We deduce that

$$H^{n-1}(\partial E) = \lim_{r \downarrow 0} \frac{\mathcal{L}^n(E_r \setminus E)}{r} \geq \lim_{r \downarrow 0} \mathcal{L}^n(B) \frac{(1+r)^n - 1}{r} =$$

$$= \alpha(n) \cdot n = H^{n-1}(\partial B).$$

PROOF OF THE BRUNN-MINKOWSKI INEQUALITY

Theorem Let $A, B \subset \mathbb{R}^n$ be nonempty. Then,

$$\mathcal{L}^n(A+B)^{\frac{1}{n}} \geq \mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}}$$

where $A+B = \{x+y \in \mathbb{R}^n;\ x \in A,\ y \in B\}$ and $\mathcal{L}^n$ is the Lebesgue outer measure in $\mathbb{R}^n$.

Proof. Let

$$P = \{P_1 \times \ldots \times P_n \subset \mathbb{R}^n;\ P_i \subset \mathbb{R} \text{ interval},\ i=1,\ldots,n\}.$$

First case: $A, B \in P$;

$$A = P_1 \times \ldots \times P_n,$$
$$B = Q_1 \times \ldots \times Q_n.$$

Then we have

$$A+B = (P_1+Q_1) \times \ldots \times (P_n+Q_n)$$

where P_i+Q_i is an interval of length

$$\mathcal{L}^1(P_i+Q_i) = \mathcal{L}^1(P_i) + \mathcal{L}^1(Q_i).$$

Let

$$u_i = \frac{\mathcal{L}^1(P_i)}{\mathcal{L}^1(P_i)+\mathcal{L}^1(Q_i)} \quad , \quad v_i = \frac{\mathcal{L}^1(Q_i)}{\mathcal{L}^1(P_i)+\mathcal{L}^1(Q_i)}$$

By the geometric - arithmetic mean inequality

$$\left(\prod_{i=1}^{n} u_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} v_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} u_i + \frac{1}{n}\sum_{i=1}^{n} v_i =$$

$$= \frac{1}{n}\sum_{i=1}^{n}(u_i+v_i) = 1 ,$$

This implies

$$\left(\prod_{i=1}^{n} \mathcal{L}^1(P_i)\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \mathcal{L}^1(Q_i)\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n} \left(\mathcal{L}^1(P_i)+\mathcal{L}^1(Q_i)\right)\right)^{\frac{1}{n}}$$

that is

$$\mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}} \leq \mathcal{L}^n(A+B)^{\frac{1}{n}} .$$

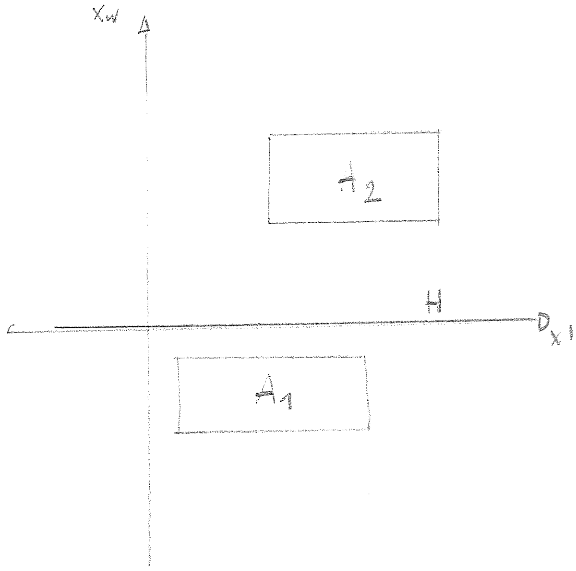
Second case: $A = \bigcup_{i=1}^{h} {}^{(\mathrm{disj.})} A_i$ and $B = \bigcup_{i=1}^{k} {}^{(\mathrm{disj.})} B_i$ with $A_i, B_i \in \mathcal{P}$ and $h, k \geq 1$. When $h = k = 1$ the claim is already proved. The proof is by induction on $m = h+k$. Assume $h \geq 2$.

