# Reduced Boundary

**Definition** (Reduced Boundary) Let $E \subset \mathbb{R}^n$ be a set with locally finite perimeter. The reduced boundary of $E$ is the set $\partial^* E \subset \mathbb{R}^n$ of all points $x \in \mathbb{R}^n$ such that:

(1) $\|\partial E\|(B_r(x)) > 0$ for all $r > 0$;

(2) $$\lim_{r \downarrow 0} \frac{1}{\|\partial E\|(B_r(x))} \int_{B_r(x)} \nu_E(y)\, d\|\partial E\| = \nu_E(x),$$ where

$\nu_E$ is the measure theoretic outward normal of $E$;

(3) $|\nu_E(x)| = 1$.

## Remarks

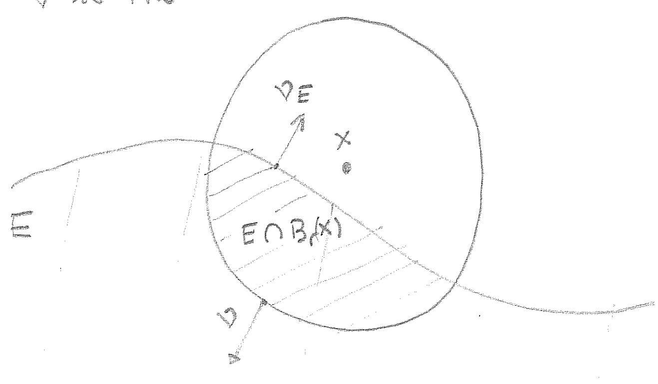
(1) We always have $\partial^* E \subset \partial E$;

(2) $\|\partial E\|(\mathbb{R}^n \setminus \partial^* E) = 0$ because $|\nu_E| = 1$ $\|\partial E\|$-a.e. and the limit (2) holds $\|\partial E\|$-a.e. by the differentiation theorem.

**Lemma 1** $E \subset \mathbb{R}^n$ loc. finite perimeter, $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $x \in \mathbb{R}^n$. Then for $\mathcal{L}^1$-a.e. $r > 0$ we have

$$\int_{E \cap B_r(x)} \operatorname{div} \varphi(y)\, dy = \int_{\underset{\text{closed ball}}{B_r(x)}} \varphi \cdot \nu_E \, d\|\partial E\| + \int_{E \cap \partial B_r(x)} \varphi \cdot \nu \, dH^{n-1}$$

where $\nu$ is the outward normal to $\partial B_r(x)$



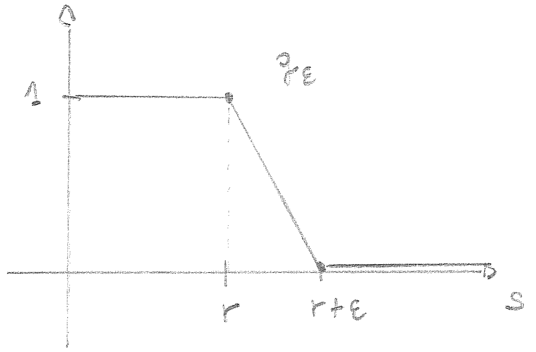
Proof. Let $h \in C^\infty(\mathbb{R}^n)$:

$$= \int_E \operatorname{div}(h\varphi)\,dy = \int_E Dh\cdot\varphi\,dy + \int_E h \operatorname{div}\varphi\,dy$$

Structure Theorem

$$= \int_{\mathbb{R}^n} h\,\varphi\cdot\nu_E \,d\|\partial E\|.$$

Fix $\varepsilon > 0$, and define $g_\varepsilon : [0,\infty) \longrightarrow [0,1]$




$$g_\varepsilon(s) = \begin{cases} 1 & 0 \leq s \leq r \\ 0 & s \geq r+\varepsilon \\ \frac{1}{\varepsilon}(r+\varepsilon-s) & r \leq s \leq r+\varepsilon \end{cases}$$

Lipschitz

The previous formula holds for $h_\varepsilon(y) = g_\varepsilon(|y-x|)$; $h_\varepsilon$ is Lipschitz, the formula is obtained by approximation.

