AREA FORMULA

Goal of this chapter is to prove the general version of the Area Formula

$$\int_A \sqrt{1+|Df|^2}\,dx = \mathcal{H}^n(\,gr(f|A))\,.$$

$f: \mathbb{R}^n \to \mathbb{R}$ Lipschitz

$A \subset \mathbb{R}^n$

Jacobians

Def. (i) We say that a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$, is orthogonal if $Tx \cdot Ty = x \cdot y \quad \forall x, y \in \mathbb{R}^n$

(ii) We say that a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, is symmetric if $x \cdot Ty = Tx \cdot y \quad \forall x, y \in \mathbb{R}^n$.

(iii) The adjoint of a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ is the linear map $T^*: \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$x \cdot Ty = T^* x \cdot y \qquad \forall x \in \mathbb{R}^m \quad \forall y \in \mathbb{R}^n$$

Theorem (Polar decomposition) Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then:

(i) If $n \leq m$ there is $S: \mathbb{R}^n \to \mathbb{R}^n$ symmetric and $T: \mathbb{R}^n \to \mathbb{R}^m$ orthogonal such that $L = T \circ S$.

(ii) If $m \leq n$ there is $S: \mathbb{R}^m \to \mathbb{R}^m$ symmetric and $T: \mathbb{R}^m \to \mathbb{R}^n$ orthogonal such that $L = S \circ T^*$.

<u>Definition</u> Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear. We let:

(i) $[\![L]\!] = |\det S|$ if $n \le m$ and $L = T \circ S$;

(ii) $[\![L]\!] = |\det S|$ if $m \le n$ and $L = S \circ T^*$.

We call $[\![L]\!]$ the Jacobian of L.

<u>Theorem</u> $L: \mathbb{R}^n \to \mathbb{R}^m$ linear. Then:

(i) If $n \le m$ then $[\![L]\!]^2 = \det(L^* \circ L)$;

(ii) If $m \le n$ then $[\![L]\!]^2 = \det(L \circ L^*)$.

Proof (i):
$$\det(L^* \circ L) = \det((T \circ S)^* \circ T \circ S)$$
$$= \det(S^* \circ T^* \circ T \circ S)$$
$$= \det(S^* \circ S) = \det(S)^2 = [\![L]\!]^2$$

<u>Definition</u> For $n \le m$ let:

(i) $\Lambda(m,n) = \{ \lambda : \{1, \dots, n\} \to \{1, \dots, m\} : \lambda \text{ is (strictly) increasing} \}$

(ii) For each $\lambda \in \Lambda(m,n)$ let $P_\lambda : \mathbb{R}^m \to \mathbb{R}^n$ be the "projection"
$$P_\lambda(x) = (x_{\lambda(1)}, \dots, x_{\lambda(n)})$$

<u>Theorem (Binet-Cauchy Formula)</u> Let $n \le m$ and $L: \mathbb{R}^n \to \mathbb{R}^m$ linear. Then
$$[\![L]\!]^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

Proof Let $L = (l_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, $L^* = (l^*_{ji})_{\substack{j=1,\dots,n \\ i=1,\dots,m}}$

with $l^*_{ji} = l_{ij}$. Moreover, let

$$A = L^* \circ L \ , \quad A = (a_{ij})_{i,j=1,\dots,n} \ ,$$

$$a_{ij} = \sum_{k=1}^{m} l^*_{ik} \, l_{kj} = \sum_{k=1}^{m} l_{ki} \, l_{kj} \ .$$

Then

$$[\![L]\!]^2 = \det(A) = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{k=1}^{m} l_{ki} \, l_{k\sigma(i)}$$

with $\Sigma_n = \{ \text{perm. of } n \text{ elements} \}$,

Now let $\Phi := \{ \lambda : \{1,\dots,n\} \to \{1,\dots,m\} \}$,
$\Phi^* := \{ \lambda \in \Phi \ ; \ \lambda \text{ injective} \}$.

