and finally

$$\left.\begin{array}{l} |\sigma| = |\sigma^*| = 1 \\ \sigma^* \leq \sigma \end{array}\right\} \Rightarrow \sigma^* = \sigma \Rightarrow \mu^* = \mu .$$

In particular, we deduce that $\mu_n^* = \mu_n$.
This implies that $K = 1$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$
in the proof of the inequality $\mu_n^*(A) \leq \mu_n(A)$
on p. 160.

Conclusion:

$$P(E^*) = P(E) \quad \Rightarrow \quad E \text{ is } x_n\text{-normal.}$$

$\square$

Now we fix $d = 1$.

<u>Theorem</u> For any $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable we have

$$(*) \qquad \min\left\{ \mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\right\} \leq c_n \, \|\partial E\|(\mathbb{R}^n)^{\frac{n}{n-1}}$$

with $\mathcal{L}^n(B) = c_n \|\partial B\|(\mathbb{R}^n)^{\frac{n}{n-1}}$, $B \subset \mathbb{R}^n$ ball.
Equality in (*) implies that E or $\mathbb{R}^n \setminus E$ is a ball.

Proof.

First case: E is bounded. (And: $\|\partial E\|(\mathbb{R}^n) < \infty$).

Fix $K>0$ such that $E \subset B_K(0)$. Let

$$\mathcal{F} = \left\{ F \subset \overline{B_K(0)} : \begin{array}{l} F\ \mathcal{L}^n\text{-measurable with} \\ \mathcal{L}^n(F) = \mathcal{L}^n(E) \end{array} \right\}.$$

By compactness and lower semicontinuity, there is $F \in \mathcal{F}$ such that

$$\|\partial F\|(\mathbb{R}^n) = \min_{G \in \mathcal{F}} \|\partial G\|(\mathbb{R}^n) \leq \|\partial E\|(\mathbb{R}^n).$$

By minimality, $F^* \in \mathcal{F}$ and

$$\|\partial F^*\|(\mathbb{R}^n) = \|\partial F\|(\mathbb{R}^n).$$

Then: F is x_n-normal. By a rotation of the system of coordinates:

F is normal w.r.t. any hyperplane in $\mathbb{R}^n$ through $0 \in \mathbb{R}^n$.

Then: F is CONVEX.

Then we have

$$F = \{ x \in \mathbb{R}^n : f_1(x') < x_n < f_2(x'),\ x' \in D \}$$

where $D \subset \mathbb{R}^{n-1}$ is convex and $f_1, -f_2 : D \to \mathbb{R}$ are convex functions.

Let $A \subset\subset \operatorname{int}(D)$ open. Then we have, by the Area Formula,

$$\|\partial F\|(A \times \mathbb{R}) = \int_A \sqrt{1+|Df_1|^2}\, dx' + \int_A \sqrt{1+|Df_2|^2}\, dx' .$$

Moreover

$$F^* = \left\{ x \in \mathbb{R}^n : |x_n| < \frac{1}{2}\left(f_2(x') - f_1(x') \right), x' \in D \right\}$$

and

$$\|\partial F^*\|(A \times \mathbb{R}) = 2 \int_A \sqrt{1 + \frac{1}{4}|D(f_2 - f_1)|^2}\, dx' .$$

By minimality we have $\|\partial F\|(A \times \mathbb{R}) = \|\partial F^*\|(A \times \mathbb{R})$ for any $A \subset\subset \operatorname{int}(D)$ open. We deduce that

$$\sqrt{1+|Df_1|^2} + \sqrt{1+|Df_2|^2} = 2\sqrt{1 + \frac{1}{4}|Df_2 - Df_1|^2} \iff$$

$$\iff 1 + |Df_1|^2 + 1 + |Df_2|^2 + 2\sqrt{1+|Df_1|^2}\sqrt{1+|Df_2|^2} = 4 + |Df_2 - Df_1|^2$$

$$\iff \sqrt{1+|Df_1|^2}\sqrt{1+|Df_2|^2} = 1 - Df_1 \cdot Df_2$$
$$= (1, Df_1) \cdot (1, -Df_2)$$

$$\iff Df_1 = -Df_2 \iff f_1 + f_2 = \text{constant}$$

Up to a translation: constant $= 0$ ($\Rightarrow f_2 = -f_1$).

Conclusion :

F is x_n- normal and x_n - symmetric.

This holds w.r.t. any hyperplane through $0 \in \mathbb{R}^n$.

Conclusion :

F is a ball.

Then we have :

$$\mathcal{L}^n(E) = \mathcal{L}^n(F) = C_n \, \|\partial F\|(\mathbb{R}^n)^{\frac{n}{n-1}} \leq C_n \, \|\partial E\|(\mathbb{R}^n)^{\frac{n}{n-1}}.$$

Moreover, if we have equality then E is a ball.

Second case : E is not bounded but $\mathcal{L}^n(E) < \infty$.

Recall that

$$\int_0^\infty H^{n-1}\left(E \cap B_r(0)\right) dr = \mathcal{L}^n(E) < \infty.$$

Fix $\varepsilon > 0$. There is $r > 0$ such that

(1) $$H^{n-1}(E \cap B_r(0)) < \varepsilon,$$

(2) $$\mathcal{L}^n(E) \leq \mathcal{L}^n(E \cap B_r(0)) + \varepsilon.$$

Moreover we have for $\mathcal{L}^1$-a.e. $r > 0$

(3) $$\|\partial(E \cap B_r(0))\|(\mathbb{R}^n) = \|\partial E\|(B_r(0)) + H^{n-1}(\partial B_r(0) \cap E).$$

We conclude that

$$\mathcal{L}^n(E) \leq \mathcal{L}^n(E \cap B_r(0)) + \varepsilon \leq$$

$$\leq C_n \|\partial(E \cap B_r(0))\|(\mathbb{R}^n)^{\frac{n}{n-1}} + \varepsilon \leq$$

$$\leq c_n\left(\|\partial E\|(B_r(0)) + H^{n-1}(\partial B_r(0) \cap E) \right)^{\frac{n}{n-1}} + \varepsilon$$

$$\leq c_n\left(\|\partial E\|(\mathbb{R}^n) + \varepsilon \right)^{\frac{n}{n-1}} + \varepsilon .$$

Letting $\varepsilon \downarrow 0$ we get the conclusion.

To finish the proof we need the following Lemma:

Proposition For any $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable we have

$$\min\{\mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\} \leq \|\partial E\|(\mathbb{R}^n).$$

In this proposition: $n \geq 2$.

This Lemma can be proved by induction on $n \geq 2$.

The proof is omitted.

$\square$