We have

$$Dh_\varepsilon(y) = \begin{cases} 0 & \text{if } |y-x|<r \text{ or } |y-x|>r+\varepsilon; \\ -\frac{1}{\varepsilon}\frac{y-x}{|y-x|}, & \text{if } r<|y-x|<r+\varepsilon. \end{cases}$$

We obtain

$$\int_{\mathbb{R}^n} h_{\varepsilon}\ \varphi \cdot \nu_E \, d\|\partial E\| = - \int_{\{r < |y-x| < r+\varepsilon\} \cap E} \frac{1}{\varepsilon} \frac{y-x}{|y-x|} \cdot \varphi \, dy + \int_E h_{\varepsilon} \operatorname{div} \varphi \, dy .$$

Now let $\varepsilon \downarrow 0$ :

$$\int_{B_r(x)} \varphi \cdot \nu_E \, d\|\partial E\| = - \int_{E \cap \partial B_r(x)} \frac{y-x}{|y-x|} \cdot \varphi \, dy + \int_{E \cap B_r(x)} \operatorname{div} \varphi \, dy .$$

closed ball

□

We used the following Lemma:

<u>Lemma 2</u> Let $g \in L^1_{loc}(\mathbb{R}^n)$. Then the function

$$r \longmapsto \phi(r) = \int_{\partial B_r(0)} g(x) \, dH^{n-1}, \quad r \geq 0,$$ is locally integrable.

Moreover, for any $r > 0$

$$\psi(r) = \int_{B_r(0)} g(x) \, dx = \int_0^r \int_{\partial B_s(0)} g(x) \, dH^{n-1}(x) \, ds .$$

In particular, $\psi$ is absolutely continuous and moreover

$$\frac{d}{dr} \int_{B_r(0)} g(x) \, dx = \int_{\partial B_r(0)} g(x) \, dH^{n-1}(x)$$

for $\mathcal{L}^1$-a.e. $r > 0$.

The proof is postponed.

**Lemma 3** There exist dimensional constants $C_1, C_2 > 0$ with the following property. Let $E \subset \mathbb{R}^n$ have locally finite perimeter and let $x \in \partial^* E$. Then:

$$(1) \quad \liminf_{r \downarrow 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{r^n} \geq C_1 > 0 ;$$

$$(2) \quad \liminf_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} \geq C_2 > 0 .$$

Proof. By Lemma 1, for $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, $|\varphi| \leq 1$, we have

$$\int_{E \cap B_r(x)} \operatorname{div} \varphi \, dy = \int_{B_r(x)} \varphi \cdot \nu_E \, d\|\partial E\| + \int_{E \cap \partial B_r(x)} \varphi \cdot \nu \, dH^{n-1},$$

$$\int_{E \cap B_r(x)} \operatorname{div} \varphi \, dy \leq \|\partial E\|(B_r(x)) + H^{n-1}(\partial B_r(x) \cap E) .$$

Taking the sup in $\varphi$:

$$\|\partial(E \cap B_r(x))\|(\mathbb{R}^n) \leq \|\partial E\|(B_r(x)) + H^{n-1}(\partial B_r(x) \cap E) .$$

Now choose $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $\varphi \underset{\text{constant}}{=} \nu_E(x)$ on $B_r(x)$:

$$0 = \nu_E(x) \cdot \int_{B_r(x)} \nu_E(y) \, d\|\partial E\| + \nu_E(x) \cdot \int_{E \cap \partial B_r(x)} \nu \, dH^{n-1}$$

We deduce that

$$1 = |\nu_E(x)|^2 = \lim_{r \downarrow 0} \nu_E(x) \cdot \fint_{B_r(x)} \nu_E(y) \, d\|\partial E\|$$

Because $x \in \partial^* E$

$$= - \lim_{r \downarrow 0} \underbrace{\frac{1}{\|\partial E\|(B_r(x))}}_{\substack{\text{denominator} > 0 \\ \text{because } x \in \partial^* E}} \nu_E(x) \cdot \int_{E \cap \partial B_r(x)} \nu \, dH^{n-1}$$

Then there is $r_0 = r_0(x) > 0$ such that for $0 < r < r_0$

$$\frac{1}{2} \leq \frac{1}{\|\partial E\|(B_r(x))} \left| \nu_E(x) \cdot \int_{E \cap \partial B_r(x)} \nu \, dH^{n-1} \right| \leq$$

$$\leq \frac{H^{n-1}(E \cap \partial B_r(x))}{\|\partial E\|(B_r(x))} .$$

We conclude that

$$\|\partial(E \cap B_r(x))\|(\mathbb{R}^n) \leq 3 \, H^{n-1}(E \cap \partial B_r(x))$$

This holds for $\mathcal{L}^1$-a.e. $0 < r < r_0$. (The a.e. comes from Lemma 1).

Now let $g(r) = \mathcal{L}^n(E \cap B_r(x))$. By Lemma 2:

$$g(r) = \int_0^r \left( \int_{\partial B_s(x) \cap E} dH^{n-1} \right) ds = \int_0^r H^{n-1}(\partial B_s(x) \cap E) \, ds$$

and moreover, for $\mathcal{L}^1$-a.e. $r > 0$:

$$g'(r) = H^{n-1}(\partial B_r(x) \cap E) .$$

Now we use the isoperimetric inequality to obtain:

$$g(r)^{\frac{n-1}{n}} = \mathcal{L}^n(E \cap B_r(x))^{\frac{n-1}{n}} \overset{\text{Isop. Inequality}}{\leq} C \, \|\partial(E \cap B_r(x))\|(\mathbb{R}^n)$$

$$\leq 3C \, H^{n-1}(E \cap \partial B_r(x)) = 3C \, g'(r)$$

for a.e. $r > 0$.

