That is:

$$T(f) \leq \|f\|_\infty \, T(J) \quad \forall f \in C_c(\mathbb{R}^n), \ spt(f) \subset K.$$

Then

$$\sup \{ T(f) : \underset{\|f\|_\infty \leq 1}{f \in C_c(\mathbb{R}^n), \ spt(f) \subset K} \} \leq T(J) < \infty,$$

The claim follows by Riesz theorem.

$\square$

HAUSDORFF MEASURES IN $\mathbb{R}^n$

Let $\alpha(n)$ denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$

$$\alpha(n) = \mathcal{L}^n(B_1(0))$$
$$= \text{Leb. measure } \{x \in \mathbb{R}^n : |x| < 1\}.$$

We claim that

$$(*) \qquad \alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-s} \, ds$ is the Gamma function.

We prove $(*)$:

$$\pi^{n/2} = \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n = \int_{\mathbb{R}^n} e^{-|x|^2} dx \overset{\text{Coarea Formula}}{=} \int_0^\infty \Big(\int_{|x|=r} e^{-r^2} \underset{\text{surface measure}}{d\sigma} \Big) dr$$

$$= \int_0^\infty e^{-r^2} \, n \, \alpha(n) \, r^{n-1} \, dr$$

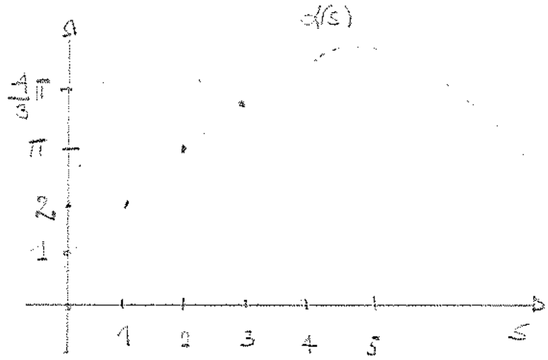
because $\sigma(\{x \in \mathbb{R}^n : |x| = r\}) = \frac{d}{dr} \mathcal{L}^n(\{|x| < r\}) =$

$$= \frac{d}{dr} \alpha(n) r^n = n \alpha(n) r^{n-1}.$$

So we get

$$\pi^{n/2} = n\alpha(n) \int_0^\infty e^{-r^2} r^{n-1} dr \overset{r^2=t}{=} \frac{n}{2} \alpha(n) \int_0^\infty e^{-t} t^{\frac{n}{2}-1} dt$$

$$= \frac{n}{2} \alpha(n) \Gamma\left(\frac{n}{2}\right) = \alpha(n) \Gamma\left(\frac{n}{2}+1\right).$$



<u>NOTATION</u> For any $s \geq 0$, we let

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$$

<u>DEFINITIONS</u> (Hausdorff premeasures and measures)

(1) Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$ and $\delta > 0$. Define:

$$H^s_\delta(A) = \inf \left\{ \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s : A \subset \bigcup_{j=1}^\infty C_j,\ C_j \subset \mathbb{R}^n \text{ sets},\ \operatorname{diam}(C_j) \leq \delta \right\}$$

(2) Let $A \subset \mathbb{R}^n$ and $s \geq 0$. Define

$$H^s(A) = \lim_{\delta \downarrow 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A).$$

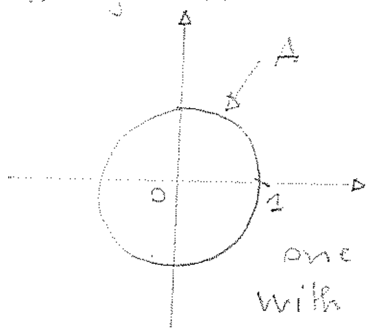
We call H^s the s-dimensional Hausdorff measure on $\mathbb{R}^n$.

Example. Let $A \subset \mathbb{R}^2$, $A = \{x \in \mathbb{R}^2 : |x| = 1\}$ the unit circle.

When $\delta \geq 1$ is big we have

$$\mathcal{H}^1_\delta(A) = 2,$$

that is not what we wish.

