<u>Rademacher Theorem</u> Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then f is (Fréchet) differentiable $\mathcal{L}^n$ - almost everywhere.

Proofs

(1) First $n=1$ case: Lebesgue theorem: $f: [0,1] \to \mathbb{R}$ monotone $\Longrightarrow$ $f'(x)$ exists finite for $\mathcal{L}^1$ a.e. $x \in [0,1]$.

This has to do with differentiation of measures.
Assume f strictly increasing, let $g = f^{-1}$ (Borel) and define $\mu = g_{\#} \mathcal{L}^1$;

$$\mu(a,b) = \mathcal{L}^1(g^{-1}(a,b)) \overset{g \text{ cont.}}{=} g^{-1}(b) - g^{-1}(a)$$

$$= f(b) - f(a).$$

Then
$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = \lim_{\delta \to 0} \frac{\mu(x, x+\delta)}{\mathcal{L}^1(x, x+\delta)}.$$

Then $n=1$ case $\Rightarrow$ $n>1$ case (Rademacher)

(2) Use Morrey inequality:

$$|f(y) - f(x)| \leq C_{n,p} |x-y|^{1-\frac{n}{p}} \left(\int_{B_r(x)} |\nabla f|^p \, dz \right)^{\frac{1}{p}}$$

$$r = |x-y|$$

for any $f \in W^{1,p}(\mathbb{R}^n)$, <u>$p > n$</u>. See [EG] p. 143

Rademacher theorem follows for any $W^{1,p}(\mathbb{R}^n)$ function with $p > n$. See [EG] p. 235

(3) "Blow up" argument for sets with finite perimeter

(4) Cheeger - Keith theorem

Approximation theorem Let $f: \mathbb{R}^n \to \mathbb{R}$ be Lipschitz.

For any $\varepsilon > 0$ there is $g \in C^1(\mathbb{R}^n)$ such that:

$$\mathcal{L}^n \{ x \in \mathbb{R}^n : f(x) \neq g(x) \text{ or } \nabla f(x) \neq \nabla g(x) \} < \varepsilon.$$

See [EG] p. 251. The proof uses Whitney Extension Theorem.

Lipschitz $\equiv W^{1,\infty}$ Let $\Omega \subset \mathbb{R}^n$ be a convex bounded open set and let $f: \Omega \to \mathbb{R}$. The following are equivalent:

(1) f is Lipschitz;

(2) $f \in W^{1,\infty}(\Omega)$ (after redefinition on a negligible set).

Some open questions

Question 1 Let $A \subset \mathbb{R}^n$ be a set such that $\mathcal{L}^n(A) = 0$.

A) • Does there exists $f: \mathbb{R}^n \to \mathbb{R}^m$, $m \geq 1$, such that f is not differentiable on A?

B) • Does there ... not differentiable precisely on A?

Answers: A) Yes if $n = 1$. Yes if $n = m = 2$. We do not know for $n \geq 3$

B) If $n = 1$: B) holds if and only if $\mathcal{L}^n(A) = 0$ AND A is a $G_{\delta\sigma}$ set; $A = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$, A_{ij} open

If $n \geq 2$: No if $m = 1$. Because of the following Example by Preiss

<u>D. Preiss' Example</u>: There exists a (Borel) set $A \subset \mathbb{R}^2$ such that $\mathcal{L}^n(A) = 0$ and any Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable in at least one point of A.

<u>Question 2</u> Let $A \subset \mathbb{R}^n$ be a set such that $\mathcal{L}^n(A) > 0$. Does there exist a Lipschitz function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $f(A) = \{x \in \mathbb{R}^n : |x| \leq 1\}$?

<u>Answer</u> Yes, if $n=1$, Exercise
Yes, if $n=2$, (Preiss)
We do not know, if $n \geq 3$.

For references see Mattila on p. 102 and p. 106

<u>Lipschitz curves in metric spaces</u>

Let (X,d) be a metric space. A curve $\gamma: [0,1] \longrightarrow X$ is Lipschitz if for some $L > 0$; $d(\gamma(t), \gamma(s)) \leq L|t-s|$ for all $t,s \in [0,1]$.

