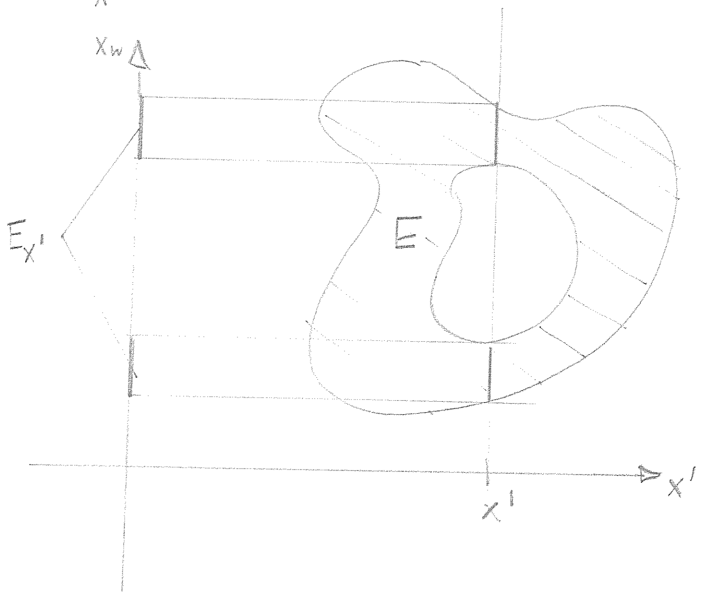
STEINER REARRANGEMENT IN $\mathbb{R}^n$

Coordinates in $\mathbb{R}^n$, $n \geq 2$: $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

For $E \subset \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$ let us consider the section

$$E_{x'} = \{ x_n \in \mathbb{R} \; ; \; (x', x_n) \in E \}.$$



Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable with $\mathcal{L}^n(E) < \infty$.

Let $\varphi : \mathbb{R}^{n-1} \longrightarrow [0, \infty)$ be the function

$$\varphi(x') = \begin{cases} \mathcal{L}^1(E_{x'}) & \text{if } \mathcal{L}^1(E_{x'}) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

By Fubini-Tonelli theorem, φ is $\mathcal{L}^n$-measurable (and in fact $\varphi \in L^1(\mathbb{R}^{n-1})$).

Definition (Steiner rearrangement) Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable with $\mathcal{L}^n(E) < \infty$. The set

$$E^* = \left\{ x \in \mathbb{R}^n ; |x_n| < \frac{1}{2} \varphi(x') \right\}$$

is called the Steiner rearrangement of E.

Remarks

(1) The set E^* is x_n-symmetric:

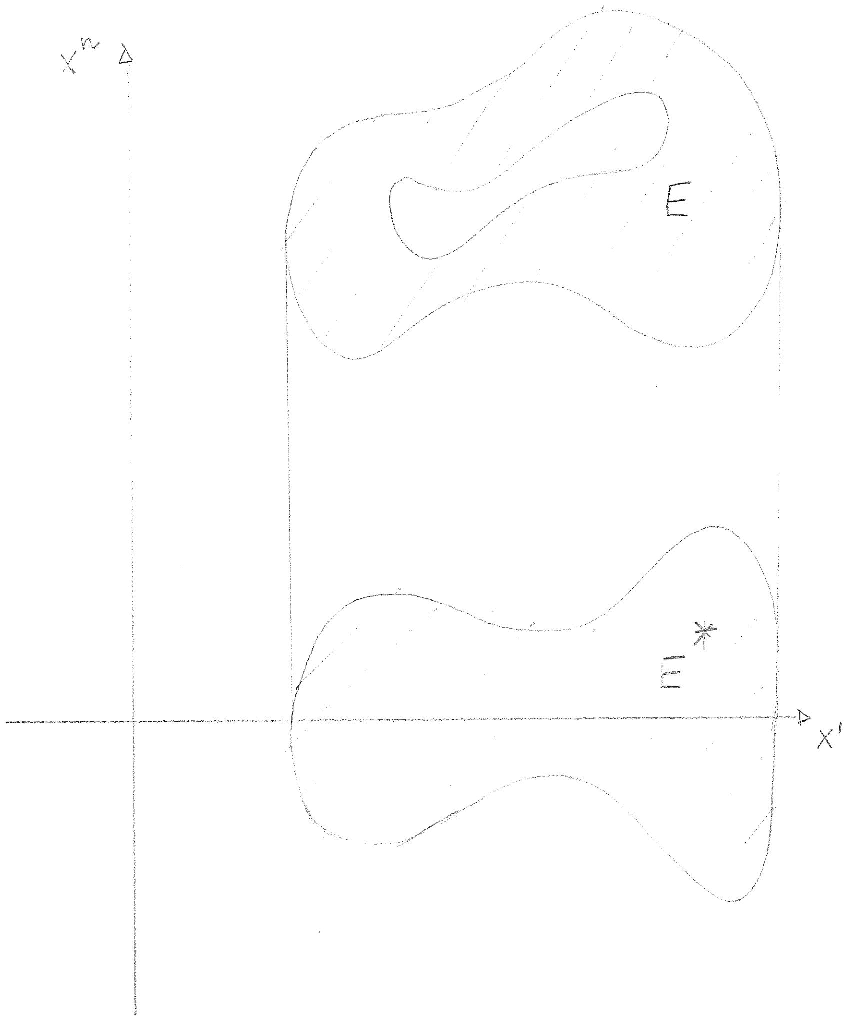
$$(x', x_n) \in E^* \iff (x', -x_n) \in E^*.$$