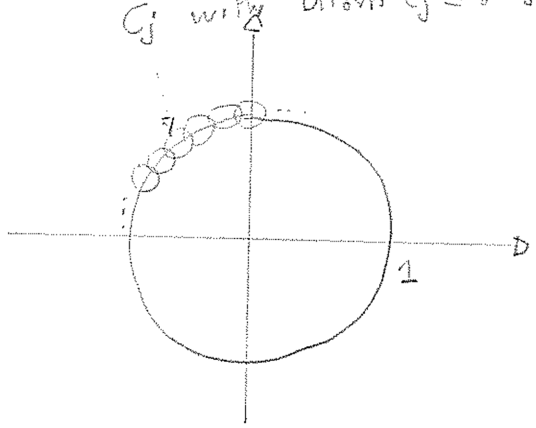


one set with diam = 2 covers A (take A itself)

However, when $0 < \delta \ll 1$ is small

$$H^1_\delta(A) \sim 2\pi$$

C_j with diam $C_j \leq \delta$ small



Remark We may assume C_j to be closed (compact), because in passing from C_j to $\overline{C_j}$ the diam does not increase.

THEOREM 1 For any $s \geq 0$, $\mathcal{H}^s$ is a Borel regular (outer) measure on $\mathbb{R}^n$.

Proof.

- Claim: H^s_δ is an outer measure for all $\delta > 0$.

Let A_k, $k \in \mathbb{N}$, be sets in $\mathbb{R}^n$ and let

$$A_k \subset \bigcup_{j=1}^{\infty} C^j_k \quad \text{with diam } C^j_k < \delta.$$

The family $\{C^j_k\}_{j,k \in \mathbb{N}}$ is a covering of $\bigcup_{k=1}^{\infty} A_k$;

$$H^s_\delta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C^j_k}{2}\right)^s$$

Taking the infimum for each fixed k;

$$H^s_\delta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} H^s_\delta(A_k)$$

- Claim: H^s is an outer measure.

For any $\delta > 0$:

$$H^s_\delta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} H^s_\delta(A_k) \leq \sum_{k=1}^{\infty} H^s(A_k)$$

Let $\delta \downarrow 0$.

- Claim: H^s is a Borel measure.

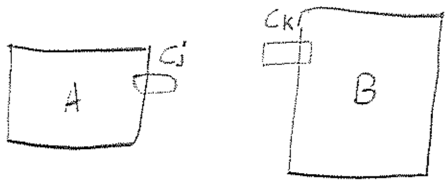
Let $A, B \subset \mathbb{R}^n$ be such that $\operatorname{dist}(A,B) > 0$ and choose $0 < \delta < \frac{1}{4}\operatorname{dist}(A,B)$.

Let $A \cup B \subset \bigcup_{j=1}^{\infty} C_j$ with $\operatorname{diam} C_j < \delta$.

Define

$$\mathcal{A} = \{ j \in \mathbb{N} : C_j \cap A \neq \emptyset \},$$
$$\mathcal{B} = \{ j \in \mathbb{N} : C_j \cap B \neq \emptyset \}.$$

Then $j \in \mathcal{A}$ and $k \in \mathcal{B} \Rightarrow C_j \cap C_k = \emptyset$



Now
$$\sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s \geq \sum_{j\in\mathcal{A}} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s + \sum_{k\in\mathcal{B}} \alpha(s)\left(\frac{\operatorname{diam} C_k}{2}\right)^s$$
$$\geq H^s_\delta(A) + H^s_\delta(B)$$

Take the infimum :

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

Let $\delta \downarrow 0$:

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

It follows $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$.

By Carathéodory II : $\mathcal{H}^s$ is a Borel (outer) measure.

