SCHWARZ REARRANGEMENT.

Let $f : \mathbb{R}^n \to [0,\infty)$ be a $\mathcal{L}^n$-measurable function such that for all $t>0$ we have

$$\mathcal{L}^n(E_t) < \infty, \quad \text{with} \quad E_t = \{x \in \mathbb{R}^n \; ; \; f(x) > t\}.$$

We have the following representation for f:

$$f(x) = \int_0^{f(x)} dt = \int_0^\infty \chi_{(0,f(x))}(t)\, dt$$

$$= \int_0^\infty \chi_{E_t}(x)\, dt .$$

In fact, $\chi_{E_t}(x) = 1 \iff \chi_{(0,f(x))}(t) = 1.$

For any $t>0$ let

$$E_t^* = B_r(0) \quad \text{with} \quad r>0$$

such that $\mathcal{L}^n(B_r(0)) = \mathcal{L}^n(E_t)$.

By the isoperimetric inequality

$$\|\partial E_t^*\|(\mathbb{R}^n) \leq \|\partial E_t\|(\mathbb{R}^n).$$

We define a function $f^* : \mathbb{R}^n \to [0,\infty)$ letting

$$f^*(x) = \int_0^\infty \chi_{E_t^*}(x)\, dt.$$

We call f^* the Schwarz rearrangement of f.

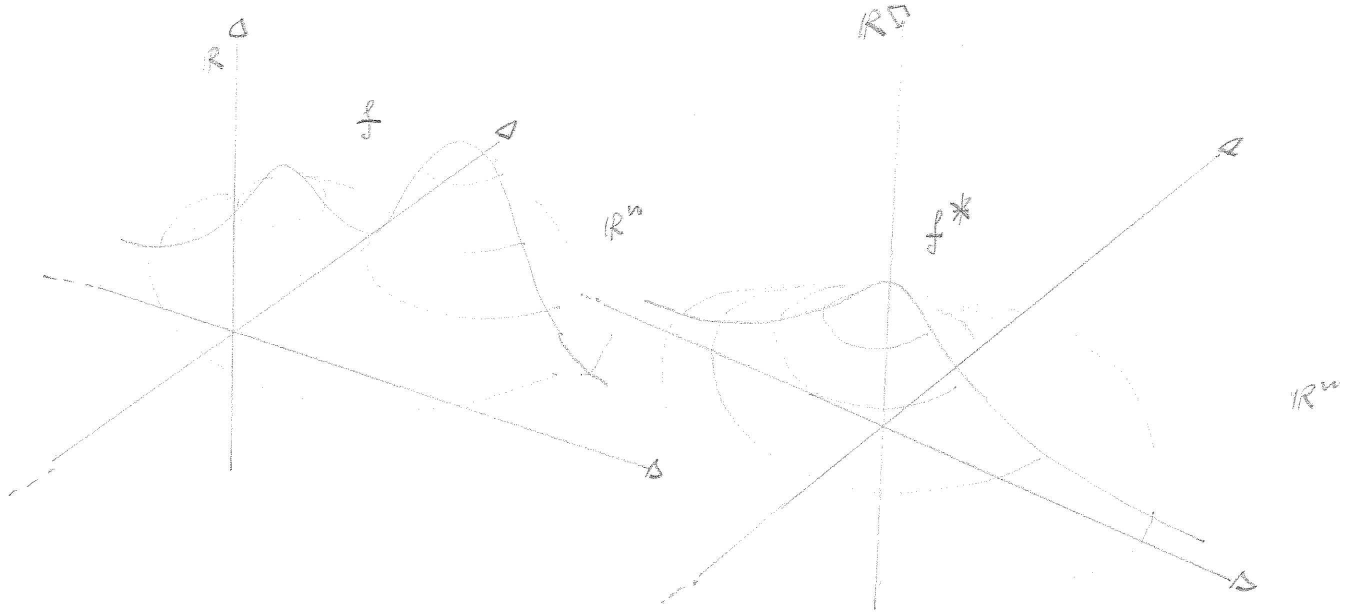
Elementary properties:

(1) f^* is $\mathcal{L}^n$-measurable;

(2) $f^*(x) = \varphi(|x|)$, $|x|>0$, for some $\varphi : (0,\infty) \longrightarrow [0,\infty)$;

(3) φ is decreasing.

(4) $E_t^* = \{ f^* > t \}$.



<u>Comment</u>:

(1) $f \in C^1 \not\Rightarrow f^* \in C^1$

(2) $f \in BV(\mathbb{R}^n) \Longrightarrow f^* \in BV(\mathbb{R}^n)$ (not easy)

(3) $f \in W^{1,p}(\mathbb{R}^n) \underset{(p>1)}{\Longrightarrow} f^* \in W^{1,p}(\mathbb{R}^n)$ (not easy).

