This is a second order elliptic, non linear equation.

Equation (*) has the following geometric meaning.

"The mean curvature of $gr(f)$ is identically zero".

<u>Definition</u> A C^2 hypersurface of $\mathbb{R}^n$, $n \geq 3$, with zero mean curvature is called <u>minimal surface</u>.

<u>Proof of the Area Formula</u>

<u>Lemma 2</u> $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Lip., $m \geq n$, $A \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable.

Then :

(i) $f(A) \subset \mathbb{R}^m$ is H^n-measurable

(ii) $y \longmapsto H^0(A \cap f^{-1}\{y\})$ is H^n-meas. on $\mathbb{R}^m$ and

(iii) $\int_{\mathbb{R}^m} H^0(A \cap f^{-1}\{y\})\, dH^n(y) \leq (\mathrm{Lip}(f))^n\, \mathcal{L}^n(A)$.

Proof. W.L.G : A is bounded ($\Rightarrow \mathcal{L}^n(A) < \infty$).

(i) $\forall i \in \mathbb{N}$ there is $K_i \subset A$ compact such that

$$\mathcal{L}^n(A) - \mathcal{L}^n(K_i) = \mathcal{L}^n(A \setminus K_i) < \frac{1}{i}.$$

Now : $f(K) \subset \mathbb{R}^m$ compact $\Rightarrow$ $f(K)$ Borel in $\mathbb{R}^m$

$\Rightarrow$ H^n-measur. in $\mathbb{R}^m$

Then $f\left(\bigcup_{i=1}^{\infty} K_i\right) = \bigcup_{i=1}^{\infty} f(K_i)$ is H^n-measur.

Moreover

$$\begin{aligned} H^n\left(f(A) \setminus f\left(\bigcup_{i=1}^{\infty} K_i\right)\right) &\leq H^n\left(f\left(A \setminus \bigcup_{i=1}^{\infty} K_i\right)\right) \\ &\leq (\mathrm{Lip}(f))^n\, \underbrace{\mathcal{L}^n\left(A \setminus \bigcup_{i=1}^{\infty} K_i\right)}_{=\,0} \\ &\leq 0. \end{aligned}$$

Then $\quad f(A) \setminus f(\bigcup_{i=1}^{\infty} K_i)$ is H^n- meas. This implies (i).

(H^n is a complete measure)

(ii) Let $\quad \mathcal{B}_k = \{ Q \subset \mathbb{R}^n \ ; \ Q = \prod_{i=1}^{n} (a_i, b_i] \ , \ a_i = \frac{c_i}{k} \ , \ b_i = \frac{c_i+1}{k} \ , \ c_i \in \mathbb{Z} \ , \ i=1,\dots,n \}$

where $k \in \mathbb{N}$ is fixed.

Clearly $\quad \mathbb{R}^n = \bigcup_{Q \in \mathcal{B}_k} Q$.

Define

$$g_k(y) = \sum_{Q \in \mathcal{B}_k} \chi_{f(A \cap Q)}(y) \quad , \quad y \in \mathbb{R}^m$$

$$= \# \{ Q \in \mathcal{B}_k \ ; \ y \in f(A \cap Q) \}.$$

Then $g_k \uparrow$ and g_k is H^n- meas. on $\mathbb{R}^m$.

Moreover

$$\lim_{k \to \infty} g_k(y) = \# \{ x \in A \ ; \ f(x) = y \}$$

$$= H^0 (A \cap f^{-1}\{y\})$$

$\uparrow_y$ is thus H^n- measurable

(iii) By Monotone convergence

$$\int_{\mathbb{R}^m} H^0 (A \cap f^{-1}\{y\}) \, dH^n(y) = \lim_{k \to \infty} \int_{\mathbb{R}^m} g_k(y) \, dH^n(y) =$$

$$= \lim_{k \to \infty} \sum_{Q \in \mathcal{B}_k} H^n(f(A \cap Q)) \leq \limsup_{k \to \infty} \sum_{Q \in \mathcal{B}_k} (\mathrm{Lip}(f))^n \mathcal{L}^n(A \cap Q) =$$

$$= (\mathrm{Lip}(f))^n \ \mathcal{L}^n(A). \qquad \square$$

__Lemma 3__ Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, be Lipschitz.
Let $t > 1$ and let
$$B = \{ x \in \mathbb{R}^n ; f \text{ is diff. at } x \text{ and } Jf(x) > 0 \}$$
There is a sequence of Borel sets $E_k \subset B$ such that:

