Proof of the theorem $\mathcal{H}^n = \mathcal{L}^n$.

- We claim that $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$ for any $A \subset \mathbb{R}^n$.

Let $\delta > 0$ and let $C_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$, be sets such that

$$\operatorname{diam}(C_j) < \delta, \quad A \subset \bigcup_{j=1}^{\infty} C_j .$$

Then we have

$$\mathcal{L}^n(A) \leq \mathcal{L}^n\Big(\bigcup_{j=1}^{\infty} C_j\Big) \leq \sum_{j=1}^{\infty} \mathcal{L}^n(C_j) \leq \text{(Isod. Ineq.)}$$

$$\leq \sum_{j=1}^{\infty} \alpha(n)\left(\frac{\operatorname{diam} C_j}{2}\right)^n .$$

Taking the infimum: $\mathcal{L}^n(A) \leq \mathcal{H}^n_\delta(A)$.

Letting $\delta \to 0$ we get the claim.

- Now we prove the opposite inequality: $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$.

Let $\delta > 0$. We use the following definition for the Lebesgue measure:

$$\mathcal{L}^n(A) = \inf\Big\{ \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) : Q_j \subset \mathbb{R}^n \text{ cubes}, \ \operatorname{diam}(Q_j) < \delta, \ A \subset \bigcup_{j=1}^{\infty} Q_j \Big\}.$$

Notice that for some constant $C_n > 0$ we have

$$C_n \mathcal{L}^n(Q_j) = \alpha(n)\left(\frac{\operatorname{diam}(Q_j)}{2}\right)^n .$$

It follows that

$$\mathcal{H}^n_\delta(A) \leq C_n \inf\{\text{as above}\} = C_n \mathcal{L}^n(A)$$

and thus $\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A)$. In particular, $\mathcal{L}^n(A) = 0 \Rightarrow \mathcal{H}^n(A) = 0$.

Let $\varepsilon, \delta > 0$ and choose cubes Q_j' such that

$$\operatorname{diam}(Q_j') < \delta, \quad A \subset \bigcup_{j=1}^{\infty} Q_j' \quad \text{and} \quad \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j') \leq \mathcal{L}^n(A) + \varepsilon .$$

$$\boxed{\begin{array}{l}\text{Now we need the Vitali covering Lemma:} \\ \\ \underline{\text{Lemma}} \text{ Let } \Omega \subset \mathbb{R}^n \text{ be open and let } \delta > 0. \text{ There exist (closed) disjoint} \\ \text{balls } B_i \subset \Omega, i \in \mathbb{N}, \text{ such that } \operatorname{diam}(B_i) < \delta \text{ and} \\ \qquad\qquad \mathcal{L}^n\left(\Omega \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0 . \qquad\qquad \text{see [EG] p. 28}\end{array}}$$

Then for any cube Q_j' there are closed disjoint balls $B_j^i \subset \operatorname{int}(Q_j')$ such that $\operatorname{diam}(B_j^i) < \delta$ and

$$\mathcal{L}^n\left(\operatorname{int}(Q_j') \setminus \bigcup_{i=1}^{\infty} B_j^i\right) = 0 .$$

It follows that
$$\mathcal{H}^n\left(\operatorname{int}(Q_j') \setminus \bigcup_{i=1}^{\infty} B_j^i\right) = 0 .$$

Then we have

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(Q_j') = \sum_{j=1}^{\infty} \mathcal{H}^n_\delta\left(\bigcup_{i=1}^{\infty} B_j^i\right) \\
&\leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mathcal{H}^n_\delta(B_j^i) \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \alpha(n)\left(\frac{\operatorname{diam}(B_j^i)}{2}\right)^n = \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mathcal{L}^n(B_j^i) = \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j') \\
&\leq \mathcal{L}^n(A) + \varepsilon .
\end{aligned}$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, we get the claim.

HAUSDORFF DIMENSION OF THE CANTOR SET

Let $0 < \lambda < 1/2$ be a parameter.

