We begin with the case $i = n$. We can assume:

$$\mu_n(A) = P_n(E; A\times\mathbb{R}) \le P(E; A\times\mathbb{R}) \le$$
$$\le P(E;\mathbb{R}^n) < \infty .$$

We assume

Otherwise there is nothing to prove:

We have:

$$\mu_n(A) = \sup_{\substack{\varphi\in C_c^1(A\times\mathbb{R})\\ |\varphi|\le 1}} \int_{(A\times\mathbb{R})\cap E} \frac{\partial}{\partial x_n}\Big(\overbrace{f(x')\varrho(x_n)}^{d(x)}\varphi(x)\Big)\,dx$$

$$= \sup_{''} \int_A f(x') \int_{E_{x'}} \frac{\partial}{\partial x_n}\Big(\varrho(x_n)\varphi(x)\Big)\,dx_n\,dx'$$

$$(*) \quad \overset{(=)}{\ge} \int_A f(x') \sup_{\substack{\varphi\in C_c^1(\mathbb{R})\\ |\varphi|\le 1}} \int_{E_{x'}} \frac{\partial}{\partial x_n}\Big(\varrho(x_n)\varphi(x_n)\Big)\,dx_n\,dx' .$$

To prove $(*)$, notice that when χ_E is replace with a function $\nu\in C^\infty(\mathbb{R}^n)$, the identity $(*)$ reads

$$\int_{A\times\mathbb{R}} \left|\frac{\partial\nu}{\partial x_n}(x)\right| d(x)\,dx = \int_A\left(\int_{\mathbb{R}} \left|\frac{\partial \nu}{\partial x_n}(x)\right| d(x)\,dx_n\right)dx',$$

that is true by Fubini-Tonelli theorem.

In general, consider a sequence $\nu_h\in C^\infty(\mathbb{R}^n)$ such that

- $$\lim_{h\to\infty} \int_{A\times\mathbb{R}} \left|\frac{\partial \nu_h}{\partial x_n}(x)\right| d(x)\,dx = \mu_n(A) = P_n(E; A\times\mathbb{R})$$

- $$\nu_h \xrightarrow[h\to\infty]{} \chi_E \quad \text{in } L^1_{loc}(A\times\mathbb{R}; d(x)\,\mathcal{L}^n) .$$

Then we also have

$$\chi_{E_h}(x',\cdot) \xrightarrow[h\to\infty]{} \chi_E(x',\cdot) \quad \text{in} \quad L^1_{loc}(\mathbb{R}\,;\, d(x',\cdot)dx_n)$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in A$. By lower semicontinuity

$$\int_A f(x') \sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \le 1}} \int_{\mathbb{R}} \chi_E \frac{\partial}{\partial x_n}\Big(\varrho(x_n)\varphi(x_n)\Big)\,dx_n\,dx' \le$$

$$\le \int_A f(x') \liminf_{h\to\infty} \int_{\mathbb{R}} \left|\frac{\partial \chi_{E_h}}{\partial x_n}\right| \varrho(x_n)\,dx_n\,dx' \le \text{(Fatou lemma)}$$

$$\le \lim_{h\to\infty}(\inf) \int_{A\times\mathbb{R}} \left|\frac{\partial \chi_{E_h}}{\partial x_n}\right| d(x)\,dx = P_n(E\,;\,A\times\mathbb{R}).$$

This finishes the proof of (*).

Next we claim that for $\mathcal{L}^{n-1}$-a.e. $x' \in A$ we have

$$(**) \quad \sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \le 1}} \int_{E_{x'}} \frac{\partial}{\partial x_n}(\varrho\,\varphi)\,dx_n \ge \sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \le 1}} \int_{E^*_{x'}} \frac{\partial}{\partial x_n}(\varrho\,\varphi)\,dx_n,$$

From (*) and (**) it follows that:

$$\mu_n(A) = \int_A f(x') \sup_{\substack{\varphi\in C^1_c(\mathbb{R})\\ |\varphi|\le 1}} \int_{E_{x'}} \frac{\partial}{\partial x_n}(\varrho\varphi)\,dx_n\,dx' \geq$$

$$\geq \int_A f(x') \sup_{\substack{\varphi\in C^1_c(\mathbb{R})\\ |\varphi|\le 1}} \int_{E^*_{x'}} \frac{\partial}{\partial x_n}(\varrho\varphi)\,dx_n\,dx' =$$

$$= \mu_n^*(A).$$

We prove (**). For $\mathcal{L}^{n-1}$-a.e. $x'\in A$ we have $\int_{E_{x'}} \varrho(x_n)\,dx_n < \infty$ and

$$\sup_{\substack{\varphi\in C^1_c(\mathbb{R})\\ |\varphi|\le 1}} \int_{E_{x'}} \frac{\partial}{\partial x_n}(\varrho\varphi)\,dx_n < \infty.$$