(i) $\bigcup_{k=1}^{\infty} E_k = B$

(ii) $f: E_k \to \mathbb{R}^m$ is 1-1 for all $k \in \mathbb{N}$.

(iii) For each $k \in \mathbb{N}$ there is $T_k : \mathbb{R}^n \to \mathbb{R}^n$ linear, symmetric, invertible such that:

- $\mathrm{Lip}(f|_{E_k} \circ T_k^{-1}) \leq t$;
- $\mathrm{Lip}(T_k \circ f|_{E_k}^{-1}) \leq t$;
- $\frac{1}{t^n} |\det T_k| \leq Jf(x) \leq t^n |\det T_k| \quad \forall x \in E_k$.

Proof. Fix $\varepsilon > 0$ s.t. $\frac{1}{t} + \varepsilon < 1 < t - \varepsilon$.

Let
- $C \subset B$ dense and countable
- $\mathcal{S} \subset \{ S: \mathbb{R}^n \to \mathbb{R}^n \text{ lin. inv. sym.} \}$ dense and countable

__Def__ For $c \in C$, $T \in \mathcal{S}$ and $i \in \mathbb{N}$ let
$$E(c, T, i) = \{ b \in B(c, \tfrac{1}{i}) \cap B ; \ (1) \text{ and } (2) \text{ hold} \}$$

(1) $\left(\frac{1}{t} + \varepsilon\right) |Tv| \leq |Df(b) v| \leq (t - \varepsilon) |Tv| \qquad \forall v \in \mathbb{R}^n$

(2) $|f(a) - f(b) - Df(b) \cdot (b-a)| \leq \varepsilon |T(b-a)| \qquad \forall a \in B(b, \tfrac{2}{i})$

Notice : • The family of all $E(c,T,i)$ is at most countable.

• Each $E(c,T,i)$ is Borel because f and Df are Borel

(1) and (2) imply

$$(3) \quad \frac{1}{t}\,|T(a-b)| \leq |f(a)-f(b)| \leq t\,|T(a-b)| \qquad \forall b \in E(c,T,i) \quad \forall a \in B\left(b,\tfrac{2}{i}\right)$$

<u>Claim</u> : For all $b \in E(c,T,i)$ we have

$$(4) \quad \left(\frac{1}{t}+\varepsilon\right)^n |\det T| \leq [\![Df(b)]\!] = Jf(b) \leq (t-\varepsilon)^n |\det(T)|$$

Proof. We have $Df(b) = \underset{\text{orthog.}}{O} \circ \underset{\text{symmetric}}{S}$, polar decomposition

And thus

$$Jf(b) = |\det S|.$$

By (1) :

$$\left(\frac{1}{t}+\varepsilon\right)|Tv| \leq \underbrace{|O\circ S(v)|}_{=|Sv|} \leq (t-\varepsilon)\,|Tv| \qquad \forall v \in \mathbb{R}^n$$

$$\Downarrow$$

$$\left(\frac{1}{t}+\varepsilon\right)|v| \leq |S\circ T^{-1} v| \leq (t-\varepsilon)\,|v|$$

$\Downarrow$ similar (...)

$$\Downarrow$$

$$S\circ T^{-1}(B(0,1)) \subset B(0,t-\varepsilon)$$

$$\Downarrow$$

$$\mathcal{L}^n(S\circ T^{-1}(B(0,1))) \leq \mathcal{L}^n(B(0,t-\varepsilon))$$

$$\Updownarrow$$

$$|\det S| \leq (t-\varepsilon)^n |\det T| \iff |\det(S\circ T^{-1})| \leq (t-\varepsilon)^n$$

This proves the right inequality in (4),
The proof of the ineq. on the left is similar.

