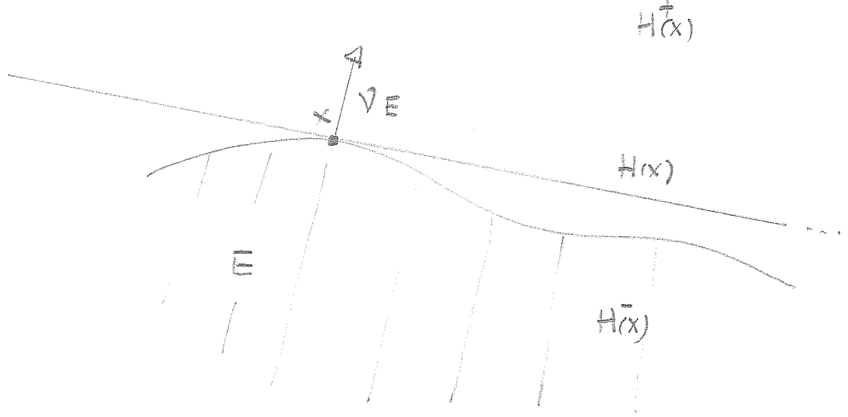
Blow-up of the reduced boundary

$E \subset \mathbb{R}^n$ has locally finite perimeter and $x \in \partial^* E$.

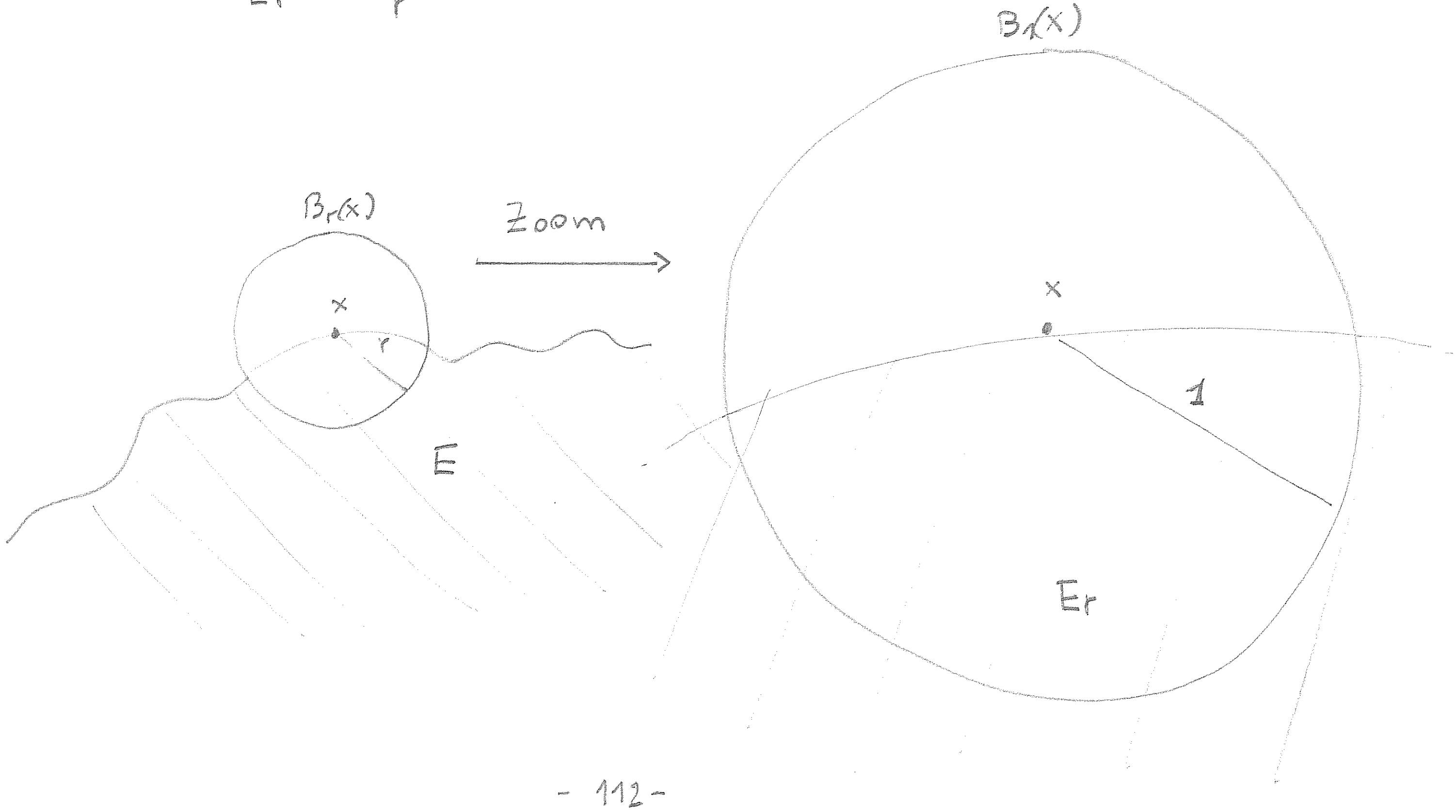
Definition We let:

