TEOREMA DI RAPPRESENTAZIONE DI RIESZ

Notazione Con $\mathcal{C}_c(\mathbb{R}^n;\mathbb{R}^m)$ indichiamo l'insieme delle funzioni continue $f : \mathbb{R}^n \to \mathbb{R}^m$ tali che

$$spt(f) := \overline{\{x \in \mathbb{R}^n \; ; \; f(x) \neq 0\}}$$

è compatto. Qui $n, m \geq 1$

TEOREMA 1 Sia $T : \mathcal{C}_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ una trasformazione lineare tale che per ogni compatto $K \subset \mathbb{R}^n$ si abbia

$$\sup \{ T(f) \; : \; f \in \mathcal{C}_c(\mathbb{R}^n;\mathbb{R}^m) \, , \, spt(f) \subset K \text{ e } \|f\|_\infty \leq 1 \} < \infty .$$

Allora esiste una misura di Radon μ su $\mathbb{R}^n$ ed esiste una funzione μ-misurabile $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ tale che

(i) $$T(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu \qquad \forall f \in \mathcal{C}_c(\mathbb{R}^n;\mathbb{R}^m) \; ;$$

(ii) $$|\sigma(x)| = 1 \quad \text{per } \mu\text{-q.o. } x \in \mathbb{R}^n .$$

Prova. Schema della dimostrazione:

(1) Costruzione di μ

(2) μ di Borel regolare finita sui compatti

(3) Esistenza di σ. Parte complessa. Useremo

$$L^1(\mathbb{R}^n;\mu)^* = L^\infty(\mathbb{R}^n,\mu)$$

(4) $|\sigma| = 1$ μ-q.o.

BV functions

$A \subset \mathbb{R}^n$ open set

DEF We say that $f \in BV(A)$ is a function of bounded variation in A if $f \in L^1(A)$ and

$$\|Df\|(A) := \sup\left\{ \int_A f \operatorname{div}\varphi \, dx \; : \; \varphi \in C^1_c(A;\mathbb{R}^n) \text{ and } \|\varphi\|_\infty \le 1 \right\} < \infty$$

DEF We say that $f \in BV_{loc}(A)$ (locally bounded variation) if $f \in L^1_{loc}(A)$ and for any open set $B \subset\subset A$ ($\overline{B}$ compact $\subset A$): $\|Df\|(B) < \infty$.

THEOREM 2 (Structure of BV_{loc} functions). Let $f \in BV_{loc}(A)$, $A \subset \mathbb{R}^n$ open. Then there exist μ Radon measure on A and $\sigma : A \to \mathbb{R}^n$ μ-measurable such that:

(i) $|\sigma(x)| = 1$ for μ-a.e. $x \in A$;

(ii) $\displaystyle\int_A f \operatorname{div}\varphi \, dx = -\int_A \varphi \cdot \sigma \, d\mu$ (integr. by parts formula).

for all $\varphi \in C^1_c(A;\mathbb{R}^n)$

Notation

$$\operatorname{div}\varphi = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial x_i} \quad \text{divergence of } \varphi = (\varphi_1, \dots, \varphi_n)$$

Comment With $\varphi = (0, \dots, 0, \underset{i}{\psi}, 0 \dots 0)$:

$$\int_A f \frac{\partial \psi}{\partial x_i} \, dx = -\int_A \psi \, \sigma_i \, d\mu \qquad \forall \psi \in C^1_c(A),$$

that is

$$\frac{\partial f}{\partial x_i} = \sigma_i \mu \quad \text{in distr. (or weak) sense.}$$

PROOF Let $T : C^1_c(A;\mathbb{R}^n) \longrightarrow \mathbb{R}$ be the linear functional

$$T(\varphi) = -\int_A f \operatorname{div}\varphi \, dx \, , \qquad \varphi \in C^1_c(A;\mathbb{R}^n).$$