We relabel: $E(c, T, i) \rightsquigarrow E_k$ with $k \in \mathbb{N}$.
If $E_k = E(c, T, i)$ we let $T_k = T$.

(*) We prove: $B = \bigcup_{k=1}^{\infty} E_k$.

Let $b \in B$. We have $Df(b) = O \circ S$.
We choose $T \in \mathcal{S}$ very close to S in the following sense

$$\mathrm{Lip}(S \circ T^{-1}) \leq t - \varepsilon \qquad \left(\Longrightarrow |Df(b)v| \leq (t-\varepsilon)|Tv| \right)$$
$$\mathrm{Lip}(T^{-1} \circ S) \leq \left(\frac{1}{t} + \varepsilon\right)^{-1}, \qquad \left(\Longrightarrow |Df(b)v| \geq \left(\frac{1}{t} + \varepsilon\right)|Tv| \right)$$

Because $\mathrm{Lip}(S \circ T^{-1}) \leq \| S \circ T^{-1} \|$, by density there is such a $T \in \mathcal{S}$.

Next we select $i \in \mathbb{N}$ such that

$$|f(a) - f(b) - Df(b)(a-b)| \leq \frac{\varepsilon}{\mathrm{Lip}(T^{-1})} |b-a| \underset{\substack{\uparrow \\ \text{Easy.}}}{\leq} \varepsilon |T(b-a)|$$
for all $a \in B(b, \frac{2}{i})$.

By differentiability of f at b, there is such an $i \in \mathbb{N}$.

Finally, we select $c \in C$ such that $|b - c| < \frac{1}{i}$.
By density of C in B, there is such a point $c \in C$.
Conclusion: $b \in E(c, T, i)$. This ends the proof of the claim (*).

We prove (ii). Let $E_K = E(c, T, i)$ with $T = T_K$.

From (3):

$$(5) \qquad \frac{1}{t} \, |T_K(a-b)| \le |f(a) - f(b)| \le t \, |T_K(a-b)| \qquad \begin{array}{l} \forall b \in E_K \\ \forall a \in B_{2/i}(b) \end{array}$$

Notice that

$$a \in E_K \Rightarrow |a-c| < \frac{1}{i} \Rightarrow |a-b| < \frac{2}{i} \Rightarrow a \in B_{2/i}(b)$$

Thus

$$(6) \qquad \frac{1}{t} \, |T_K(a-b)| \le |f(a) - f(b)| \le t \, |T_K(a-b)| \qquad \forall a, b \in E_K,$$

Conclusion: $f|_{E_K}$ is 1-1.

We prove (iii). From (6) it follows

$$Lip(f_{|E_K} \circ T_K^{-1}) \le t,$$

$$Lip(T_K \circ f_{|E_K}^{-1}) \le t.$$

From (4)

$$\left(\frac{1}{t}\right)^n |\det T_K| \le Jf \le t^n |\det(T_K)| \quad \text{on } E_K.$$

□

Proof of the area formula.

By Rademacher theorem and Lemma 2 part (iii) we can assume that f is differentiable on A.

We can also assume $\mathcal{L}^n(A) < \infty$.

Case 1: $A \subset \{x \in \mathbb{R}^n ; Jf(x) > 0\}$

Case 2: $A \subset \{x \in \mathbb{R}^n ; Jf(x) = 0\}$

We start with the Case 1. We have

$$A \subset B = \{ x \in \mathbb{R}^n : \; Jf(x) > 0 \} .$$

Fix $t > 1$ and let $\{E_j\}_{j \in \mathbb{N}}$ be the sets of Lemma 3.
Fix $k \in \mathbb{N}$ and let

$$\mathcal{B}_k = \{ Q \; ; \; Q \text{ cubes of side length } 1/k \text{ of Lemma 2, proof} \}$$

$$= \{ Q_i : \; i \in \mathbb{N} \} .$$

Let

$$F_j^i = A \cap E_j \cap Q_i \; , \quad i \in \mathbb{N}, \; j \in \mathbb{N} .$$