Let $H \subset \mathbb{R}^n$ be a hyperplane parallel to one of the coordinate hyperplanes that separates A_1 and A_2. For instance,

$$A_1 \subset \{x \in \mathbb{R}^n : x_n \leq 0\}$$
$$A_2 \subset \{x \in \mathbb{R}^n : x_n \geq 0\}$$

Let $A^+ = A \cap \{x_n \geq 0\}$,

$A^- = A \cap \{x_n \leq 0\}$.



Also let $B^+ = B \cap \{x_n \geq 0\}$,

$B^- = B \cap \{x_n \leq 0\}$.

Because the Brunn-Minkowski inequality is invariant by translations of the sets, we can assume that

$$\frac{\mathcal{L}^n(B^-)}{\mathcal{L}^n(B)} = \frac{\mathcal{L}^n(A^-)}{\mathcal{L}^n(A)}.$$

Then we also have

$$\frac{\mathcal{L}^n(B^+)}{\mathcal{L}^n(B)} = \frac{\mathcal{L}^n(B)-\mathcal{L}^n(B^-)}{\mathcal{L}^n(B)} = 1 - \frac{\mathcal{L}^n(A^-)}{\mathcal{L}^n(A)} = \frac{\mathcal{L}^n(A^+)}{\mathcal{L}^n(A)}.$$

By induction, the inequality holds for the pairs A^-, B^- and A^+, B^+. Then we have

$$\mathcal{L}^n(A+B) \geq \mathcal{L}^n(A^- + B^-) + \mathcal{L}^n(A^+ + B^+) \geq$$

$$\geq \left(\mathcal{L}^n(A^-)^{\frac{1}{n}} + \mathcal{L}^n(B^-)^{\frac{1}{n}}\right)^n + \left(\mathcal{L}^n(A^+)^{\frac{1}{n}} + \mathcal{L}^n(B^+)^{\frac{1}{n}}\right)^n =$$

$$= \mathcal{L}^n(A^-)\left(1 + \left(\frac{\mathcal{L}^n(B)}{\mathcal{L}^n(A)}\right)^{\frac{1}{n}}\right)^n + \mathcal{L}^n(A^+)\left(1 + \left(\frac{\mathcal{L}^n(B)}{\mathcal{L}^n(A)}\right)^{\frac{1}{n}}\right)^n$$

$$= \mathcal{L}^n(A)\left(1 + \left(\frac{\mathcal{L}^n(B)}{\mathcal{L}^n(A)}\right)^{\frac{1}{n}}\right)^n = \left(\mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}}\right)^n.$$

Third case: A, B are open. Let

$$\mathcal{F} = \left\{ \bigcup_{i=1}^{h} E_i \; ; \; E_i \in P, \; h \in \mathbb{N} \right\}.$$

There are increasing sequences $A_k, B_k \in \mathcal{F}$ such that

$$A = \bigcup_{k=1}^{\infty} A_k, \qquad A_k \subset A_{k+1},$$

$$B = \bigcup_{k=1}^{\infty} B_k, \qquad B_k \subset B_{k+1}.$$

Then we also have $A + B = \bigcup_{k=1}^{\infty} A_k + B_k$.

Passing to the limit in the inequality

$$\mathcal{L}^n(A_k + B_k)^{\frac{1}{n}} \geq \mathcal{L}^n(A_k)^{\frac{1}{n}} + \mathcal{L}^n(B_k)^{\frac{1}{n}}, \qquad k \in \mathbb{N}$$

we get the conclusion.

Fourth case: A and B are compact. Then $A+B$ is compact.

Let $A_\varepsilon = \{ x \in \mathbb{R}^n : \text{dist}(x, A) < \varepsilon \}$ open, and
$B_\varepsilon = \{ x \in \mathbb{R}^n : \text{dist}(x, B) < \varepsilon \}$ open.

We have
$$\mathcal{L}^n(A_\varepsilon + B_\varepsilon)^{\frac{1}{n}} \geq \mathcal{L}^n(A_\varepsilon)^{\frac{1}{n}} + \mathcal{L}^n(B_\varepsilon)^{\frac{1}{n}}$$
$$\geq \mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}}$$

Because
$$\bigcap_{\varepsilon > 0} (A_\varepsilon + B_\varepsilon) = A + B$$

letting $\varepsilon \downarrow 0$ we get the conclusion.

Fifth case: A, B and $A+B$ are measurable.

By approximation from inside by compact sets.

General case: omitted