ISOPERIMETRIC INEQUALITY, INTRODUCTION.

Theorem 1 (Sobolev Embedding) For any $f \in C^1_c(\mathbb{R}^n)$, $n \geq 2$, we have

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |Df(x)| \, dx .$$

Proof. (Gagliardo) For $x \in \mathbb{R}^n$ we have, $x = (x_1, \dots, x_n)$,

$$f(x) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n) \, dt_i ,$$

and hence

$$|f(x)| \leq \int_{-\infty}^{x_i} |Df(x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)| \, dt_i .$$

It follows that

$$|f(x)|^{\frac{n}{n-1}} \leq \left(\prod_{i=1}^{n} \int_{-\infty}^{+\infty} |Df(\dots)| \, dt_i \right)^{\frac{1}{n-1}} ,$$

Integrate in x_1:

$$(\square) \quad \int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{\mathbb{R}} |Df(\dots)| \, dt_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\prod_{i=2}^{n} \int_{\mathbb{R}} |Df(\dots)| dt_i \right)^{\frac{1}{n-1}} dx_1$$

Now we use the following Lemma:

Lemma Let $\phi_1, \dots, \phi_k \geq 0$ be functions on $\mathbb{R}$, $k \in \mathbb{N}$. Then

$$\int_{\mathbb{R}} (\phi_1 \cdot \ldots \cdot \phi_k)^{\frac{1}{k}} dt \leq \left(\prod_{i=1}^{k} \int_{\mathbb{R}} \phi_i \, dt \right)^{\frac{1}{k}} ,$$

Proof by induction on $k \geq 1$. For $k = 1$ we have equality.
Assume the inequality holds for $k-1$.

By the Lemma we get

$$\int_{\mathbb{R}^2} |f(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \Big(\int_{\mathbb{R}^2} |Df(\dots)| \, dx_1 dt_2 \cdot \int_{\mathbb{R}^2} |Df(\dots)| \, dt_1 dx_2 \cdot$$

$$\cdot \prod_{i=3}^{n} \int_{\mathbb{R}^3} |Df(\dots)| \, dx_1 dx_2 dt_i \Big)^{\frac{1}{n-1}}$$

We continue in the same way and we find:

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \leq \Big(\int_{\mathbb{R}^n} |Df(x)| \, dx \Big)^{\frac{n}{n-1}} .$$

$\square$

<u>Corollary</u> For any $f \in BV(\mathbb{R}^n)$ there holds

$$\Big(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} \leq \|Df\|(\mathbb{R}^n) .$$

Proof. 1^{st} case: $f \in BV(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$

For any $R > 0$ let $\zeta_R \in C^\infty(\mathbb{R}^n)$ be such that

- $0 \leq \zeta_R \leq 1$;
- $\zeta_R(x) = 1$ if $|x| \leq R$;
- $\zeta_R(x) = 0$ if $|x| \geq R+1$.
- $|D\zeta_R(x)| \leq 1$ for all $x \in \mathbb{R}^n$

Then

$$\Big(\int_{\mathbb{R}^n} \zeta_R(x)^{\frac{n}{n-1}} |f(x)|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |D(\zeta_R f)| \, dx \leq$$

$$\leq \int_{\mathbb{R}^n} \{ \zeta_R |Df| + |f| \, |D\zeta_R| \} dx$$

By Hölder inequality we have:

$$\int_{\mathbb{R}} (\phi_1 \cdots \phi_K)^{\frac{1}{K}} dt = \int_{\mathbb{R}} (\phi_1 \cdots \phi_{K-1})^{\frac{1}{K}} \phi_K^{\frac{1}{K}} dt \leq$$

$$\leq \left(\int_{\mathbb{R}} (\phi_1 \cdots \phi_{K-1})^{\frac{p}{K}} dt \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \phi_K^{\frac{q}{K}} dt \right)^{\frac{1}{q}} = (*)$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Choose $q = K$ and then

$$\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q} = \frac{K-1}{K} \Rightarrow p = \frac{K}{K-1}$$

$$(*) = \left(\int_{\mathbb{R}} (\phi_1 \cdots \phi_{K-1})^{\frac{1}{K-1}} dt \right)^{\frac{K-1}{K}} \left(\int_{\mathbb{R}} \phi_K \, dt \right)^{\frac{1}{K}} \overset{\text{Induction}}{\leq}$$

$$\leq \left(\prod_{i=1}^{K-1} \int_{\mathbb{R}} \phi_i \, dt \right)^{\frac{1}{K}} \left(\int_{\mathbb{R}} \phi_K \, dt \right)^{\frac{1}{K}} =$$

$$= \left(\prod_{i=1}^{K} \int_{\mathbb{R}} \phi_i \, dt \right)^{\frac{1}{K}} . \qquad \square$$