- Claim : $\mathcal{H}^s$ is Borel regular

Let $A \subset \mathbb{R}^n$
We can assume that $\mathcal{H}^s(A) < \infty$ (and so $\mathcal{H}^s_\delta(A) < \infty \quad \forall \delta > 0$).
Choose $\delta = \frac{1}{k}$, $k \in \mathbb{N}$. There exists <u>closed</u> sets $C_j^k \subset \mathbb{R}^n$ such that

$$\bigcup_{j=1}^\infty C_j^k \supset A, \qquad \operatorname{diam}(C_j^k) < \frac{1}{k} \quad \text{and}$$

$$\mathcal{H}^s_{1/k}(A) \geq \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2}\right)^s - \frac{1}{k}.$$

Let $B_k = \bigcup_{j=1}^\infty C_j^k$ Borel set and $B = \bigcap_{k=1}^\infty B_k$ Borel set.

Then $A \subset B$

$$\mathcal{H}^s_{1/k}(B) \leq \sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j^k}{2}\right)^s \leq \mathcal{H}^s_{1/k}(A) + \frac{1}{k}$$

Let $k \to \infty$:

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A) \qquad \text{with} \quad A \subset B,$$

It follows : $\mathcal{H}^s(B) = \mathcal{H}^s(A)$. $\square$

THEOREM 2 (Properties of H^s)

(i) $H^0(A) = \mathrm{Card}(A)$ for any $A \subset \mathbb{R}^n$ (H^0 is the counting measure);

(ii) $H^1 = \mathcal{L}^1$ on $\mathbb{R}$ ($\mathcal{L}^1$ = Lebesgue measure on $\mathbb{R}$);

(iii) $H^s = 0$ on $\mathbb{R}^n$ for all $s > n$,

(iv) $H^s(\lambda A) = \lambda^s H^s(A)$ for all $\lambda > 0$ and $A \subset \mathbb{R}^n$;

(v) $H^s(T(A)) = H^s(A)$ for any $A \subset \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ isometry.

(i) $\alpha(0) = \left.\dfrac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}\right|_{s=0} = 1 \implies \underset{\text{with } s=0}{H^s(\{x\})} = 1 \quad \forall x \in \mathbb{R}^n$.

(ii) Let $\delta > 0$ and $A \subset \mathbb{R}$;

$$\mathcal{L}^1(A) = \inf\left\{ \sum_{j=1}^{\infty} \mathrm{diam}\, C_j \,;\ A \subset \bigcup_{j=1}^{\infty} C_j, \quad C_j \subset \mathbb{R} \text{ interval} \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \mathrm{diam}\, C_j \,;\ A \subset \bigcup_{j=1}^{\infty} C_j,\ C_j \subset \mathbb{R} \text{ interval},\ \mathrm{diam}\, C_j < \delta \right\}$$

$$= \inf\left\{ \sum_{j=1}^{\infty} \mathrm{diam}\, C_j \,;\ A \subset \bigcup_{j=1}^{\infty} C_j,\ C_j \subset \mathbb{R},\ \mathrm{diam}\, C_j < \delta \right\}$$

$$= H^1_\delta(A).$$

(iii) Let $Q = [0,1]^n$. We have $[0,1] = \bigcup_{i=1}^{K} \underbrace{\left[\frac{i-1}{K}, \frac{i}{K}\right]}_{I_K^i}$, $K \in \mathbb{N}$.

Let

$$Q_K = \left\{ I_K^{i_1} \times \dots \times I_K^{i_n} \,;\ i_1, \dots, i_n = 1, \dots, K \right\}$$

$$\mathrm{Card}\, Q_K = K^n$$

$$C \in Q_K \implies \mathrm{diam}\, C = \frac{1}{K}\sqrt{n}$$

For $0 < \frac{\sqrt{n}}{K} < \delta$;

$$H^s_\delta(Q) \le \sum_{C \in Q_K} \alpha(s) \left(\frac{\mathrm{diam}(C)}{2}\right)^s = \alpha(s) \frac{1}{2^s} \left(\frac{\sqrt{n}}{K}\right)^s \cdot K^n$$

$$= \frac{\alpha(s)}{2^s} n^{s/2} \frac{1}{K^{s-n}} \xrightarrow[K \to \infty]{} 0 \quad \text{because } s > n.$$

Then $H^s_\delta(Q) = 0 \ \forall \delta > 0 \implies H^s(Q) = 0 \implies H^s(\mathbb{R}^n) = 0$.

