \infty} P(E_k) = \inf\{ P(E) : E \in \mathcal{A} \} < \infty .$$

We have

$$\sup_{k \in \mathbb{N}} \mathcal{L}^n(E_k) + \|\partial E_k\|(A) < \infty .$$

Then the sequence of $BV(A)$-functions $f_k = \chi_{E_k}$ has a subsequence f_{k_j}, $j \in \mathbb{N}$, converging in $L^1(A)$ to a function $f \in BV(A)$:

$$f_{k_j} \xrightarrow[j \to \infty]{} f \quad \text{in } L^1(A).$$

Up to a further sub-sequence we also have

$$\lim_{j \to \infty} f_{k_j}(x) = f(x) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A .$$

Because $f_{k_j}(x) \in \{0,1\}$, we conclude that $f(x) \in \{0,1\}$ for $\mathcal{L}^n$-a.e. $x \in A$. This means that

$$f = \chi_F \quad \text{for some } F \subset A, \ \mathcal{L}^n\text{-measurable.}$$

Moreover we have

$$\left. \begin{array}{l} \mathcal{L}^n(E_{k_j}) = m \quad \forall j \in \mathbb{N} \\ \chi_{E_{k_j}} \xrightarrow[j \to \infty]{} \chi_F \quad \text{in } L^1(A) \end{array} \right\} \Longrightarrow \mathcal{L}^n(F) = m .$$

and we conclude that $F \in \mathcal{A}$.

The functional $P : \mathcal{A} \longrightarrow [0,\infty]$ is lower semicontinuous for the $L^1(A)$-convergence:

$$P(F) = \|\partial F\|(A) \leq \liminf_{j\to\infty} \|\partial E_{k_j}\|(A) =$$

$F \in \mathcal{A}$

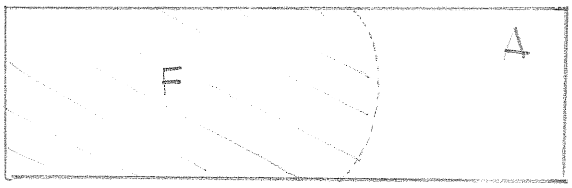
$$= \liminf_{j\to\infty} P(E_{k_j})$$

$$= \inf\{P(E) ; E \in \mathcal{A}\}.$$

We conclude that $P(F) = \min\{P(E) ; E \in \mathcal{A}\}$.

Remarks. The solution is not unique, in general.

- When $n=2$, the boundary $\partial F \cap A$ of a solution F is made of pieces of arcs with the same curvature.



- When $n \geq 3$, the boundary $\partial F \cap A$ of a solution F is made of hypersurfaces with constant mean curvature.

An open problem. It is conjectured that when $A \subset \mathbb{R}^n$ is a bounded convex set then any solution of the problem

$$\min\{\|\partial E\|(A) ; E \subset A \ \mathcal{L}^n\text{-meas}, \ \mathcal{L}^n(E) = m\},$$

with $0 < m < \mathcal{L}^n(A)$, is convex. This conjecture is still unsolved.

<u>Example 2</u> (Weak version of the Plateau problem)

Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Fix a Borel set $B \subset \partial A$. Let

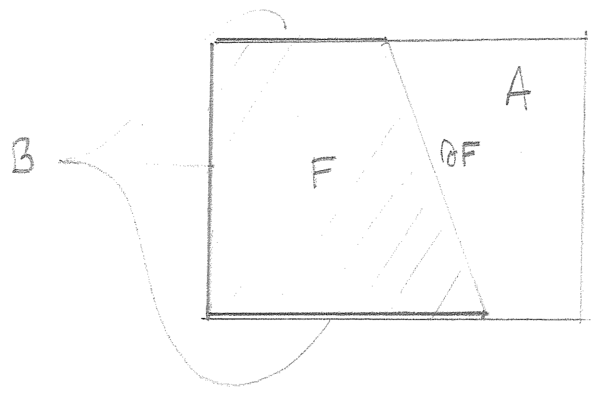
$$\mathcal{A} = \{ E \subset A ; E \ \mathcal{L}^n\text{-measurable}, \ T(\chi_E) = \chi_B \}$$

where $T : BV(A) \longrightarrow \mathcal{L}^1(\partial A ; \mathcal{H}^{n-1})$ is the trace operator.

Assume: there is $E \in \mathcal{A}$ with $\|\partial E\|(A) < \infty$.

Prove: there is $F \in \mathcal{A}$ such that

$$\|\partial F\|(A) = \min \{ \|\partial E\|(A) ; E \in \mathcal{A} \}.$$



<u>Comment</u> The surface $\partial F \cap A$ is a "minimal surface".