We apply the Lemma to $(\square)$. We obtain:

$$\int_{\mathbb{R}} |f(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{\mathbb{R}} |Df(\ldots)| dt_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |Df(\ldots)| dt_i \, dx_1 \right)^{\frac{1}{n-1}}$$

Integrate in x_2:

$$\int_{\mathbb{R}^2} |f(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |Df(\ldots)| dx_1 dt_2 \right)^{\frac{1}{n-1}} \cdot$$

$$\cdot \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} |Df(\ldots)| dt_1 \right)^{\frac{1}{n-1}} \cdot \left(\prod_{i=3}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |Df(\ldots)| dt_i \, dx_1 \right)^{\frac{1}{n-1}} \right] dx_2$$

Now we have

$$\lim_{R \to \infty} \int_{\mathbb{R}^n} \zeta_R(x)\ |Df(x)|\ dx = \int_{\mathbb{R}^n} |Df(x)|\ dx$$

E.g.: dominated convergence with $|Df| \in L^1(\mathbb{R}^n)$.

$$\lim_{R \to \infty} \int_{\mathbb{R}^n} |f|\ \underset{\text{if } |x|<R}{\underset{0}{\underset{|||}{|D\zeta_R|}}}\ dx = 0$$

E.g. dominated convergence with $f \in L^1(\mathbb{R}^n)$

We conclude that

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} = \left(\int_{\mathbb{R}^n} \lim_{R \to \infty}(\inf) |\zeta_R(x) f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq$$

$$\overset{\text{Fatou lemma}}{\leq} \liminf_{R \to \infty} \int_{\mathbb{R}^n} |D(\zeta_R f)|\ dx = \int_{\mathbb{R}^n} |Df(x)|\ dx.$$

If we only have $f \in BV(\mathbb{R}^n)$, we can find a sequence $(f_k)_{k \in \mathbb{N}}$ such that:

- $f_k(x) \xrightarrow[k \to \infty]{} f(x)$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$;
- $\lim_{k \to \infty} \int_{\mathbb{R}^n} |Df_k(x)|\ dx = \|Df\|(\mathbb{R}^n)$.

Passing to the limit in the inequality

$$\left(\int_{\mathbb{R}^n} |f_k|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |Df_k|\ dx$$

we get the claim.

$\square$

THEOREM 2 For any set $E \subset \mathbb{R}^n$ measurable with $\mathcal{L}^n(E) < \infty$ there holds

$$\mathcal{L}^n(E)^{\frac{n-1}{n}} \leq \|\partial E\|(\mathbb{R}^n)$$

Proof. Take $f = \chi_E$ in the previous Corollary. $\square$

We shall prove later the sharp version of this theorem:

THEOREM 3 (Isoperimetric Inequality) For any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, $n \geq 2$, there holds

$$(*) \quad \min\left\{\mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\right\}^{\frac{n-1}{n}} \leq \frac{1}{n\,\alpha(n)^{1/n}} \|\partial E\|(\mathbb{R}^n).$$

Moreover, equality holds if and only if E (or its complement) is (equivalent to) a ball of $\mathbb{R}^n$.

Remarks (1) In the plane $\mathbb{R}^2$, the isoperimetric inequality for a bounded set $E \subset \mathbb{R}^2$ with Lipschitz boundary ∂E is

$$\mathcal{L}^2(E)^{\frac{1}{2}} \leq \frac{1}{2\pi^{1/2}} \operatorname{Length}(\partial E),$$

that is

$$4\pi \, \mathcal{L}^2(E) \leq \operatorname{Length}(\partial E)^2.$$

(2) The isoperimetric inequality $*$) is "invariant" under isometries of $\mathbb{R}^n$ and scalings $E \mapsto \lambda E = \{\lambda x \in \mathbb{R}^n : x \in E\}$ for fixed $\lambda > 0$.