Lemma 1 Let $A \subset \mathbb{R}^n$, $s>0$ and $\delta>0$. Then:

$$\mathcal{H}^s_\delta(A) = 0 \implies \mathcal{H}^s(A) = 0 .$$

Proof. Let $\varepsilon>0$. There exist sets $C_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$, such that:

$$A \subset \bigcup_{j=1}^{\infty} C_j , \quad \operatorname{diam} C_j < \delta , \quad \sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^s < \varepsilon$$

In particular, we have:

$$\operatorname{diam}(C_j) < 2\left(\frac{\varepsilon}{\alpha(s)}\right)^{\frac{1}{s}} := \delta(\varepsilon) .$$

Notice that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Then we conclude

$$\mathcal{H}^s_{\delta(\varepsilon)}(A) < \varepsilon ,$$

and letting $\varepsilon \to 0$ we get $\mathcal{H}^s(A) = 0$.

Lemma 2 Let $A \subset \mathbb{R}^n$ and $0 \leq s < t < \infty$. Then:

(1) $\mathcal{H}^s(A) < \infty \implies \mathcal{H}^t(A) = 0$;

(2) $\mathcal{H}^t(A) > 0 \implies \mathcal{H}^s(A) = \infty$.

Proof (1) $\Rightarrow$ (2) is clear. We prove (1).

For any $\delta>0$ there exist sets $C_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$, such that

$$\operatorname{diam}(C_j) < \delta , \quad A \subset \bigcup_{j=1}^{\infty} C_j , \quad \sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^s < \mathcal{H}^s_\delta(A) + 1 \leq \mathcal{H}^s(A) + 1$$

Then we have

$$\mathcal{H}^t_\delta(A) \leq \sum_{j=1}^{\infty} \alpha(t)\left(\frac{\operatorname{diam} C_j}{2}\right)^t \leq \frac{\alpha(t)}{\alpha(s)} \frac{\delta^{t-s}}{2^{t-s}} \sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s ,$$

that is

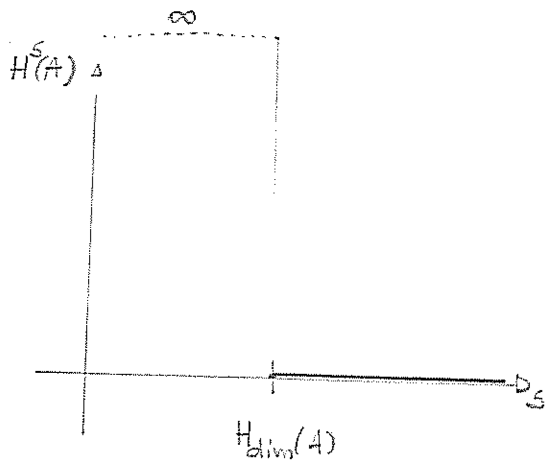
$$H^t_\delta(A) \leq \delta^{t-s} \frac{\alpha(t)}{\alpha(s)} \frac{1}{2^{t-s}} \left(H^s(A) + 1 \right),$$

and letting $\delta \to 0$ we get $H^t(A) = 0$.

$\square$

<u>DEFINITION</u> The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is

$$H_{dim}(A) = \dim_H(A) = \inf\{ t > 0 \; ; \; H^t(A) = 0 \}$$
$$= \sup\{ s > 0 \; ; \; H^s(A) = \infty \}.$$



<u>Remark</u> Let $s = H_{dim}(A)$, $A \subseteq \mathbb{R}^n$. Then $0 \leq s \leq n$.
It can be: $H^s(A) = 0$; $0 < H^s(A) < \infty$; $H^s(A) = \infty$.

Next, we shall prove the following theorem

<u>THEOREM</u> On $\mathbb{R}^n$ we have $H^n = \mathcal{L}^n$, where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.

We need some preparation.