<u>Remark</u> For any $p \geq 1$ we have

$$\int_{\mathbb{R}^n} |f^*(x)|^p \, dx = \int_{\mathbb{R}^n} |f(x)|^p dx .$$

Proof :

$$\int_{\mathbb{R}^n} f(x)^p \, dx = \int_{\mathbb{R}^n} \int_0^{f(x)^p} dt \; dx = \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{(0, f(x)^p)}(t) \, dt \, dx$$

$$= \int_0^{\infty} \int_{\{f(x)^p > t\}} dx \quad dt = \int_0^{\infty} \mathcal{L}^n \{ x \in \mathbb{R}^n : f(x) > t^{1/p} \} \, dt$$

$$= \int_0^{\infty} \mathcal{L}^n \{ x \in \mathbb{R}^n : f^*(x) > t^{1/p} \} \, dt = \int_{\mathbb{R}^n} f^*(x)^p \, dx .$$

<u>Proposition</u> Let $f \in \mathrm{Lip}(\mathbb{R}^n)$, $f \geq 0$, be rearrangeable. Then

$$\mathrm{Lip}(f^*) \leq \mathrm{Lip}(f).$$

Proof. Assume $\left| f(x) - f(y) \right| \leq L |x-y|$ for all $x, y \in \mathbb{R}^n$.

Let $x, y \in \mathbb{R}^n$ and let

$$t_1 = f^*(x),$$

$$t_2 = f^*(y) .$$

We may assume that $t_1 > t_2 (\geq 0)$. Let $\varepsilon > 0$ be such that

$$2\varepsilon < t_1 - t_2 .$$

Then we have

$$x \in \{ f^* > t_1 - \varepsilon \} = \{ f > t_1 - \varepsilon \}^* = B_{r_1}$$

$$y \notin \{ f^* > t_2 + \varepsilon \} = \{ f > t_2 + \varepsilon \}^* = B_{r_2}$$

for suitable $r_2 \geq r_1 \geq 0$. We deduce that

$$|x - y| \geq r_2 - r_1 .$$

Let

$$E_1 = \{ f > t_1 - \varepsilon \},$$

$$E_2 = \{ f > t_2 + \varepsilon \}.$$

Because $2\varepsilon < t_1 - t_2$, we infer that $E_1 \subset E_2$ and

$$\mathcal{L}^n(E_1) = \mathcal{L}^n(B_{r_1}),$$

$$\mathcal{L}^n(E_2) = \mathcal{L}^n(B_{r_2}).$$

Consider the number

$$r = \operatorname{dist}(E_1, \partial E_2) = \inf_{\substack{\bar{x} \in E_1 \\ \bar{y} \in \partial E_2}} |\bar{x} - \bar{y}| ,$$

Using the inclusion

$$E_1 + rB \subset E_2 ,$$

we obtain

$$\begin{array}{l} \mathcal{L}^n(B_{r_2}) = \mathcal{L}^n(E_2) \geq \mathcal{L}^n(E_1 + rB) \geq \\ \quad \parallel \\ r_2^n \, \mathcal{L}^n(B) \end{array}$$

$$\geq \left(\mathcal{L}^n(E_1)^{\frac{1}{n}} + \mathcal{L}^n(rB)^{\frac{1}{n}} \right)^n = \left(\mathcal{L}^n(B_{r_1})^{\frac{1}{n}} + \mathcal{L}^n(B_r)^{\frac{1}{n}} \right)^n =$$

$$= (r_1 + r)^n \, \mathcal{L}^n(B) .$$

The conclusion is $\quad r_2 \geq r_1 + r$, that is

$$r \leq r_2 - r_1 \leq |x-y| .$$

Take $\bar{x} \in E_1$ and $\bar{y} \in \partial E_2$ such that

$$|\bar{x} - \bar{y}| \leq r + \varepsilon .$$

In particular,

$$f(\bar{x}) > t_1 - \varepsilon ,$$

$$f(\bar{y}) \underset{(=)}{\leq} t_2 + \varepsilon$$

Thus

$$\left| f^*(x) - f^*(y) \right| = |t_1 - t_2| = t_1 - t_2 < f(\bar{x}) + \varepsilon - f(\bar{y}) + \varepsilon \leq$$

$$\leq L\,|\bar{x} - \bar{y}| + 2\varepsilon \leq L\,(r+\varepsilon) + 2\varepsilon \leq L\,(|x-y| + \varepsilon) + 2\varepsilon ,$$

Letting $\varepsilon \downarrow 0$ we obtain $\left| f^*(x) - f^*(y) \right| \leq L\,|x-y|$.

$\square$

<u>THEOREM</u> Let $p \geq 1$. For any $f \in \mathrm{Lip}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |\nabla f^*(x)|^p dx \leq \int_{\mathbb{R}^n} |\nabla f(x)|^p dx .$$

Proof. We assume w.l.o.g. $f \geq 0$. We know that $f^* \in \mathrm{Lip}(\mathbb{R}^n)$.