Let $T_1 : [0,1] \longrightarrow [0,1]$ be the map $T_1(x) = \lambda x$,

Let $T_2 : [0,1] \longrightarrow [0,1]$ be the map $T_2(x) = \lambda x + (1-\lambda)$.

Both T_1 and T_2 are contractions by a factor λ.

Let $T : \mathcal{P}([0,1]) \longrightarrow \mathcal{P}([0,1])$ be the map

$$T(E) = T_1(E) \cup T_2(E).$$

Let $E_0 = [0,1]$ and define by induction $E_{n+1} = T(E_n)$, $n \in \mathbb{N}$.
It can be checked that

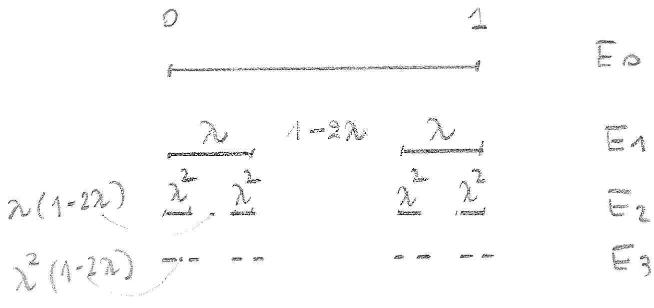
$$E_{n+1} \subset E_n \quad , \quad n \in \mathbb{N}.$$

The intersection

$$K = \bigcap_{n=0}^{\infty} E_n$$

is a nonempty compact set in $[0,1]$. In fact, $K = T(K)$ is the unique fixed point of T in the class of nonempty compact subsets of $[0,1]$. (Proof omitted)

The picture is as follows:



In general, we have

$$E_n = E_n^1 \cup \dots \cup E_n^{2^n} \quad \text{(ordered in the natural way)}$$

with $\operatorname{diam}(E_n^i) = \lambda^n$ and $\operatorname{dist}(E_n^1, E_n^2) = \lambda^{n-1}(1-2\lambda)$.

Let $s \in (0,1)$ the solution of the equation

$$2\lambda^s = 1 \iff \log 2 + s \log \lambda = 0$$
$$\iff s = \frac{\log(2)}{\log(1/\lambda)},$$

<u>THEOREM</u> $\mathcal{H}_{\dim}(K) = s$ and $\mathcal{H}^s(K) = \dfrac{\alpha(s)}{2^s}$.

Proof. Let $\delta > 0$ and choose $n \in \mathbb{N}$ s.t. $\lambda^n < \delta$.
Then
$$K \subset \bigcup_{i=1}^{2^n} E_n^i \implies \mathcal{H}^s_\delta(K) \le \sum_{i=1}^{2^n} \alpha(s) \left(\frac{\operatorname{diam} E_n^i}{2} \right)^s =$$
$$= \frac{\alpha(s)}{2^s} \lambda^{ns} 2^n = \frac{\alpha(s)}{2^s} \underbrace{(\lambda^s 2)^n}_{=1}$$

Letting $\delta \to 0$: $\mathcal{H}^s(K) \le \dfrac{\alpha(s)}{2^s}$.

We claim that $\mathcal{H}^s(K) \ge \dfrac{\alpha(s)}{2^s}$.
It is enough to prove that if $\bigcup_{j=1}^\infty C_j \supset K$ then
$$\frac{\alpha(s)}{2^s} \le \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s .$$
We can assume that each C_j is an (open) interval.
By compactness, we can assume to have a finite covering.
We have to show:
$$K \subset \bigcup_{I \in \mathcal{F}} I \implies 1 \le \sum_{I \in \mathcal{F}} \operatorname{diam}(I)^s$$
with $\mathcal{F}$ finite family of intervals.
We can also assume that each $I \in \mathcal{F}$ is of the form

$$I = \underset{\overset{\shortparallel}{A}}{I_n^i} \cup \underset{\text{interval}}{J} \cup \underset{\overset{\shortparallel}{B}}{I_m^j} \qquad \text{for some } n, m \in \mathbb{N} \text{ and } i, j .$$