$$H(x) = \{ y \in \mathbb{R}^n ; (y-x) \cdot \nu_E(x) = 0 \},$$
$$H^+(x) = \{ y \in \mathbb{R}^n ; (y-x) \cdot \nu_E(x) \geq 0 \},$$
$$H^-(x) = \{ y \in \mathbb{R}^n ; (y-x) \cdot \nu_E(x) \leq 0 \}.$$



Notation. For any $r>0$ we let

$$E_r = \frac{1}{r}(E-x) + x = \{ y \in \mathbb{R}^n ; r(y-x)+x \in E \}.$$



Theorem (Blow-up of $\partial^* E$) $E \subset \mathbb{R}^n$ of locally finite perimeter, $x \in \partial^* E$

Then

$$\chi_{E_r} \xrightarrow[r \downarrow 0]{} \chi_{H^-(x)} \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Proof. Without loss of generality: $x = 0$ and $\nu_E(0) = e_n = (0, \dots, 0, 1)$;

$H(0) = \{ y \in \mathbb{R}^n ; y_n = 0 \}$

$H^+(0) = \{ y_n \geq 0 \}$

$H^-(0) = \{ y_n \leq 0 \}$

Claim: For any sequence $r_k \downarrow 0$ there is a subsequence $r_{k_j} \downarrow 0$ such that

$$\chi_{E_{r_{k_j}}} \xrightarrow[j \to \infty]{} \chi_{H^-(0)} \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

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Fix $L > 0$ and let $D_r = E_r \cap \underset{\text{open ball}}{B_L(0)} = \frac{1}{r} E \cap B_L$.

Then we have:

$$\mathcal{L}^n(D_r) \leq \mathcal{L}^n(B_L(0)) < \infty \quad \text{indep. of } r > 0$$

Moreover, by scale-invariance we have

$$\| \partial D_r \| (\mathbb{R}^n) = \| \partial (\tfrac{1}{r} E \cap B_L) \| (\mathbb{R}^n) = \frac{1}{r^{n-1}} \| \partial (E \cap B_{Lr}) \| (\mathbb{R}^n)$$

$$\leq C < \infty \quad \text{indep. of } r > 0,$$

by Lemma 4 part (3).

By the compactness theorem, for any sequence $r_k \downarrow 0$ there is a sub-sequence $s_j = r_{k_j}$ such that

$$\chi_{E_{s_j}} \xrightarrow[j \to \infty]{} f \in BV_{loc}(\mathbb{R}^n), \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

We can also assume that $\chi_{E_{s_j}}(x) \xrightarrow[j\to\infty]{} f(x)$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.
We deduce that $f = \chi_F$ for some $F \subset \mathbb{R}^n$ with loc. finite perimeter. Then

$$\int_F \operatorname{div}\varphi \, dy = \int_{\mathbb{R}^n} \varphi \cdot \nu_F \, d\|\partial F\| \quad , \quad \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

<u>Notation</u>: $E_{s_j} = E_j$ and $\nu_{E_{s_j}} = \nu_j$

Claim: $\nu_F = e_n = (0,\dots,0,1)$ $\|\partial F\|$-a.e. on $\mathbb{R}^n$.