This is

$$\frac{1}{3C} \leq \varrho(r)^{\frac{1-n}{n}} \varrho'(r) = n \left( \varrho(r)^{\frac{1}{n}} \right)' .$$

We integrate

$$\varrho(r)^{\frac{1}{n}} = \int_0^r \left( \varrho(s)^{\frac{1}{n}} \right)' ds \geq \frac{r}{3nC} ,$$

and this leads to

$$\mathcal{L}^n (E \cap B_r(x)) \geq \frac{r^n}{(3nC)^n} \qquad \text{for } 0 < r < r_0 .$$

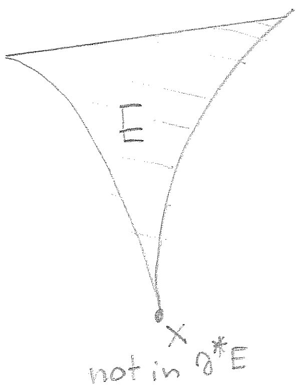
The claim (1) follows. Claim (2) follows from

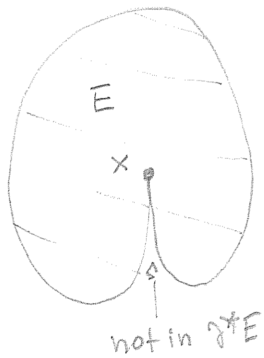
$$\|\partial E\| = \|\partial (\mathbb{R}^n \setminus E)\|$$

$$\nu_E = - \nu_{\mathbb{R}^n \setminus E}$$

□

<u>Remarks</u> :



Lemma 4 There exist dimensional constants $c_1, c_2, c_3 > 0$ with the following property. Let $E \subset \mathbb{R}^n$ have locally finite perimeter and let $x \in \partial^* E$. Then:

$$(1) \quad \liminf_{r \downarrow 0} \frac{\|\partial E\|(B_r(x))}{r^{n-1}} \geq c_1 > 0;$$

$$(2) \quad \limsup_{r \downarrow 0} \frac{\|\partial E\|(B_r(x))}{r^{n-1}} \leq c_2 < \infty;$$

$$(3) \quad \limsup_{r \downarrow 0} \frac{\|\partial(E \cap B_r(x))\|(\mathbb{R}^n)}{r^{n-1}} \leq c_3 < \infty.$$

Proof.

(1) This follows from the relative isoperimetric inequality

$$\|\partial E\|(B_r(x)) \geq \underset{\substack{\text{independent}\\ \text{of } r>0 \text{ and}\\ \text{of } x \in \mathbb{R}^n}}{C} \min\Big\{ \underbrace{\mathcal{L}^n(E \cap B_r(x))}_{\substack{\geq \\ c_1 r^n}}, \underbrace{\mathcal{L}^n(B_r(x) \setminus E)}_{\substack{\geq \\ c_2 r^n}} \Big\}^{\frac{n-1}{n}}.$$

Lemma 3

(2) In the proof of Lemma 3 we had:

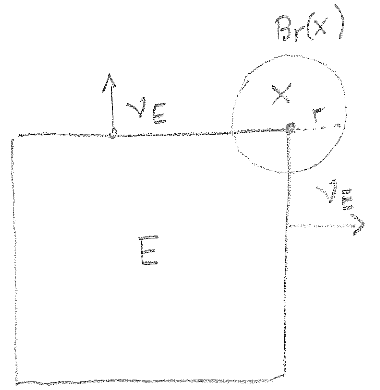
$$\|\partial E\|(B_r(x)) \leq 3\, \mathcal{H}^{n-1}(\partial B_r(x) \cap E) \leq 3\, n\alpha(n)\, r^{n-1}.$$

(3) In the proof of Lemma 3 we had:

$$\|\partial(E \cap B_r(x))\|(\mathbb{R}^n) \leq \|\partial E\|(B_r(x)) + \mathcal{H}^{n-1}(E \cap \partial B_r(x))$$

and the claim follows. $\square$

Remark The corner in the following picture is not in the reduced boundary




$$\fint_{B_r(x)} \nu_E \, d\|\partial E\| = \frac{1}{2r}\left\{ (1,0)\cdot r + (0,1)\cdot r \right\} = \frac{(1,1)}{2}, \quad \forall r > 0.$$

But $\left| \frac{1}{2}(1,1) \right| \neq 1.$