Since $f \in BV_{loc}(A)$, for any open $B \subset\subset A$ we have

$$\|Df\|(B) = \sup\left\{ \int_B f \operatorname{div}\varphi \; : \; \varphi \in C^1_c(B;\mathbb{R}^n), \; \|\varphi\|_\infty \leq 1 \right\} < \infty$$

It follows that

$$(*) \qquad |T(\varphi)| \leq \|Df\|(B) \cdot \|\varphi\|_\infty \qquad \forall \varphi \in C^1_c(B;\mathbb{R}^n).$$

Let $K \subset A$ be compact and let open $B \subset\subset A$ be such that $K \subset B$. There exists for any $\varphi \in C_c(A;\mathbb{R}^n)$ with $spt(\varphi) \subset K$ $(\varphi_k)_{k\in\mathbb{N}}$ with $\varphi_k \in C^1_c(B;\mathbb{R}^n)$ $\forall k$ and

$$\lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^\infty(B)} = 0 \qquad \text{(unif. conv.)}$$

From $(*)$:

$(\varphi_k)_k$ Cauchy in $L^\infty(B)$ $\Longrightarrow$ $(T(\varphi_k))_{k\in\mathbb{N}}$ Cauchy seq. in $\mathbb{R}$

The limit

$$T(\varphi) := \lim_{k\to\infty} T(\varphi_k)$$

does exist and is independent of $(\varphi_k)_{k\in\mathbb{N}}$.

Now we have $T : C_c(A;\mathbb{R}^n) \longrightarrow \mathbb{R}$ linear functional with

$$\sup\left\{ T(\varphi) \; : \; \varphi \in C_c(A;\mathbb{R}^n), \; spt(\varphi) \subset K, \; \|\varphi\|_\infty \leq 1 \right\} < \infty$$

for any compact set $K \subset A$. This follows from $(*)$.

The claim follows from Riesz Theorem.

□

PROOF OF RIESZ THEOREM

(1) Per $A \subset \mathbb{R}^n$ aperto definiamo

$$\mu(A) = \sup\left\{ T(f) : f \in C_c(\mathbb{R}^n;\mathbb{R}^m),\ spt(f) \subset A,\ \|f\|_\infty \leq 1 \right\}$$

Per $E \subset \mathbb{R}^n$ insieme definiamo

$$\mu(E) = \inf\{\mu(A) : E \subset A \text{ aperto}\}.$$

Affermo: μ misura esterna.
Siano $A_K \subset \mathbb{R}^n$ aperti, $K \in \mathbb{N}$, e sia $spt(f) \subset \bigcup_{K=1}^{\infty} A_K$, $|f| \leq 1$.
compatto

Allora $\exists N \in \mathbb{N}$ t.c.

$$spt(f) \subset \bigcup_{K=1}^{N} A_K .$$

Esistono funzioni $\gamma_K \in C_c(A_K)$, $K = 1 \dots N$, tali che

- $0 \leq \gamma_K \leq 1$
- $\sum_{K=1}^{N} \gamma_K = 1$ su $spt(f)$

("Partizione dell'unità subordinata al ricoprimento".)

Allora

$$f = f \sum_{K=1}^{N} \gamma_K = \sum_{K=1}^{N} f\gamma_K$$

e

$$T(f) = \sum_{K=1}^{N} T(f\gamma_K), \quad \text{per linearità}.$$

Siccome $f\gamma_K \in C_c(A_K;\mathbb{R}^m)$ e $|f\gamma_K| \leq 1$

$$T(f) \leq \sum_{K=1}^{N} \mu(A_K) \leq \sum_{K=1}^{\infty} \mu(A_K)$$

e passando al sup su f:

$$\mu\left(\bigcup_{K=1}^{\infty} A_K\right) \leq \sum_{K=1}^{\infty} \mu(A_K).$$

In modo analogo si ottiene la tesi per insiemi arbitrari.

Prova omessa.

(2) Affermo: μ misura di Borel.