Because $\chi_{E_j} \longrightarrow \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$:

$$\begin{array}{ccccc} \displaystyle\int_{E_j} \operatorname{div}\varphi \, dy & = & \displaystyle\int_{\mathbb{R}^n} \varphi\cdot\nu_j \, d\|\partial E_j\| & & \forall \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n) \\ \downarrow & \Longrightarrow & \downarrow & & \Downarrow \\ \displaystyle\int_F \operatorname{div}\varphi \, dy & = & \displaystyle\int_{\mathbb{R}^n} \varphi\cdot\nu_F \, d\|\partial F\| & & \forall \varphi \in C_c(\mathbb{R}^n;\mathbb{R}^n) \end{array}$$

That is: $\nu_j \|\partial E_j\| \longrightarrow \nu_F \|\partial F\|$ weakly in the sense of Radon measures.

We know that $\|\partial F\|(\partial B_L(0)) = 0$ for all $L>0$ but a countable set. We fix such an $L>0$.
By the characterization of the weak convergence:

$$\int_{B_L(0)} \nu_j \, d\|\partial E_j\| \xrightarrow[j\to\infty]{} \int_{B_L(0)} \nu_F \, d\|\partial F\|$$

Now we use the following relations;

- $$\|\partial E_j\|(B_L(0)) = \|\partial(\tfrac{1}{s_j}E)\|(B_L(0)) = \frac{1}{s_j^{n-1}}\|\partial E\|(B_{s_jL}(0))$$

- $$\int_{B_L(0)} \nu_j \, d\|\partial E_j\| = \frac{1}{s_j^{n-1}} \int_{B_{s_jL}(0)} \nu_E \, d\|\partial E\|$$

The proof is left as an exercise, Whence we find

$$\lim_{j\to\infty} \fint_{B_L(0)} \nu_j \, d\|\partial E_j\| = \lim_{j\to\infty} \frac{\int_{B_{s_jL}(0)} \nu_E \, d\|\partial E\|}{\|\partial E\|(B_{s_jL}(0))} \overset{0\in\partial^* E}{=} \nu_E(0) = e_n = (0,\dots,0,1)$$

and thus

$$\lim_{j\to\infty} \frac{\int_{B_L(0)} e_n\cdot\nu_j \, d\|\partial E_j\|}{\|\partial E_j\|(B_L(0))} = 1.$$

By lower semicontinuity, as $\chi_{E_j} \to \chi_F$ in L^1_{loc}

$$\|\partial F\|(B_L(0)) \le \liminf_{j\to\infty} \|\partial E_j\|(B_L(0)) =$$

$$= \lim_{j\to\infty} \int_{B_L(0)} e_n\cdot\nu_j \, d\|\partial E_j\| = \int_{B_L(0)} e_n\cdot e_F \, d\|\partial F$$

Conclusion:

$$\|\partial F\|(B_L(0)) \leq \int_{B_L(0)} e_n \cdot \nu_F \, d\|\partial F\| \leq$$

$$\leq \|\partial F\|(B_L(0))$$

$$\Downarrow$$

$$e_n \cdot \nu_F = 1 \quad \|\partial F\|\text{-a.e.}$$

$$\Downarrow$$

$$\nu_F = e_n \quad \|\partial F\|\text{-a.e.}$$

As a byproduct:

$$\|\partial F\|(B_L(0)) = \lim_{j\to\infty} \|\partial E_j\|(B_L(0)).$$

Claim: F is a half space and in fact $F = H^-(0)$.