(3) Let $\mathcal{A} = \{ E \subset \mathbb{R}^n : E$ is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(E) = 1\}$.
Consider $P : \mathcal{A} \to [0,\infty]$, $P(E) = \|\partial E\|(\mathbb{R}^n)$.

Theorem 3 is equivalent to the following

THEOREM 4 The minimum problem

$$\min \{ P(E) : E \in \mathcal{A} \}$$

has solutions and the solutions are precisely the balls of $\mathbb{R}^n$ with measure 1.

The existence of solutions is nontrivial, because the compact embedding theorem for BV cannot be applied.

(4) There are several proofs of the (sharp) isoperimetric inequality:

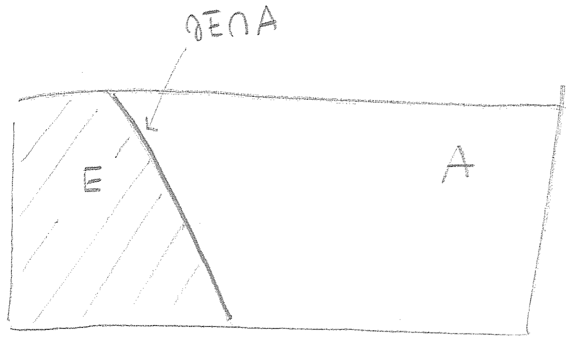
- Fourier Series (only $n=2$)
- Complex variable (only $n=2$)
- Rearrangments
- Gromov proof
- Optimal Transportation theory
- Brunn-Minkowski ineq. $\Rightarrow$ Isop. Inequality

We mention without proof the following:

THEOREM 5 (Relative Isoperimetric Inequality) Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. There exists a constant $C > 0$ such that

$$\min \{ \mathcal{L}^n(E), \mathcal{L}^n(A \setminus E) \}^{\frac{n-1}{n}} \leq C \, \| \partial E \| (A)$$

for all $\mathcal{L}^n$-measurable sets $E \subset A$.

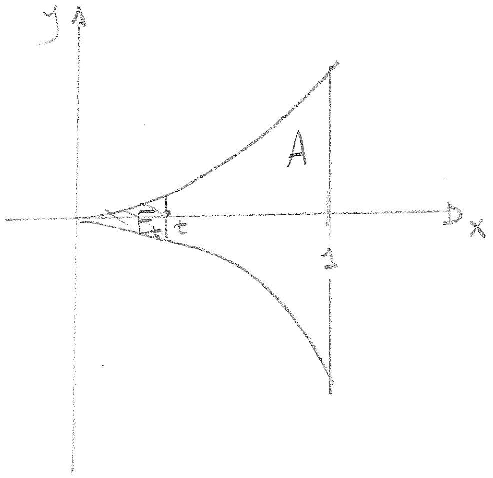


Theorem 5 follows from

$$\left(\int_A |f - f_A|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq C \int_A |Df| .$$

Example Let $A = \{ (x,y) \in \mathbb{R}^2 : \underset{(x>0)}{|y| < x^\alpha < 1} \}$, where $\alpha > 1$ is a parameter. Let $E_t \subset A$ be the set

$$E_t = \{ \underset{(x>0)}{|y| < x^\alpha < t} \} \quad \text{for } 0 < t << 1 .$$



Then we have $\mathcal{L}^2(E_t) = 2\int_0^t x^\alpha \, dx = \frac{2}{\alpha+1} t^{\alpha+1}$.

Moreover

$$\|\partial E_t\|(A) = 2t^\alpha$$

The quotient

$$R(t) = \frac{\|\partial E_t\|(A)}{\mathcal{L}^2(E_t)^{1/2}} = \frac{\sqrt{\alpha+1}\; 2t^\alpha}{\sqrt{2}\; t^{\alpha/2+1/2}} = \sqrt{2(\alpha+1)}\; t^{\frac{\alpha}{2}-\frac{1}{2}}$$

satisfies

$$\lim_{t\downarrow 0} R(t) = 0 \qquad \text{when } \alpha > 1.$$

The (relative) isoperimetric inequality does not hold in A.