Case $p = 1$. By the coarea formula

$$\int_{\mathbb{R}^n} |\nabla f(x)| dx = \int_0^\infty \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > t\}) \, dt .$$

Let

$$E_t = \{x \in \mathbb{R}^n : f(x) > t\},$$

$$E_t^* = \{x \in \mathbb{R}^n ; f^*(x) > t\} . \qquad t > 0$$

Ball

We know that

$$\mathcal{L}^n(E_t) = \mathcal{L}^n(E_t^*)$$

$$\|\partial E_t\|(\mathbb{R}^n) \geq \|\partial E_t^*\|(\mathbb{R}^n).$$

It follows that

$$\int_{\mathbb{R}^n} |\nabla f| dx = \int_0^\infty \mathcal{H}^{n-1}(\partial\{f > t\}) dt \overset{\circledast}{\geq} \int_0^\infty \mathcal{H}^{n-1}(\partial\{f^* > t\}) \, dt = \int_{\mathbb{R}^n} |\nabla f^*| dx .$$

Remark. We have equality in $\circledast$ if and only if $E_t \subset \mathbb{R}^n$ is a ball for a.e. $t > 0$. These balls however need not be centered at 0 (need not have all the same center)

In this case: $\int_{\mathbb{R}^n} |\nabla f| = \int_{\mathbb{R}^n} |\nabla f^*|$ but f is not radially symmetric.

Case $p > 1$. In this case, the Coarea Formula gives:

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx = \int_0^\infty \int_{\underbrace{\partial\{f>t\}}_{= \partial E_t}} |\nabla f(x)|^{p-1} \, d\mathcal{H}^{n-1}(x) \, dt .$$

We need some preliminary observations. Notice that

$$\mathcal{L}^n(E_t) = \int_{E_t} \frac{1}{|\nabla f|} |\nabla f| \, dx$$

$$= \int_t^\infty \int_{\partial E_s} \frac{1}{|\nabla f|} \, d\mathcal{H}^{n-1}(x) \, ds$$

$$\mathcal{L}^n(E_t^*) = \int_t^\infty \int_{\partial E_s^*} \frac{1}{|\nabla f^*|} \, d\mathcal{H}^{n-1}(x) \, ds$$

$$= \int_t^\infty \underset{(\text{on } \partial E_s^*)}{\frac{1}{|\nabla f^*|}} \mathcal{H}^{n-1}(\partial E_s^*) \, ds$$

In fact, $|\nabla f^*|$ is constant on the spheres ∂E_s^*:

$$\left|\nabla f^*(x)\right| = \left|\varphi'(|x|)\frac{x}{|x|}\right| = |\varphi'(|x|)|.$$

$$f^*(x) = \varphi(|x|)$$

Differentiating the identity $\mathcal{L}^n(E_t) = \mathcal{L}^n(E_t^*)$ we obtain:

$$\int_{\partial E_t} \frac{1}{|\nabla f|}\, dH^{n-1} = \frac{1}{|\nabla f^*|} H^{n-1}(\partial E_t^*),$$

for a.e. $t>0$.

A second observation is the following:

$$H^{n-1}(\partial E_t) = \int_{\partial E_t} dH^{n-1} = \int_{\partial E_t} \frac{1}{|\nabla f|^{\frac{1}{p'}}} \cdot |\nabla f|^{\frac{1}{p'}}\, dH^{n-1} \leq$$

$$\overset{\text{Hölder Inequality}}{\leq} \left(\int_{\partial E_t} \frac{1}{|\nabla f|}\, dH^{n-1}\right)^{\frac{1}{p'}} \left(\int_{\partial E_t} |\nabla f|^{\frac{p}{p'}}\, dH^{n-1}\right)^{\frac{1}{p}} =$$

$$= \left(\int_{\partial E_t} \frac{1}{|\nabla f|}\, dH^{n-1}\right)^{\frac{p-1}{p}} \left(\int_{\partial E_t} |\nabla f|^{p-1}\, dH^{n-1}\right)^{\frac{1}{p}}.$$

Recall that $\frac{1}{p'} + \frac{1}{p} = 1 \iff p' = \frac{p}{p-1}$.

We deduce the inequality

$$\int_{\partial E_t} |\nabla f|^{p-1} \, dH^{n-1} \geq H^{n-1}(\partial E_t)^{p} \left(\int_{\partial E_t} \frac{1}{|\nabla f|} \, dH^{n-1} \right)^{1-p} \geq$$

$$\geq H^{n-1}(\partial E_t)^{p} \, |\nabla f^{*}|^{p-1} \, H^{n-1}(\partial E_t^{*})^{1-p} \qquad \text{Isop. inequality}$$

$$\geq |\nabla f^{*}|^{p-1} \, H^{n-1}(\partial E_t^{*}) =$$

$$= \int_{\partial E_t^{*}} |\nabla f^{*}|^{p-1} \, dH^{n-1} .$$

Integrating on $(0,\infty)$ we obtain the claim:

$$\int_{\mathbb{R}^n} |\nabla f|^{p} \, dx = \int_0^{\infty} \int_{\partial E_t} |\nabla f|^{p-1} \, dH^{n-1} \, dt \geq$$

$$\geq \int_0^{\infty} \int_{\partial E_t^{*}} |\nabla f^{*}|^{p-1} \, dH^{n-1} \, dt = \int_{\mathbb{R}^n} |\nabla f^{*}|^{p} \, dx .$$

$\square$