We find

$$[\![L]\!]^2 = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varphi \in \Phi} \prod_{i=1}^{n} l_{\varphi(i) i} \, l_{\varphi(i) \sigma(i)}$$

$$= \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \sum_{\varphi \in \Phi^*} \prod_{i=1}^{n} l_{\varphi(i) i} \, l_{\varphi(i) \sigma(i)} \ .$$

Now $\varphi \in \Phi^*$ is of the form $\varphi = \lambda \circ \theta$, unique, with $\theta \in \Sigma_n$ and $\lambda \in \Lambda(m,n)$, so

$$[\![L]\!]^2 = \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \sum_{\theta \in \Sigma_n} \sum_{\lambda \in \Lambda(m,n)} \prod_{i=1}^{n} l_{\lambda(\theta(i)), i} \, l_{\lambda(\theta(i)), \sigma(i)}$$

and thus

$$[\![L]\!] = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \sum_{\theta\in\Sigma_n} \sum_{\lambda\in\Lambda(m,n)} \prod_{i=1}^{n} l_{\lambda(i),\theta(i)}\, l_{\lambda(i),\sigma(\theta(i))}$$

$$\sigma\circ\theta = \eta \in \Sigma_n \qquad \operatorname{sgn}(\sigma) = \operatorname{sgn}(\theta)\operatorname{sgn}(\eta)$$

$$= \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta,\eta\in\Sigma_n} \operatorname{sgn}(\theta)\operatorname{sgn}(\eta) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)}\, l_{\lambda(i),\eta(i)}$$

$$= \sum_{\lambda\in\Lambda(m,n)} \left(\sum_{\theta\in\Sigma_n} \operatorname{sgn}(\theta) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)} \right)^2$$

$$= \sum_{\lambda\in\Lambda(m,n)} \Big(\det(P_\lambda \circ L) \Big)^2 .$$

□

<u>Lemma 1</u> Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $n \le m$, be linear and let $A \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then

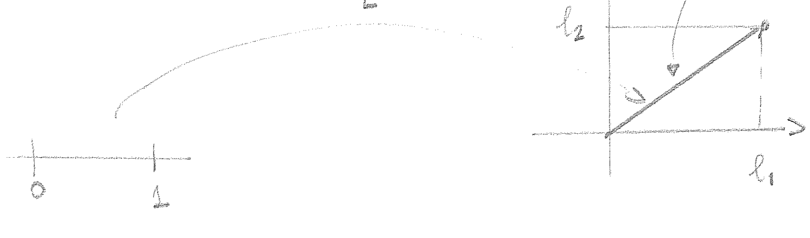
$$\mathcal{H}^n(L(A)) = [\![L]\!]\, \mathcal{L}^n(A)$$

Proof: Well known. It is omitted

<u>Example</u> $L : \mathbb{R} \longrightarrow \mathbb{R}^2$ linear, $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$, $A = [0,1]$

Then

$$\mathcal{H}^1(L(A)) = \sqrt{l_1^2 + l_2^2}$$



Def (Jacobian) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $n, m \in \mathbb{N}$, be Lipschitz.
The Jacobian of f is

$$Jf(x) = [\![Df(x)]\!] , \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n .$$

Theorem (Area Formula) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be Lipschitz, $n \leq m$, and let $A \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then

$$\int_A Jf(x)\, dx = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n(y) .$$

In particular, when f is 1-1 we have

$$(*) \qquad \int_A Jf(x)\, dx = \mathcal{H}^n(f(A)) .$$

Comment. $y \longmapsto \mathcal{H}^0(A \cap f^{-1}(\{y\})) = \# \{ x \in A : f(x) = y \}$ is the multiplicity function

Example 1 When $n = 1$, let $\gamma : [0,1] \longrightarrow \mathbb{R}^m$ be Lip. and 1-1.
Formula $(*)$ reads:

$$\int_{[0,1]} |\dot{\gamma}(t)|\, dt = \mathcal{H}^1(\gamma([0,1])) .$$

Example 2 Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lip. and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be the graph

$$F(x) = (x, f(x)) .$$

The differential of F is

$$DF(x) = \begin{pmatrix} & I_n & \\ \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