Siano $A_1, A_2 \subset \mathbb{R}^n$ aperti tali che

$$dist(A_1, A_2) > 0 .$$ (Qui basta: A_1, A_2 <u>aperti</u> disgiunti)

Allora

$$\varrho \in C_c(A_1 \cup A_2; \mathbb{R}^m), \ |\varrho| \le 1 \iff \varrho = \varrho_1 + \varrho_2 \quad \text{con } \varrho_i \in C_c(A_i; \mathbb{R}^m), \ |\varrho_i| \le 1$$

Passando al sup in

$$T(\varrho) = T(\varrho_1 + \varrho_2) = T(\varrho_1) + T(\varrho_2)$$

si ottiene

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) .$$

Passando all'inf. la tesi segue per insiemi "<u>staccati</u>" qualsiasi.

Teor. Carathéodory II $\Rightarrow$ μ mis Borel.

Inoltre: μ Borel-regolare per costruzione.

Infine: μ finita sui compatti - segue dall'ipotesi del Teorema.

(3) Definiamo $C_c^+(\mathbb{R}^n) = \{ f \in C_c(\mathbb{R}^n) : f \ge 0 \}$.

Definiamo $\lambda : C_c^+(\mathbb{R}^n) \longrightarrow [0, \infty)$

$$\lambda(f) = \sup \{ T(\varrho) : \varrho \in C_c(\mathbb{R}^n; \mathbb{R}^m) \ \text{e} \ |\varrho| \le f \}$$

Prop. elementari:

(i) $0 \le f_1 \le f_2 \Rightarrow \lambda(f_1) \le \lambda(f_2)$

(ii) $c \ge 0$ costante $\Rightarrow \lambda(cf) = c\,\lambda(f)$

• Affermo che $\lambda(f_1+f_2) = \lambda(f_1)+\lambda(f_2)$

"$\leq$" Sia $|\varrho| \leq f_1+f_2$, Definiamo:
$\varrho \in C_c(\mathbb{R}^n;\mathbb{R}^m)$

$$\varrho_i = \begin{cases} \dfrac{\varrho f_i}{f_1+f_2} & f_1+f_2>0 \\ 0 & \text{altrimenti} \end{cases} \qquad i=1,2$$

Allora $\varrho_i \in C_c(\mathbb{R}^n;\mathbb{R}^m)$ e $|\varrho_i| \leq f_i$. Dunque

$$T(\varrho) = T(\varrho_1+\varrho_2) = T(\varrho_1)+T(\varrho_2) \leq \lambda(f_1)+\lambda(f_2)$$

Al sup su ϱ:

$$\lambda(f_1+f_2) \leq \lambda(f_1)+\lambda(f_2)$$

"$\geq$" Analogo (e più facile).

Osservazione:

$$\int_{\mathbb{R}^n} \chi_A \, d\mu = \mu(A) = \sup\{T(\varrho) : \varrho\in C_c(\mathbb{R}^n;\mathbb{R}^m), |\varrho|\leq \chi_A\}$$
$$A \text{ aperto} \qquad = \lambda(\chi_A)$$

• Affermo che per ogni $f \in C_c^+(\mathbb{R}^n)$ si ha

$$\int_{\mathbb{R}^n} f \, d\mu = \lambda(f). \qquad (**)$$

We begin the proof of (**). Fix $\varepsilon > 0$ and let

$$0 = t_0 < t_1 < \ldots < t_N = 2\|f\|_\infty$$

be such that:

(i) $t_j - t_{j-1} < \varepsilon$ for all $j = 1, 2, \ldots, N$;

(ii) $\mu(f^{-1}(t_j)) = 0$ for all $j = 1, 2, \ldots, N$.

Notice that the set $\{t > 0 : \mu(f^{-1}(t)) > 0\}$ is at most countable.

