COAREA FORMULA

Theorem (Coarea Formula) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m \geq 1$, be Lipschitz. Then for any $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$ we have

$$\int_A Jf(x)\, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, dy .$$

For the proof see [EG] pp. 104-116.

Example When $m=1$ we have $Jf(x) = |Df(x)|$.

The coarea formula reads

$$\int_{\mathbb{R}^n} |Df(x)|\, dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{x \in \mathbb{R}^n ; f(x) = t\})\, dt$$

Theorem (Change of variable) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, be Lipschitz and let $g \in L^1(\mathbb{R}^n)$. Then:

(i) $g|_{f^{-1}\{y\}}$ is $\mathcal{H}^{n-m}$-integrable for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$;

(ii)
$$\int_{\mathbb{R}^n} g(x)\, Jf(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}\{y\}} g(x)\, d\mathcal{H}^{n-m}(x) \right) dy .$$

Proof (Sketch) Assume $g \geq 0$. There are $\mathcal{L}^n$-measurable sets $A_i \subset \mathbb{R}^n$, $i \in \mathbb{N}$, such that

$$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i} .$$

Then

$$\int_{\mathbb{R}^n} g(x)\, Jf(x)\, dx = \sum_{i=1}^{\infty} \frac{1}{i} \int_{A_i} Jf(x)\, dx =$$

$$= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} H^{n-m}(A_i \cap f^{-1}\{y\})\, dy$$

$$= \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \int_{f^{-1}\{y\}} \chi_{A_i}(x)\, dH^{n-m}\, dy$$

$$= \int_{\mathbb{R}^m} \left(\int_{f^{-1}\{y\}} g(x)\, dH^{n-m}(x) \right) dy .$$

$\square$

Example When $m=1$ and $f(x) = |x|$ we have $Df(x) = \dfrac{x}{|x|}$ and $Jf(x) = |Df(x)| \equiv 1$.

The change of variable formula is

$$\int_{\mathbb{R}^n} g(x)\, dx = \int_0^{\infty} \left(\int_{|x|=t} g(x)\, dH^{n-1} \right) dt .$$

SHARP SOBOLEV INEQUALITY

If $A \subset \mathbb{R}^n$ is a bounded set with "regular" boundary, the isoperimetric inequality reads

$$\mathcal{L}^n(A)^{\frac{n-1}{n}} \leq C_n \, \mathcal{H}^{n-1}(\partial A)$$

where the sharp constant is determined by

$$\alpha(n)^{\frac{n-1}{n}} = C_n \, n \, \alpha(n)$$

$$\Updownarrow$$

$$C_n = \frac{1}{n} \cdot \frac{1}{\alpha(n)^{1/n}} .$$

Theorem For any function $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 2$, we have the sharp inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq C_n \int_{\mathbb{R}^n} |Df(x)| \, dx .$$

Proof. Assume w.l.g. that $f \geq 0$. By the coarea formula we have

$$\int_{\mathbb{R}^n} |Df(x)| \, dx = \int_0^\infty \mathcal{H}^{n-1}(\{x \in \mathbb{R}^n : f(x) = t\}) \, dt .$$

Let $A_t = \{x \in \mathbb{R}^n : f(x) > t\}$. By Sard's Lemma

$$\partial A_t = \{f = t\} \quad \text{is a } C^\infty\text{-hypersurface}$$

for $\mathcal{L}^1$-a.e. $t > 0$.

Fix a parameter $t \geq 0$ and let $f_t : \mathbb{R}^n \longrightarrow [0, \infty)$ be the function:

$$f_t(x) = \begin{cases} t & \text{if } f(x) > t, \\ f(x) & \text{if } f(x) \leq t. \end{cases}$$

f_t is the truncation of f at level t.

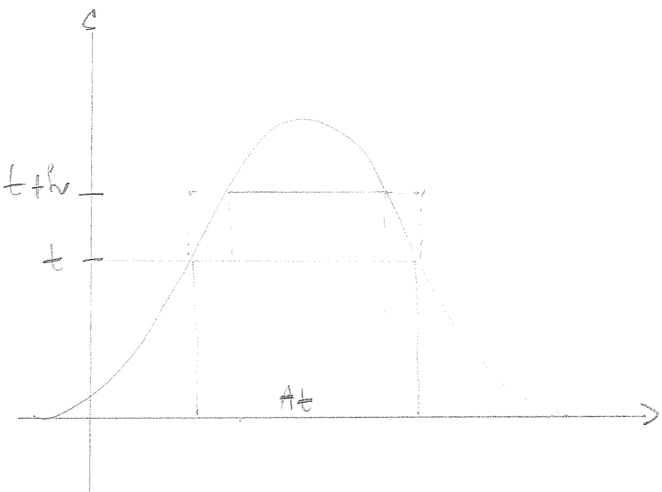
Also let

$$u(t) = \left(\int_{\mathbb{R}^n} |f_t|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}}.$$