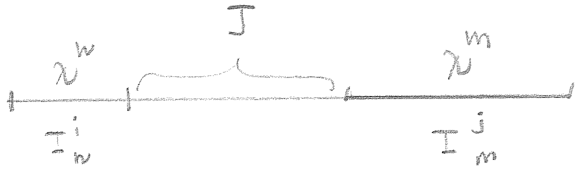
with essentially disjoint union.

Assume that $n \geq m$. We have

$$\operatorname{diam} A = \operatorname{diam} I_n^i = \lambda^n$$
$$\operatorname{diam} B = \operatorname{diam} I_m^j = \lambda^m \geq \operatorname{diam} A$$

We estimate $\operatorname{diam} J$. By construction, we have

$$\operatorname{diam} J \geq \lambda^{m-1}(1-2\lambda) = \frac{1-2\lambda}{\lambda} \operatorname{diam} B$$



Then

$$\begin{aligned} \operatorname{diam} I &= \operatorname{diam} A + \operatorname{diam} J + \operatorname{diam} B \\ &\geq \operatorname{diam} A \left\{ 1 + \frac{1}{2} \frac{1-2\lambda}{\lambda} \right\} + \operatorname{diam} B \left\{ 1 + \frac{1}{2} \frac{1-2\lambda}{\lambda} \right\} = \\ &= \frac{1}{2\lambda} \left(\operatorname{diam} A + \operatorname{diam} B \right) \end{aligned}$$

and

$$(\operatorname{diam} I)^s \geq \underset{\substack{\| \\ 2}}{\left(\frac{1}{\lambda^s}\right)} \left(\frac{\operatorname{diam} A + \operatorname{diam} B}{2} \right)^s \underset{\substack{\text{concavity} \\ \text{of } t \mapsto t^s,\, t \geq 0. \\ (0<s<1)}}{\geq} (\operatorname{diam} A)^s + (\operatorname{diam} B)^s$$

Then in $\sum_{I \in \mathcal{F}} \operatorname{diam}(I)^s$ we may replace I with A and B, the sum gets smaller.

We can repeat the argument several times and we can assume that $\mathcal{F} = \{E_n^1, \dots, E_n^{2^n}\}$ for some $n \in \mathbb{N}$.

In this case, we know that $\sum_{I \in \mathcal{F}} (\operatorname{diam} I)^s = 1$.

This finishes the proof.

LIPSCHITZ FUNCTIONS

Let $n, m \geq 1$ and let $A \subset \mathbb{R}^n$.
A function $f : A \longrightarrow \mathbb{R}^m$ is Lipschitz if there is $L \geq 0$ such that

$$(*) \qquad \left| f(x) - f(y) \right| \leq L \, |x-y| \qquad \forall x, y \in A.$$

The Lipschitz constant of f is

$$\operatorname{Lip}(f) = \inf \{ L \geq 0 \, ; \, (*) \text{ holds} \}.$$

Extension theorem: $f : A \longrightarrow \underset{(m=1)}{\mathbb{R}}$ Lipschitz $\Longrightarrow \exists \hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lip. with $\hat{f} = f$ on A and $\operatorname{Lip}(\hat{f}) = \operatorname{Lip}(f)$.

Proof:
$$\hat{f}(x) = \inf_{y \in A} \left\{ f(y) + \operatorname{Lip}(f) \, |x-y| \right\}$$

Kirzbraun Theorem $f : A \longrightarrow \mathbb{R}^m$ Lip. $\Longrightarrow \exists \hat{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Lip. with $\hat{f} = f$ on A and $\operatorname{Lip}(\hat{f}) = \operatorname{Lip}(f)$

See Federer 2.10.43