The sets

$$A_j = f^{-1}((t_{j-1}, t_j)) \qquad j = 1 \ldots N$$

are open and bounded. It follows that $\mu(A_j) < \infty$ -
for any $j = 1 \ldots N$ there exists $K_j \subset A_j$ compact such that

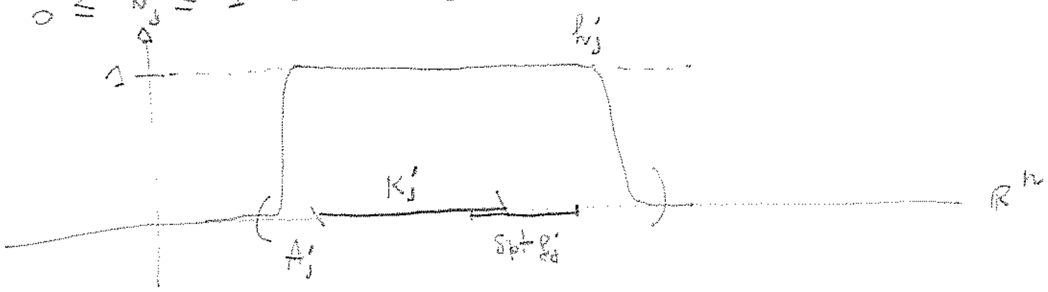
$$\mu(A_j \setminus K_j) < \frac{\varepsilon}{N}.$$

There exists functions $g_j \in C_c(A_j; \mathbb{R}^m)$ such that

a) $|g_j| \leq 1$

b) $T(g_j) \geq \mu(A_j) - \frac{\varepsilon}{N}$.

There exist functions $h_j \in C_c(A_j)$ such that:
$0 \leq h_j \leq 1$ and $h_j = 1$ on $K_j \cup \operatorname{spt}(g_j)$.



We have:

$$\lambda(h_j) \geq T(g_j) \underset{b)}{\geq} \mu(A_j) - \frac{\varepsilon}{N}$$

(from the def. of λ)

On the other hand

$$\lambda(h_j) \overset{\text{def.}}{=} \sup\{ T(g) : g \in C_c(\mathbb{R}^n;\mathbb{R}^m), \quad |g| \leq h_j \}$$
$$\leq \sup\{ T(g) : g \in C_c(\mathbb{R}^n;\mathbb{R}^m), \ \|g\|_\infty \leq 1, \ \mathrm{spt}(g) \subset A_j \}$$
$$= \mu(A_j).$$

We conclude:

$$\mu(A_j) - \frac{\varepsilon}{N} \leq \lambda(h_j) \leq \mu(A_j).$$

Let us define the open set:

$$A = \left\{ x \in \mathbb{R}^n : f(x)\left(1 - \sum_{j=1}^{N} h_j(x)\right) > 0 \right\}.$$

We have

$$\lambda\left(f - f\sum_{j=1}^{N} h_j\right) = \sup\left\{ T(g) : g \in C_c(\mathbb{R}^n;\mathbb{R}^m) \text{ with } |g| \leq f - f\sum_{j=1}^{N} h_j \right\}$$
$$\leq \sup\{ T(g) : g \in C_c(\mathbb{R}^n;\mathbb{R}^m) \text{ with } |g| \leq \|f\|_\infty \chi_A \}$$
$$= \|f\|_\infty \sup\{ T(g) : g \in C_c(\mathbb{R}^n;\mathbb{R}^m), \ |g| \leq \chi_A \}$$
$$= \|f\|_\infty \mu(A) = \|f\|_\infty \mu\left(\bigcup_{j=1}^{N} A_j \setminus \{h_j = 1\} \right)$$
$$\leq \|f\|_\infty \sum_{j=1}^{N} \mu(A_j \setminus K_j)$$
$$\leq \varepsilon \|f\|_\infty$$

It follows that

$$\lambda(f) = \lambda\left(f - f\sum_{j=1}^{N} h_j\right) + \lambda\left(f\sum_{j=1}^{N} h_j\right)$$
$$\leq \varepsilon\|f\|_\infty + \sum_{j=1}^{N} \lambda(f h_j) \leq \varepsilon\|f\|_\infty + \sum_{j=1}^{N} t_j \lambda(h_j)$$
$$\leq \varepsilon\|f\|_\infty + \sum_{j=1}^{N} t_j \mu(A_j)$$