We claim that $u : [0,\infty) \longrightarrow \mathbb{R}$ is absolutely continuous. (it is Lipschitz).
But we do not prove this claim, here. Exercise.

Notice that

$$f_{t+h}(x) \leq f_t(x) + h\, \chi_{A_t}(x), \quad x \in \mathbb{R}^n,$$



It follows that

$$u(t+h) = \left(\int_{\mathbb{R}^n} | f_{t+h}(x) |^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq$$

$$\leq \left(\int_{\mathbb{R}^n} \left| f_t(x) + h\chi_{A_t}(x) \right|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \quad \|\cdot\|_{\frac{n}{n-1}} \text{ is a norm}$$

$$\leq \left(\int_{\mathbb{R}^n} |f_t(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} + \left(\int_{\mathbb{R}^n} | h \chi_{A_t}(x) |^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} =$$

$$= u(t) + h \left(\mathcal{L}^n (A_t) \right)^{\frac{n-1}{n}}.$$

Then, for $\mathcal{L}^1$-a.e. $t \geq 0$ we have

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \leq \lim_{h \to 0} \mathcal{L}^n(A_t)^{\frac{n-1}{n}} = \mathcal{L}^n(A_t)^{\frac{n-1}{n}}.$$

We deduce that

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} = u(\infty) - u(0) = \int_0^\infty u'(t)\, dx \leq$$

$$\leq \int_0^\infty \mathcal{L}^n(A_t)^{\frac{n-1}{n}} dt \leq c_n \int_0^\infty H^{n-1}(\partial A_t)\, dt \overset{\text{coarea}}{=}$$

$$= c_n \int_{\mathbb{R}^n} |Df(x)|\, dx.$$

The constant $c_n = \frac{1}{n\, \alpha(n)^{1/n}}$ is sharp.

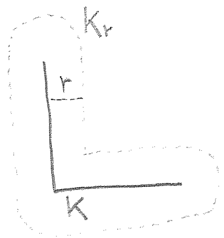
This proof is due to Federer and Fleming, Normal and Integral Currents, 1959 Ann. Math, p. 487.

$\square$

A proof of the isoperimetric inequality.

Def. The r-tubular neighborhood, $r>0$, of a compact set $K \subset \mathbb{R}^n$ is

$$K_r = \{ x \in \mathbb{R}^n : \operatorname{dist}(x; K) < r \}.$$



Definition (Minkowski content) Let $1 \leq k \leq n-1$. The k-dimensional lower and upper Minkowski contents of a compact set $K \subset \mathbb{R}^n$ are

$$\mathcal{M}_k^-(K) = \liminf_{r \downarrow 0} \frac{\mathcal{L}^n(K_r)}{\alpha(n-k)\, r^{n-k}},$$

$$\mathcal{M}_k^+(K) = \limsup_{r \downarrow 0} \frac{\mathcal{L}^n(K_r)}{\alpha(n-k)\, r^{n-k}}.$$

If $\mathcal{M}_k^-(K) = \mathcal{M}_k^+(K)$, we call the common value $\mathcal{M}_k(K)$ the k-Minkowski content of K.

Theorem Let $E \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\mathcal{M}_{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial E).$$

We do not prove this theorem, here, see [AFP] p. 108.

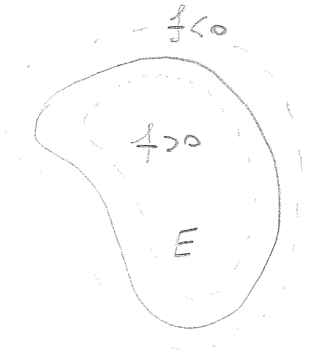
Even though we did not give a formal proof of this, we also know that

$$H^{n-1}(\partial E) = \|\partial E\|(\mathbb{R}^n).$$

We suggest an euristic proof when ∂E is a C^2-hypersurface.
In this case, the function

$$f(x) = \begin{cases} \operatorname{dist}(x, \partial E) & \text{if } x \in E \\ -\operatorname{dist}(x, \partial E) & \text{if } x \in \mathbb{R}^n \setminus E \end{cases}$$

is of class C^2 in $(\partial E)_r$, for $r>0$ small enough.