Lipschitz curves are rectifiable:

$$\mathrm{Var}(\gamma) = \sup_{0 = t_0 < t_1 < \ldots < t_n = 1} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) \leq L < \infty .$$

<u>Theorem</u> Let $\gamma: [0,1] \to X$ be Lipschitz. The <u>metric derivative</u>

$$|\dot{\gamma}|(t) := \lim_{\delta \to 0} \frac{d(\gamma(t+\delta), \gamma(t))}{|\delta|}$$

exists for $\mathcal{L}^1$-a.e. $t \in [0,1]$. The function $t \longmapsto |\dot{\gamma}|(t)$ is measurable and

$$\mathrm{Var}(\gamma) = \int_0^1 |\dot{\gamma}|(t)\, dt .$$

(This a special case of the "Area Formula").

Definition For any $K \subset X$ let

$$\mathcal{H}^1(K) = \sup_{\delta>0} \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(C_j) \ : \ \begin{array}{l} C_j \subset X \text{ set}, \ K \subset \bigcup_{j=1}^{\infty} C_j, \\ \operatorname{diam}(C_j) < \delta \end{array} \right\}.$$

Theorem Let $\gamma : [0,1] \longrightarrow X$ be Lipschitz and injective. Then

$$\operatorname{Var}(\gamma) = \mathcal{H}^1(\gamma([0,1])).$$

For these results see Ambrosio - Tilli, Topics on Analysis in metric spaces -

We have three different notions of "length of a curve" and they coincide.

Lipschitz functions on metric spaces

When the "source space" of Lipschitz functions is not finite dimensional there is no "clear" measure measuring the exceptional set of nondifferentiability.

Theorem (D, Preiss) Let $(X, \|\cdot\|)$ be a Banach space such that $x \longmapsto \|x\|$ is differentiable on $X \setminus \{0\}$. Then any Lipschitz function $f : X \longrightarrow \mathbb{R}$ is differentiable on a dense subset of X.

See Preiss,

Now we illustrate the Cheeger - Keith theory.

Let (X, d) be a metric space and let μ be a Borel measure such that for some $\delta > 0$ there holds

$$0 < \mu(B_{2r}(x)) \leq \delta \, \mu(B_r(x)) < \infty \qquad \forall x \in X, \ \forall r > 0$$

Such measures are called "doubling".

Let $f: X \to \mathbb{R}$ be a Lipschitz function. For any $x \in X$ let

$$\operatorname{lip} f(x) = \liminf_{r \downarrow 0} \sup_{d(x,y)<r} \frac{|f(x)-f(y)|}{r},$$

$$\operatorname{Lip} f(x) = \limsup_{r \downarrow 0} \sup_{d(x,y)<r} \frac{|f(x)-f(y)|}{r}.$$

<u>Def (Lip-lip condition)</u> The measure metric space (X, d, μ) satisfies the Lip-lip condition if there exists $K>0$ such that

$$\operatorname{Lip} f \leq K \operatorname{lip} f \qquad \mu\text{-a.e.}$$

for any Lipschitz function $f: X \to \mathbb{R}$.

<u>Theorem</u> Let (X, d, μ) be a measure metric space.

Assume:

1) X is locally compact;
2) μ is doubling;
3) Lip-lip condition.

Then there exists a number $N \in \mathbb{N}$, $N = N(\delta, K)$, there exist measurable sets $A_\alpha \subset X$, $\alpha \in \mathbb{N}$, with $\mu(X \setminus \bigcup_{\alpha \in \mathbb{N}} A_\alpha) = 0$, there exist functions $\varphi_\alpha : A_\alpha \xrightarrow{\text{Lip.}} \mathbb{R}^{N_\alpha}$, $N_\alpha \in \mathbb{N}$ with $N_\alpha \leq N$, such that:

for any $f: X \to \mathbb{R}$ Lipschitz, for μ-a.e. $x \in A_\alpha$ there exists a unique $df_\alpha(x) \in \mathbb{R}^{N_\alpha}$ such that

$$\lim_{\substack{d(y,x)\to 0 \\ y \in A_\alpha}} \frac{f(y) - f(x) - df_\alpha(x)\cdot(\varphi_\alpha(y) - \varphi_\alpha(x))}{d(x,y)} = 0.$$

Reference: S. Keith, A differentiable structure for metric measure spaces.