On the other hand

$$\lambda(f) \geq \lambda\Big(\sum_{j=1}^{N} h_j f\Big) = \sum_{j=1}^{N} \lambda(h_j f) \geq$$

$$\geq \sum_{j=1}^{N} t_{j-1}\, \lambda(h_j') \geq \sum_{j=1}^{N} t_{j-1}\Big(\mu(A_j') - \frac{\varepsilon}{N}\Big) \geq$$

$$\geq \sum_{j=1}^{N} t_{j-1}\, \mu(A_j') - \frac{\varepsilon}{N} t_N \cdot N .$$

Finally, we also have

$$\sum_{j=1}^{N} t_{j-1}\, \mu(A_j) \leq \int_{\mathbb{R}^n} f \, d\mu \leq \sum_{j=1}^{N} t_j\, \mu(A_j) .$$

Taking differences:

$$(\bullet) \qquad \lambda(f) - \int_{\mathbb{R}^n} f \, d\mu \leq \varepsilon \|f\|_\infty + \sum_{j=1}^{N} (t_j - t_{j-1})\mu(A_j')$$

$$(\bullet\bullet) \qquad \lambda(f) - \int_{\mathbb{R}^n} f \, d\mu \geq -\varepsilon t_N + \sum_{j=1}^{N} (t_{j-1} - t_j)\mu(A_j')$$

Eventually, we obtain the estimate:

$$\Big|\lambda(f) - \int_{\mathbb{R}^n} f \, d\mu\Big| \leq 2\varepsilon \|f\|_\infty + \sum_{j=1}^{N} \underbrace{(t_j - t_{j-1})}_{\overset{\wedge}{\varepsilon}} \mu(A_j')$$

$$\leq 2\varepsilon \|f\|_\infty + \varepsilon\, \mu(\operatorname{spt}(f)) .$$

Letting $\varepsilon \downarrow 0$, we get the claim.

- Next, we claim that there exists $\sigma \in L^\infty_\mu(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$T(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu \qquad \forall f \in C_c(\mathbb{R}^n; \mathbb{R}^m) .$$

For any $i = 1, \dots, m$ let

$$\lambda_i(f) = T(f e_i) \qquad \text{for } f \in C_c(\mathbb{R}^n) .$$

where $e_i = (0 - 0 \underset{i}{1} 0 - 0)$

We have

$$|\lambda_i(f)| = |T(f e_i)| \leq \lambda(|f|) = \int_{\mathbb{R}^n} |f| \, d\mu$$

for all $f \in C_c(\mathbb{R}^n)$. We can extend λ_i to get linear and bounded.

$$\lambda_i : L^1(\mathbb{R}^n; \mu) \longrightarrow \mathbb{R}$$

In other words, we have $\lambda_i \in L^1(\mathbb{R}^n; \mu)^* \cong L^\infty(\mathbb{R}^m; \mu)$. [Duality Theorem]

Then there exists $\sigma_i \in L^\infty(\mathbb{R}^n; \mu)$ such that

$$T(f e_i) = \lambda_i(f) = \int_{\mathbb{R}^n} \sigma_i f \, d\mu \qquad \forall f \in L^1(\mathbb{R}^n; \mu).$$

Now take $f \in C_c(\mathbb{R}^n; \mathbb{R}^m)$:

$$T(f) = T\left(\sum_{i=1}^m (f \cdot e_i) e_i \right)$$

$$= \sum_{i=1}^m T((f \cdot e_i) e_i)$$

$$= \sum_{i=1}^m \int_{\mathbb{R}^n} \sigma_i \, f \cdot e_i \, d\mu$$

$$= \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu, \qquad \sigma := (\sigma_1, \dots, \sigma_m) \in L^\infty_\mu.$$

- We show that $|\sigma| = 1$ μ-a.e. Let $A \subset \mathbb{R}^n$ be an open set with $\mu(A) < \infty$. There exists a sequence $(f_k)_{k \in \mathbb{N}}$ such that $f_k \in C_c(A; \mathbb{R}^m)$, $|f_k| \leq 1$ and

$$f_k \cdot \sigma \underset{k \to \infty}{\longrightarrow} |\sigma| \quad \mu\text{-a.e. on } A$$

We do not prove this fact, here. See [EG].