The proof of this fact is not completely elementary.
Moreover, we have

$$|Df(x)| = 1 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n$$

By the coarea formula

$$\mathcal{L}^n((\partial E)_r) = \mathcal{L}^n(\{x \in \mathbb{R}^n : -r < f(x) < r\}) =$$

$$= \int_{\{-r<f<r\}} |Df(x)|\, dx = \int_{-r}^{r} H^{n-1}(\{x \in \mathbb{R}^n : f(x) = t\})\, dt$$

It follows that

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^n((\partial E)_r)}{2r} = \lim_{r\downarrow 0} \frac{1}{2r} \int_{-r}^{r} H^{n-1}(\{x\in\mathbb{R}^n : f(x)=t\})\,dt$$

$$= H^{n-1}(\{x\in\mathbb{R}^n ; f(x)=0\})$$

$$= H^{n-1}(\partial E),$$

because $t \longmapsto H^{n-1}(\{x\in\mathbb{R}^n : f(x)=t\})$ is continuous.

Exercise.

<u>Theorem</u> Let $B\subset\mathbb{R}^n$ be the unit ball and let $E\subset\mathbb{R}^n$ be a bounded set with Lipschitz boundary such that $\mathcal{L}^n(E)=\mathcal{L}^n(B)$. Then

$$H^{n-1}(\partial B) \leq H^{n-1}(\partial E).$$

Proof. Let

$$E_r = \{x\in\mathbb{R}^n : \operatorname{dist}(x,E)<r\} = \underbrace{E + B_r(0)}_{\text{Minkowski sum}}, \quad B_r(0) = B_r.$$

We use the Brunn-Minkowski inequality

$$\mathcal{L}^n(E_r)^{\frac{1}{n}} = \mathcal{L}^n(E+B_r)^{\frac{1}{n}} \geq \mathcal{L}^n(E)^{\frac{1}{n}} + \mathcal{L}^n(B_r)^{\frac{1}{n}}$$

Thus we have

$$\mathcal{L}^n(E_r\setminus E) = \mathcal{L}^n(E_r)-\mathcal{L}^n(E) \geq \left(\mathcal{L}^n(E)^{\frac{1}{n}} + \mathcal{L}^n(B_r)^{\frac{1}{n}}\right)^n - \mathcal{L}^n(E)$$

$$= \left(\mathcal{L}^n(B)^{\frac{1}{n}} + r\,\mathcal{L}^n(B)^{\frac{1}{n}}\right)^n - \mathcal{L}^n(B)$$

$$= \mathcal{L}^n(B)\left\{(1+r)^n - 1\right\}$$

We deduce that

$$\mathcal{H}^{n-1}(\partial E) = \lim_{r \downarrow 0} \frac{\mathcal{L}^n(E_r \setminus E)}{r} \geq \lim_{r \downarrow 0} \mathcal{L}^n(B) \frac{(1+r)^n - 1}{r} =$$

$$= \alpha(n) \cdot n = \mathcal{H}^{n-1}(\partial B).$$

PROOF OF THE BRUNN-MINKOWSKI INEQUALITY

Theorem Let $A, B \subset \mathbb{R}^n$ be nonempty. Then,

$$\mathcal{L}^n(A+B)^{\frac{1}{n}} \geq \mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}}$$

where $A+B = \{x+y \in \mathbb{R}^n;\ x \in A,\ y \in B\}$ and $\mathcal{L}^n$ is the Lebesgue outer measure in $\mathbb{R}^n$.

Proof. Let

$$P = \{P_1 \times \dots \times P_n \subset \mathbb{R}^n;\ P_i \subset \mathbb{R} \text{ interval},\ i=1,\dots,n\}.$$

First case: $A, B \in P$;

$$A = P_1 \times \dots \times P_n,$$
$$B = Q_1 \times \dots \times Q_n.$$

Then we have

$$A+B = (P_1 + Q_1) \times \dots \times (P_n + Q_n)$$

where $P_i + Q_i$ is an interval of length

$$\mathcal{L}^1(P_i + Q_i) = \mathcal{L}^1(P_i) + \mathcal{L}^1(Q_i).$$