THE "BEST DRUM"

We sketch a proof of the following statement:

"The circular shape of a planar membrane with fixed area produces the lowest fundamental tone"

Let $A \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, $n \geq 2$ and in fact $n = 2$.

Consider the wave equation with 0 boundary condition

$$\Delta u = u_{tt} \qquad x \in A,\ t > 0,$$

$$u(x,t) = 0 \qquad x \in \partial A,\ t > 0.$$

We look for a solution

$$u(x,t) = f(x)\, g(t).$$

By separation of variables:

$$\Delta f(x) = -\lambda f(x) \qquad x \in A$$

$$f(x) = 0 \qquad x \in \partial A$$

and

$$g_{tt}(t) = -\lambda g(t),$$

where $\lambda > 0$.

The eigenvalues of the Δ-operator with 0-Dirichlet boundary value form a sequence

$$\{\lambda_h : h \in \mathbb{N}\} \qquad \lambda_h > 0, \quad \lambda_h \uparrow \infty \text{ as } h \to \infty$$

Let $\lambda(A) = \lambda_1$ be the first eigenvalue.
This is the square of the fundamental frequence of A.

Consider the equation $\Delta f = -\lambda f$, Multiply by
f and integrate

$$\int_A f(x)\, \Delta f(x)\, dx = -\lambda \int_A |f(x)|^2 dx .$$

Integrating by parts and using $f = 0$ on ∂A:

$$\int_A |\nabla f|^2 dx = \lambda \int_A |f|^2 dx$$

This is a Poincaré inequality with "=".
The variational characterization of the first eigenvalue
is

$$\lambda(A) = \inf \left\{ \frac{\int_A |\nabla f|^2 dx}{\int_A |f|^2 dx} \ ; \ f \in H_0^1(A),\ f \neq 0 \right\}$$

Direct method of the calculus of variation; the "inf" is a "min".
Then we have the sharp Poincaré Inequality:

$$\int_A |f|^2 dx \leq \frac{1}{\lambda(A)} \int_A |\nabla f|^2 dx .$$

Let $f \neq 0$ be the eigenfunction of $\lambda(A) = \lambda_1$. We may assume $f > 0$ in A.

Let $B = A^*$ be the ball with center $0 \in \mathbb{R}^n$ such that $\mathcal{L}^n(B) = \mathcal{L}^n(A)$.

Let f^* be the Schwarz rearrangement of f.

We have:

$$(1) \quad \int_B |f^*|^2 \, dx = \int_A |f|^2 \, dx ,$$

$$(2) \quad \int_B |\nabla f^*|^2 \, dx \leq \int_A |\nabla f|^2 \, dx .$$

It follows that:

$$\lambda_1(B) \leq \lambda_1(A).$$

$\square$

QUANTITATIVE ISOPERIMETRIC INEQUALITY

Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a set such that $\mathcal{L}^n(E) = \mathcal{L}^n(B) = \alpha(n)$, where $B = B_1(0)$ is the unit ball.

The isoperimetric deficit of E is

$$D(E) = \frac{\|\partial E\|(\mathbb{R}^n) - \|\partial B\|(\mathbb{R}^n)}{\|\partial B\|(\mathbb{R}^n)} .$$

We have:

(1) $D(E) \geq 0$;

(2) $D(E) = 0$ if and only if E is a unit ball.

The asymmetry index of E is

$$\lambda(E) = \min \left\{ \frac{\mathcal{L}^n(E \,\Delta\, B_1(x))}{\mathcal{L}^n(B_1)} \;;\; x \in \mathbb{R}^n \right\}.$$

The asymmetry index measures how far E is from a ball.

<u>THEOREM (Fusco, Maggi, Pratelli 2008)</u> There is a dimensional constant $c(n) > 0$ such that for any Borel set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) = \alpha(n)$ we have

$$D(E) \geq c(n)\, \lambda(E)^2 .$$