Then

$$\int_A |\sigma| \, d\mu = \int_A \lim_{k \to \infty} f_k \cdot \sigma \, d\mu = \lim_{k \to \infty} \int_A f_k \cdot \sigma \, d\mu \leq \mu(A)$$

because $\int_A f_k \cdot \sigma \, d\mu = T(f_k) \leq \mu(A)$, by the def. of $\mu(A)$.

On the other hand: if $f \in C_c(A; \mathbb{R}^m)$ and $|f| \leq 1$:

$$\int_{\mathbb{R}^n} f \cdot 6 \, d\mu \leq \int_A |6| \, d\mu$$

and taking the supremum:

$$\mu(A) \leq \int_A |6| \, d\mu .$$

The identity

$$\int_A |6| \, d\mu = \mu(A) \qquad \text{for open sets}$$

implies the same identity for any μ-measurable set.

It follows that

$$|6| = 1 \qquad \mu\text{-a.e.}$$

□

THEOREM 3 Let $T: C_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$ be linear and positive:

$$f \geq 0 \implies T(f) \geq 0 , \qquad \text{for all } f \in C_c(\mathbb{R}^n).$$

Then there exists a Radon measure μ on $\mathbb{R}^n$ such that

$$T(f) = \int_{\mathbb{R}^n} f \, d\mu \qquad \text{for all } f \in C_c(\mathbb{R}^n).$$

Proof. Let $K \subset \mathbb{R}^n$ be compact and fix $\zeta \in C_c(\mathbb{R}^n)$ such that

$$\zeta = 1 \text{ on } K \quad \text{and} \quad \zeta \geq 0 .$$

Then for any $f \in C_c(\mathbb{R}^n)$ with $\operatorname{spt}(f) \subset K$ there holds:

$$\|f\|_\infty \zeta - f \geq 0 .$$

It follows that

$$0 \leq T(\|f\|_\infty \zeta - f) = \|f\|_\infty T(\zeta) - T(f)$$

That is:

$$T(f) \leq \|f\|_\infty \, T(J) \quad \forall f \in C_c(\mathbb{R}^n), \ spt(f) \subset K.$$

Then

$$\sup \{ T(f) : f \in C_c(\mathbb{R}^n), \ spt(f) \subset K, \ \|f\|_\infty \leq 1 \} \leq T(J) < \infty,$$

The claim follows by Riesz theorem. □

HAUSDORFF MEASURES IN $\mathbb{R}^n$

Let $\alpha(n)$ denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$

$$\alpha(n) = \mathcal{L}^n(B_1(0))$$
$$= \text{Leb. measure } \{x \in \mathbb{R}^n : |x| < 1\}.$$

We claim that

$$(*) \qquad \alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-s} \, ds$ is the Gamma function.

We prove (*):

$$\pi^{n/2} = \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n = \int_{\mathbb{R}^n} e^{-|x|^2} dx \overset{\text{Coarea Formula}}{=} \int_0^\infty \left(\int_{|x|=r} e^{-r^2} d\sigma \right) dr$$

($d\sigma$: surface measure)

$$= \int_0^\infty e^{-r^2} \, n\,\alpha(n) \, r^{n-1} \, dr$$

because $\sigma(\{x \in \mathbb{R}^n : |x| = r\}) = \frac{d}{dr} \mathcal{L}^n(\{|x| < r\}) =$