(2) By Fubini-Tonelli theorem:

$$\mathcal{L}^n(E^*) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E^*_{x'})\, dx'$$

$$= \int_{\mathbb{R}^{n-1}} \varphi(x')\, dx' = \mathcal{L}^n(E)$$

(3) The set E^* is normal in the x_n-direction, i.e., each section $E^*_{x'}$ is an interval.



Theorem Let $E \subset \mathbb{R}^n$ be a measurable set with $\mathcal{L}^n(E) < \infty$. Then we have

$$\|\partial E^*\|(\mathbb{R}^n) \leq \|\partial E\|(\mathbb{R}^n).$$

Moreover, if $\|\partial E^*\|(\mathbb{R}^n) = \|\partial E\|(\mathbb{R}^n)$ then E is (equivalent to) a normal set in the x_n-direction.

We shall prove a more general version of this theorem.

We fix a density $d : \mathbb{R}^n \longrightarrow \mathbb{R}$ with the following properties:

(1) $d \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$;

(2) $d > 0$ on $\mathbb{R}^n$;

(3) $d(x) = f(x') \, g(x_n)$ for functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$;

(4) g is even ($g(x_n) = g(-x_n)$) and log-convex, i.e. $x_n \longmapsto \log g(x_n)$ is convex.

Examples

(1) $d \equiv 1$ satisfies (1)-(4).

(2) $d(x) = e^{|x|^2}$, $x \in \mathbb{R}^n$, satisfies (1)-(4).

Def. The (d-) volume of a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ is

$$V(E) = \int_E d(x) \, dx.$$

Def. The (d-) perimeter of a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ is

$$P(E) = \sup \left\{ \int_E \operatorname{div}(d\,\varphi) \, dx \; : \; \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), \; \|\varphi\|_\infty \leq 1 \right\}.$$

Remark When $E \subset \mathbb{R}^n$ is a bounded set with regular (Lipschitz) boundary, then we have:

$$P(E) = \int_{\partial E} \varrho(x) \, d\mathcal{H}^{n-1}(x).$$

Define the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$

$$\psi(t) = \int_0^t \varrho(s) \, ds.$$

This function satisfies:

(1) $\psi(0) = 0$, $\psi'(t) = \varrho(t) > 0 \Rightarrow \psi$ is strictly increasing;

(2) ϱ even $\Rightarrow \psi$ is odd;

(3) ψ is surjective (because ϱ is log-convex).

Define the function $F : \mathbb{R} \to \mathbb{R}$, $F = \varrho \circ \psi^{-1} = \psi' \circ \psi^{-1}$.

Notice that (if $F \in C^2$)

$$F'(s) = \varrho'(\psi^{-1}(s)) \cdot \frac{1}{\psi'(t)}\Big|_{t = \psi^{-1}(s)} = \frac{\varrho'(\psi^{-1}(s))}{\varrho(\psi^{-1}(s))} = \frac{d}{dt} \log \varrho(t) \Big|_{t = \psi^{-1}(s)}$$

$$F''(s) = \frac{d^2}{dt^2} \log \varrho(t) \Big|_{t = \psi^{-1}(s)} \cdot \frac{1}{\varrho(\psi^{-1}(s))}$$

Conclusion:

$$F \text{ convex} \iff \log \varrho \text{ convex}.$$

Generalized Steiner rearrangement.