It follows that $E_{x'}$ is (equivalent to) a finite union of open intervals

$$E_{x'} = \bigcup_{j=1}^{k} (a_j', b_j'), \qquad k\geq 1$$

with

$$-\infty < a_1 < b_1 < a_2 < b_2 < \dots < a_k < b_k < \infty.$$

Moreover, we have

$$\sup_{\substack{\varphi\in C^1_c(\mathbb{R})\\ |\varphi|\le 1}} \int_{E_{x'}} \frac{\partial}{\partial x_n}(\varrho\varphi)\,dx_n = \sum_{j=1}^{K} \varrho(a_j') + \varrho(b_j')$$

On the other hand $E^*_{x'} = (-h(x'), h(x'))$ and thus

$$\sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ |\varphi| \leq 1}} \int_{E^*_{x'}} \frac{\partial}{\partial x_n} (\rho \varphi)\, dx_n = \rho(-h(x')) + \rho(h(x'))$$

$$= 2\rho(h(x')).$$

Recall that, by definition, we have

$$h(x') = \psi^{-1}\left(\frac{1}{2} \int_{E_{x'}} \rho(t)\, dt \right)$$

$$= \psi^{-1}\left(\frac{1}{2} \sum_{j=1}^{K} \int_{a_j}^{b_j} \rho(t)\, dt \right) \qquad (\psi' = \rho)$$

$$= \psi^{-1}\left(\frac{1}{2} \sum_{j=1}^{K} \left(\psi(b_j) - \psi(a_j) \right) \right).$$

The claim (**) then reads

$$2\, \rho\left(\psi^{-1}\left(\frac{1}{2} \sum_{j=1}^{K} \left(\psi(b_j) - \psi(a_j) \right) \right) \right) \leq \sum_{j=1}^{K} \rho(a_j) + \rho(b_j)$$

with the notation

$$\alpha_j = \psi(a_j)$$

$$\beta_j = \psi(b_j),$$

and with $F = \rho \circ \psi^{-1}$, we obtain the equivalent inequality

$$F\left(\frac{1}{2} \sum_{j=1}^{K} \beta_j - \alpha_j \right) \leq \frac{1}{2} \sum_{j=1}^{K} F(\alpha_j) + F(\beta_j).$$

The function $t \mapsto F(t)$ is increasing for $t \geq 0$.

Moreover, we have

$$0 \leq \sum_{j=1}^{K} \underbrace{\beta_j - \alpha_j}_{\geq 0} \leq \beta_K - \alpha_1 .$$

The convexity of F implies

$$F\left(\frac{1}{2}\sum_{j=1}^{K} \beta_j - \alpha_j\right) \leq F\left(\frac{\beta_K - \alpha_1}{2}\right) \leq$$

$$\leq \frac{F(\beta_K) + \underset{F(\alpha_1)}{\underset{\shortparallel}{F(-\alpha_1)}}}{2} \overset{(*)}{\leq} \frac{1}{2} \sum_{j=1}^{K} F(\alpha_j) + F(\beta_j) .$$

Important!

the inequality $(*)$ is strict if $K > 1$.

Now we prove the claim on p. 155 when $i = 1, 2, \ldots, n-1$:

$$\mu_i(A) = P_i(E; A \times \mathbb{R})$$

$$= \sup_{\substack{\varphi \in C_c^1(A \times \mathbb{R}) \\ |\varphi| \leq 1}} \int_E \frac{\partial}{\partial x_i} \left(f(x') g(x_n) \varphi(x) \right) dx$$

$$\geq \sup_{\substack{\varphi \in C_c^1(A) \\ |\varphi| \leq 1}} \int_A \frac{\partial}{\partial x_i} \left(f(x') \varphi(x') \right) \underbrace{\left(\int_{E_{x'}} g(x_n)\, dx_n \right)}_{= 2K(x')} dx' = \blacksquare$$

Assume for a while that K is smooth. Then by the coarea formula (generalized version):

$$\circledast = 2\int_A \left|\frac{\partial K}{\partial x_i}\right| f(x')\,dx'$$

Partial perimeter of $\{K>t\}$

$$= 2\int_0^\infty \int_{A\cap\partial\{K>t\}} f(x')\,d\lambda_i\,dt$$

$$= 2\int_0^\infty \sup_{\substack{\varphi\in C_c^1(A)\\ |\varphi|\le 1}} \int_{\{x'\in\mathbb{R}^{n-1};\,K(x')>t\}} \frac{\partial}{\partial x_i}\left(f(x')\varphi(x')\right)dx'\,dt\,.$$