The sets F_j^i are disjoint.
The same argument as in the proof of Lemma 2 shows that

$$\lim_{k \to \infty} \sum_{i,j=1}^{\infty} \mathcal{H}^n(f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y) ,$$

Now consider

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(f \circ T_j^{-1} \circ T_j (F_j^i)) \overset{\mathrm{Lip}(f_{|E_j} \circ T_j^{-1}) \le t}{\le}$$

$$\le t^n \mathcal{H}^n(T_j(F_j^i)) = t^n |\det(T_j)| \mathcal{L}^n(F_j^i) .$$

Similarly,

$$\mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j \circ f_{|E_j}^{-1} \circ f_{|E_j}(F_j^i)) \overset{\mathrm{Lip}(T_j \circ f_{|E_j}^{-1}) \le t}{\le}$$

$$\le t^n \mathcal{H}^n(f(F_j^i)) .$$

We bring the estimates together:

$$\begin{aligned}
H^n(f(F_j^i)) &\leq t^n |\det(T_j^i)|\, \mathcal{L}^n(F_j^i) \leq \quad [\, |\det(T_j^i)| \leq t^n Jf \text{ on } E_j^i \,] \\
&\leq t^{2n} \int_{F_j^i} Jf(x)\,dx \leq \\
&\leq t^{3n} \int_{F_j^i} |\det(T_j^i)|\,dx = t^{3n} |\det(T_j^i)|\, \mathcal{L}^n(F_j^i) = \\
&= t^{3n} \mathcal{L}^n(T_j(F_j^i)) \\
&\leq t^{4n} H^n(f(F_j^i)) .
\end{aligned}$$

Sum up in i and j:

$$\sum_{i,j=1}^{\infty} H^n(f(F_j^i)) \leq t^{2n} \int_{\underset{\bigcup_{i,j=1}^{\infty} F_j^i}{\overset{A}{\shortparallel}}} Jf(x)\,dx \leq t^{4n} \sum_{i,j=1}^{\infty} H^n(f(F_j^i)) .$$

Let first $k \to \infty$ and then $t \downarrow 1$ to obtain

$$\int_{\mathbb{R}^m} H^0(A \cap f^{-1}\{y\})\, dH^n(y) = \int_A Jf(x)\,dx .$$

Case 2. Now let $A \subset \{x \in \mathbb{R}^n \,;\, Jf(x) = 0\}$.
Fix $\varepsilon > 0$ and let $g: \mathbb{R}^n \to \mathbb{R}^{m+n}$ be

$$g(x) = (f(x), \varepsilon x) .$$

Also, let $p: \mathbb{R}^{m+n} \to \mathbb{R}^m$ be $p(y,z) = y$. Then $f = p \circ g$.

We have

$$Dg(x) = \begin{array}{c} \left.\begin{pmatrix} Df(x) \\ \\ \varepsilon I_n \end{pmatrix}\right\} n+m \\ \underbrace{\hphantom{Df(x)}}_{n} \end{array} .$$

Then

$$[Dg(x)]^2 = \varepsilon^{2n} + \text{positive terms} \geq \varepsilon^{2n} > 0$$

$$[Dg]^2 = \underbrace{[Jf(x)]^2}_{\substack{= \\ 0 \\ \text{on } A}} + \text{terms cont. } \varepsilon^2 \leq \underbrace{C^2}_{\text{dep. on Lip} f} \varepsilon^2$$

Then we have $\quad p$ is 1-Lip

$$\mathcal{H}^n(f(A)) \leq \mathcal{H}^n(g(A)) = \int_A Jg(x)\,dx \leq C\varepsilon\,\mathcal{L}^n(A).$$

Letting $\varepsilon \downarrow 0$ we find $\mathcal{H}^n(f(A)) = 0$ and thus

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,d\mathcal{H}^n(y) = 0$$

because $\quad \mathcal{H}^0(A \cap f^{-1}\{y\}) > 0 \implies y \in f(A)$.

We also have

$$0 = \int_A Jf(x)\,dx .$$

The area formula trivially holds on A.

$\square$