Let $E \subset \mathbb{R}^n$ be such that $V(E) < \infty$.

Define the function $K: \mathbb{R}^{n-1} \to [0,\infty)$

$$K(x') = \begin{cases} \frac{1}{2}\int_{E_{x'}} g(x_n)\, dx_n, & \text{if the integral converges} \\ 0 & \text{otherwise} \end{cases}$$

By Fubini-Tonelli theorem $x' \mapsto K(x')$ is $\mathcal{L}^{n-1}$-measurable.

Let $h: \mathbb{R}^{n-1} \to [0,\infty)$ be the function

$$h(x') = \psi^{-1}(K(x')) = \psi^{-1}\left(\frac{1}{2}\int_{E_{x'}} g(x')\, dx_n\right).$$

Definition Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable with $V(E) < \infty$.

We call the set

$$E^* = \{x \in \mathbb{R}^n ; |x_n| < h(x')\}$$

the (d-)Steiner rearrangement of E.

Theorem Let d be a density with the properties stated above.
Let $E \subset \mathbb{R}^n$ be a $\mathcal{L}^n$-measurable set such that $V(E) < \infty$.
Then we have:

(1) $V(E^*) = V(E)$;

(2) $P(E^*) \leq P(E)$.

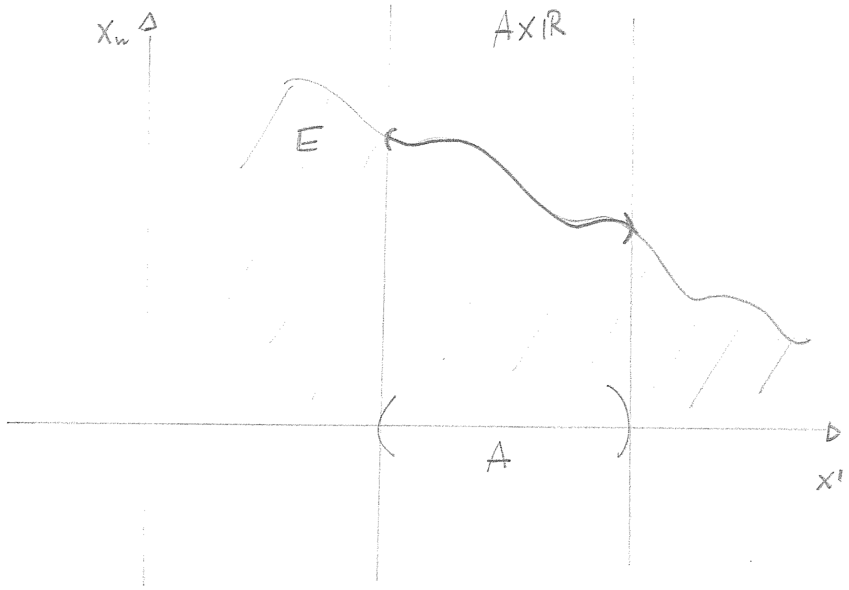
Moreover, if $P(E^*) = P(E)$ ($< \infty$), then E^* is (equivalent to) a x_n-normal set.

Proof. For $i=1,\dots,n$ and $A\subset\mathbb{R}^n$ open, let us define the partial perimeters

$$P_i(E;A) = \sup\left\{ \int_E \frac{\partial}{\partial x_i}(d\varphi)\,dx \;:\; \varphi\in C_c^1(A;\mathbb{R}^n),\ \|\varphi\|_\infty\le 1\right\}.$$

Let $\mu_1,\dots,\mu_n$ be the (finite) Borel measures in $\mathbb{R}^{n-1}$ such that for open sets $A\subset\mathbb{R}^{n-1}$ we have

$$\mu_i(A) = P_i(E;A\times\mathbb{R}).$$



The measures μ_i^*, $i=1,\dots,n$, are defined in a similar way

$$\mu_i^*(A) = P_i(E^*;A\times\mathbb{R}), \qquad i=1,\dots,n,\ A\subset\mathbb{R}^{n-1} \text{ open}.$$

<u>CLAIM</u>: $\mu_i^*(A)\le\mu_i(A)$ for all $A\subset\mathbb{R}^{n-1}$ open, $i=1,\dots,n$.