This formula can be proved in a formal way. We omit the details. We perform the change of variable

$$t = \psi(x_n)\,,\qquad dt = \psi'(x_n)\,dx_n = \varrho(x_n)\,dx_n$$

$$K(x')>t \iff h(x') = \psi^{-1}(K(x')) > x_n\,.$$

We obtain

$$\circledast = 2\int_0^\infty \sup_{\substack{\varphi\in C_c^1(A)\\ |\varphi|\le 1}} \int_{\{x'\in\mathbb{R}^{n-1};\,h(x')>x_n\}} \frac{\partial}{\partial x_i'}\left(f(x')\varphi(x')\right)dx'\;\varrho(x_n)\,dx_n$$

$$\geq 2 \sup_{\substack{\varphi\in C_c^1(A\times\mathbb{R})\\ |\varphi|\le 1}} \int_{\{x\in\mathbb{R}^n;\,h(x')>x_n>0\}=E^*\cap\{x_n>0\}} \frac{\partial}{\partial x_i}\left(d(x)\,\varphi(x)\right)dx \;=$$

$$= \sup_{\substack{\varphi \in C_c^1(A \times \mathbb{R}) \\ |\varphi| \leq 1}} \int_{E^*} \frac{\partial}{\partial x^i} \left(a(x) \varphi(x) \right) dx$$

$$= P_i(E^*; A \times \mathbb{R}) = \mu_i^*(A).$$

This ends the proof of the main claim on p. 155.

To finish the proof, we need the following argument.

Let $\mu = (\mu_1, \dots, \mu_n)$ be a $\mathbb{R}^n$-valued Borel measure on $\mathbb{R}^{n-1}$. The total variation $|\mu|$ is the real Borel measure

$$|\mu|(A) = \sup \left\{ \sum_{k=1}^{\infty} |\mu(A_k)| : A = \bigcup_{\substack{k=1 \\ \text{disj.}}}^{\infty} A_k, \ A_k \text{ Borel sets of } \mathbb{R}^{n-1} \right\}$$

$A \subset \mathbb{R}^{n-1}$ Borel set.

In our case (Exercise)

$$|\mu|(\mathbb{R}^{n-1}) = \sup_{\substack{\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n) \\ |\varphi| \leq 1}} \int_E \operatorname{div}(a(x) \varphi(x)) \, dx$$

$$= P(E).$$

Because

$$\mu^* \leq \mu \quad \text{componentwise}$$

we deduce that $|\mu|(A) \geq |\mu^*|(A)$, $A \subset \mathbb{R}^{n-1}$ Borel. When $A = \mathbb{R}^{n-1}$ we deduce

$$P(E^*) \leq P(E).$$

Now assume that $P(E^*) = P(E)$. This means

$$|\mu^*|(\mathbb{R}^{n-1}) = |\mu|(\mathbb{R}^{n-1}).$$

This in turn implies

$$|\mu^*|(A) = |\mu|(A)$$

for any $A \subset \mathbb{R}^{n-1}$ Borel, i.e., $|\mu^*| = |\mu|$.

By Radon-Nycodim theorem:

$$\mu = \sigma\,|\mu|, \qquad \sigma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$$
$$\mu^* = \sigma^*\,|\mu^*| = \sigma^*\,|\mu|, \qquad \sigma^* : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n$$

with $|\sigma| = 1$ and $|\sigma^*| = 1$ $|\mu|$-a.e. on $\mathbb{R}^{n-1}$.

On the other hand

$$\mu^* \leq \mu \implies \sigma^* \leq \sigma \quad \text{componentwise}$$

and finally

$$\left.\begin{array}{l} |\sigma| = |\sigma^*| = 1 \\ \sigma^* \leq \sigma \end{array}\right\} \Rightarrow \sigma^* = \sigma \Rightarrow \mu^* = \mu .$$

In particular, we deduce that $\mu_n^* = \mu_n$.
This implies that $K = 1$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$
in the proof of the inequality $\mu_n^*(A) \leq \mu_n(A)$
on p. 160.

Conclusion:

$$P(E^*) = P(E) \quad \Rightarrow \quad E \text{ is } x_n\text{-normal}.$$

□

Now we fix $d = 1$.

<u>Theorem</u> For any $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable we have

$$(*) \qquad \min\left\{ \mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\right\} \leq c_n \, \|\partial E\|(\mathbb{R}^n)^{\frac{n}{n-1}}$$

with $\mathcal{L}^n(B) = c_n \|\partial B\|(\mathbb{R}^n)^{\frac{n}{n-1}}$, $B \subset \mathbb{R}^n$ ball.
Equality in (*) implies that E or $\mathbb{R}^n \setminus E$ is a ball.