SOME GEOMETRIC INEQUALITIES

In the next weeks, we shall prove the Brunn-Minkowski inequality:
The Minkowski sum of two sets $A, B \subset \mathbb{R}^n$ is

$$A + B = \{ x + y \in \mathbb{R}^n \; : \; x \in A \text{ and } y \in B \}.$$

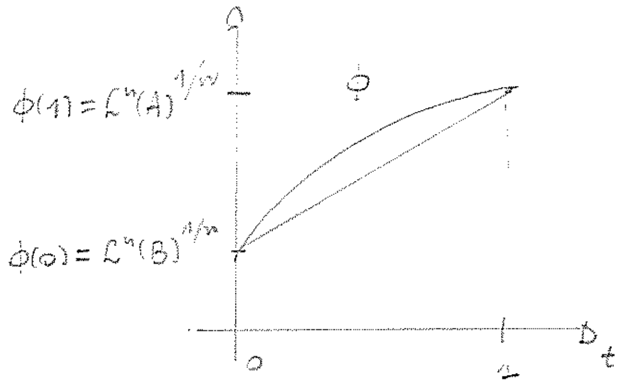
THEOREM For all sets $A, B \subset \mathbb{R}^n$, $n \geq 1$, there holds

$$\mathcal{L}^n(A+B)^{1/n} \geq \mathcal{L}^n(A)^{1/n} + \mathcal{L}^n(B)^{1/n}.$$

Proof: Postponed.

Remark In other words, the function $t \mapsto \phi(t) = \mathcal{L}^n(tA + (1-t)B)^{1/n}$, $t \in [0,1]$ is concave:

$$\phi(t) \geq \mathcal{L}^n(tA)^{\frac{1}{n}} + \mathcal{L}^n((1-t)B)^{\frac{1}{n}} = t\,\phi(1) + (1-t)\,\phi(0)$$



Here, we are interested in the isodiametric inequality:

THEOREM (Isodiametric Inequality) For any set $A \subset \mathbb{R}^n$ we have

$$(*) \qquad \mathcal{L}^n(A) \leq \alpha(n) \left(\frac{\operatorname{diam}(A)}{2} \right)^n.$$

Remark:
The theorem can be improved: in (*) there is equality if and only if A is a ball.

Proof of (*). Let us consider the set

$$B = \frac{1}{2}A - \frac{1}{2}A = \left\{ \frac{x}{2} - \frac{y}{2} \in \mathbb{R}^n : x, y \in A \right\}.$$

Then we have:

(1) $\operatorname{diam}(B) \leq \operatorname{diam}(A)$;

(2) $x \in B \implies -x \in B$;

(3) The Brunn-Mink. inequality implies:

$$\mathcal{L}^n(B)^{1/n} \geq \mathcal{L}^n\left(\frac{1}{2}A\right)^{1/n} + \mathcal{L}^n\left(-\frac{1}{2}A\right)^{\frac{1}{n}} =$$

$$= \frac{1}{2}\mathcal{L}^n(A)^{1/n} + \frac{1}{2}\mathcal{L}^n(A)^{1/n} = \mathcal{L}^n(A)^{1/n},$$

that implies $\mathcal{L}^n(B) \geq \mathcal{L}^n(A)$.

From (2) we deduce:

$$\operatorname{diam}(B) \geq |x-(-x)| = 2|x| \quad \text{for all } x \in B.$$

In other words:

$$B \subset \left\{ x \in \mathbb{R}^n ; \ |x| \leq \frac{1}{2}\operatorname{diam}(B) \right\}$$

and thus:

$$\mathcal{L}^n(B) \leq \alpha(n) \left(\frac{\operatorname{diam}(B)}{2} \right)^n,$$

Finally, we get

$$\mathcal{L}^n(A) \leq \mathcal{L}^n(B) \leq \alpha(n)\left(\frac{\operatorname{diam}(B)}{2}\right)^n \leq \alpha(n)\left(\frac{\operatorname{diam}(A)}{2}\right)^n.$$

$\square$