By the Binet-Cauchy formula:

$$JF(x)^2 = [DF(x)]^2 = 1 + |Df(x)|^2 .$$

Let $A \subset \mathbb{R}^n$ be meas. (and with $\mathcal{L}^n(A) < \infty$).
By the Area Formula we have

$$\int_A \sqrt{1 + |Df(x)|^2}\, dx = \mathcal{H}^n\left(g_f|_A\right).$$

<u>Example 3</u> Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be Lip. and 1-1 on $A \subset \mathbb{R}^n$
coordinates $(u,v) \in \mathbb{R}^2$. Then

$$\mathcal{H}^2(f(A)) = \int_A |f_u \wedge f_v| \, du dv .$$

<u>Minimal surface Equation</u>

Let $A \subset \mathbb{R}^n$ be a bounded open set and let $g : \partial A \to \mathbb{R}$ be a continuous function. Consider the set

$$\mathcal{A} = \{ f \in C(\bar{A}) : f \in \mathrm{Lip}_{loc}(A) \}.$$

Let $J : \mathcal{A} \to [0, \infty]$ be the Area functional:

$$J(f) = \int_A \sqrt{1 + |Df|^2}\, dx$$

Assume that the following Plateau problem has a solution

$$\min \{ J(f) \; ; \; f \in \mathcal{A} \} \in \mathbb{R}.$$

Let $f \in \mathcal{A}$ be a minimum. The function

$$\phi(\varepsilon) = J(f+\varepsilon\varphi), \qquad \varphi \in C^1_c(A) \text{ fixed},$$

has a minimum at $\varepsilon = 0$. Compute

$$\varphi'(\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_A \sqrt{1+|Df|^2+2\varepsilon Df\cdot D\varphi + \varepsilon^2|D\varphi|^2}\, dx \Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\int_A \frac{2\, Df\cdot D\varphi}{\sqrt{1+|Df|^2}}\, dx$$

We deduce that

$$\int_A \frac{Df\cdot D\varphi}{\sqrt{1+|Df|^2}}\, dx = 0 \qquad \forall \varphi \in C^1_c(A).$$

Assume that $f \in C^2(A)$ and integrate by parts:

$$\int_A \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) \varphi(x)\, dx = 0 \qquad \forall \varphi \in C^1_c(A).$$

We conclude that

$$(*) \qquad \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = 0 \qquad \text{in } A.$$

This is the <u>minimal surface equation</u>.

This is a second order elliptic, nonlinear equation.

Equation (*) has the following geometric meaning.

"The mean curvature of $gr(f)$ is identically zero".

<u>Definition</u> A C^2 hypersurface of $\mathbb{R}^n$, $n \geq 3$, with zero mean curvature is called <u>minimal surface</u>.

<u>Proof of the Area Formula</u>

<u>Lemma 2</u> $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Lip., $m \geq n$, $A \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable.

Then:

(i) $f(A) \subset \mathbb{R}^m$ is $\mathcal{H}^n$-measurable

(ii) $y \longmapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ is $\mathcal{H}^n$-meas. on $\mathbb{R}^m$ and

(iii) $\displaystyle\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n(y) \leq (\mathrm{Lip}(f))^n \mathcal{L}^n(A)$.

Proof. W.L.G: A is bounded ($\Rightarrow \mathcal{L}^n(A) < \infty$).

(i) $\forall i \in \mathbb{N}$ there is $K_i \subset A$ compact such that

$$\mathcal{L}^n(A) - \mathcal{L}^n(K_i) = \mathcal{L}^n(A \setminus K_i) < \frac{1}{i}.$$

Now: $f(K) \subset \mathbb{R}^m$ compact $\Rightarrow$ $f(K)$ Borel in $\mathbb{R}^m$

$\Rightarrow$ $\mathcal{H}^n$-measur. in $\mathbb{R}^m$

Then $f\left(\bigcup_{i=1}^{\infty} K_i\right) = \bigcup_{i=1}^{\infty} f(K_i)$ is $\mathcal{H}^n$-measur.

Moreover

$$\begin{aligned}\mathcal{H}^n\left(f(A) \setminus f\left(\bigcup_{i=1}^{\infty} K_i\right)\right) &\leq \mathcal{H}^n\left(f\left(A \setminus \bigcup_{i=1}^{\infty} K_i\right)\right) \\ &\leq (\mathrm{Lip}(f))^n \underbrace{\mathcal{L}^n\left(A \setminus \bigcup_{i=1}^{\infty} K_i\right)}_{=0} \\ &\leq 0.\end{aligned}$$