For $\varepsilon > 0$ let $f_\varepsilon = \chi_F * \eta_\varepsilon$ where η_ε is a standard convolution Kernel. From

$$\int_{\mathbb{R}^n} f_\varepsilon \operatorname{div} \varphi \, dy = \int_{\mathbb{R}^n} \overset{f}{\overset{\|}{\chi_F}} \operatorname{div} \varphi_\varepsilon \, dy$$

$$= \int_{\mathbb{R}^n} \nu_F \cdot \varphi_\varepsilon \, dy \,, \quad \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n$$

we deduce that

$$(*) \quad \int_{\mathbb{R}^n} f_\varepsilon \frac{\partial \psi}{\partial x_i} \, dy = 0 \qquad \forall \psi \in C^1_c(\mathbb{R}^n), \quad i = 1, \dots, n-1,$$

and

$$(**) \quad \int_{\mathbb{R}^n} f_\varepsilon \frac{\partial \psi}{\partial x_n} \, dy = \int_{\mathbb{R}^n} \psi \, dy \qquad \forall \psi \in C^1_c(\mathbb{R}^n).$$

Now

$$(*) \Rightarrow \frac{\partial f_\varepsilon}{\partial x_i} = 0 \qquad \forall i = 1, \dots, n-1;$$

$$(**) \Rightarrow \frac{\partial f_\varepsilon}{\partial x_n} \leq 0 \qquad \text{on } \mathbb{R}^n.$$

From the convergence

$$f_\varepsilon(y) \xrightarrow[\varepsilon \downarrow 0]{} \chi_F(y) \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in \mathbb{R}^n$$

we conclude that

$$F = \{ y \in \mathbb{R}^n \, ; \, y_n \leq \gamma \}$$

for some $\gamma \in \mathbb{R}$.

Finally,

$$\|\partial F\|(B_L(0)) = \lim_{j \to \infty} \|\partial E_j\|(B_L(0)) = \lim_{j \to \infty} \frac{\|\partial E\|(B_{s_j L}(0))}{s_j^{n-1}} \geq C > 0,$$

for "a.e." $L > 0$, implies $\gamma = 0 \Rightarrow F = \{ y_n \leq 0 \} = H^-$.

$\square$

Exercise $E \subset \mathbb{R}^n$ has loc. finite perimeter, and $x \in \partial^* E$.

Prove that:

(1) $$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(E \cap B_r(x) \cap H^+(x))}{r^n} = 0,$$

(2) $$\lim_{r \downarrow 0} \frac{\mathcal{L}^n((B_r(x) \setminus E) \cap H^-(x))}{r^n} = 0,$$

(3) $$\lim_{r \downarrow 0} \frac{\|\partial E\|(B_r(x))}{\alpha(n-1)\, r^{n-1}} = 1$$

Structure Theorem and characterization

Theorem Assume that $E \subset \mathbb{R}^n$ has loc. finite perimeter.

Then:

(1) $\partial^* E$ is H^{n-1}-rectifiable;

$$\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N,$$

where $\|\partial E\|(N) = 0$ and K_k is a compact subset of a C^1-hypersurface $S_k \subset \mathbb{R}^n$, $k \in \mathbb{N}$

(2) $\nu_E |_{K_k}$ is normal to S_k;

(3) $\|\partial E\| = H^{n-1} \llcorner \partial^* E$.

For the proof: see [EG] on p. 205.

<u>Definition</u> (Measure theoretic boundary) Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. The m.t.b. $\partial_* E$ of E is the set of all points $x \in \mathbb{R}^n$ such that

$$\limsup_{r \downarrow 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{r^n} > 0 \quad \text{and}$$

$$\limsup_{r \downarrow 0} \frac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} > 0 .$$

<u>Lemma</u> $\partial^* E \subset \partial_* E$ and $H^{n-1}(\partial_* E \setminus \partial^* E) = 0$

Proof: [EG] p. 208

<u>Theorem (Divergence theorem)</u> Let $E \subset \mathbb{R}^n$ be of loc. finite perimeter and let $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$. Then:

$$\boxed{\int_E \operatorname{div} \varphi \, dx = \int_{\partial_* E} \varphi \cdot \nu_E \, dH^{n-1}.}$$

The proof follows from the previous discussion

<u>Theorem (Federer)</u> Let $E \subset \mathbb{R}^n$ be measurable. The following statements are equivalent:

A) E has locally finite perimeter;

B) $H^{n-1}(K \cap \partial_* E) < \infty$ for any $K \subset \mathbb{R}^n$ compact.

Proof: [EG] p. 222