Differentiation of Radon measures

Let μ, ν be Radon measures on $\mathbb{R}^n$.

Notation: $B(x,r) = \{y \in \mathbb{R}^n : d(y,x) \leq r\}$ closed balls, in this section,

$$\boxed{B_r(x) = B(x,r)}$$

Definition For any $x \in \mathbb{R}^n$ let

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r)) > 0 \ \forall r > 0 \\ \infty & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0 \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r \to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{As above.} \\ \infty \end{cases}$$

Definition If $\underline{D}_\mu \nu(x) = \overline{D}_\mu \nu(x) < \infty$ we say that ν is differentiable w.r.t. μ at $x \in \mathbb{R}^n$ and we call

$$D_\mu \nu(x) = \underline{D}_\mu \nu(x) = \overline{D}_\mu \nu(x),$$

the derivative of ν w.r.t. μ at $x \in \mathbb{R}^n$.

We are going to prove the following theorems:

Theorem 1 Let μ, ν be Radon measures on $\mathbb{R}^n$. Then $D_\mu \nu(x)$ exists and is finite for μ-a.e. $x \in \mathbb{R}^n$. Moreover, the function $x \longmapsto D_\mu \nu(x)$ is μ-measurable.

<u>Definition</u> Let μ, ν be outer measures on $\mathbb{R}^n$. We say that ν is absolutely continuous w.r.t. μ if

$$\mu(A) = 0 \implies \nu(A) = 0 \quad \text{for all } A \subset \mathbb{R}^n.$$

In this case, we write $\nu \ll \mu$.

<u>Theorem 2</u> Let μ, ν be Radon measures on $\mathbb{R}^n$ such that $\nu \ll \mu$. Then for any μ-measurable set $A \subset \mathbb{R}^n$ we have

$$\nu(A) = \int_A D_\mu \nu(x) \, d\mu(x).$$

In the proof, we need a corollary of Besicovitch covering theorem.

<u>Theorem</u> (Besicovitch) There is a dimensional constant N depending on $n \in \mathbb{N}$ with the following property. Let $\mathcal{F}$ be a family of nondegenerate closed balls in $\mathbb{R}^n$ such that

$$\sup\{\operatorname{diam} B \,;\, B \in \mathcal{F}\} < \infty.$$

Let A be the set of the centers of the balls in $\mathcal{F}$.
Then there are $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}$ subfamilies such that each $\mathcal{G}_i$ is countable and disjoint and

$$A \subset \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_i} B.$$

For the proof see [EG] on page 30.

Besicovitch's covering theorem has the following corollary:

<u>Theorem</u> Let μ be a Borel measure on $\mathbb{R}^n$, let $\mathcal{F}$ be a family of nondegenerate closed balls in $\mathbb{R}^n$, Let A be the set of the centers. Assume that:

(1) $\mu(A) < \infty$

(2) $\inf\{r>0 : B(x,r) \in \mathcal{F}\} = 0$ for all $x \in A$, ("Fine covering")

Then for any open set $U \subset \mathbb{R}^n$ there exists a disjoint, countable subfamily $\mathcal{G} \subset \mathcal{F}$ such that $\bigcup_{B\in\mathcal{G}} B \subset U$ and

$$\mu\Big((A\cap U)\setminus \bigcup_{B\in\mathcal{G}} B\Big) = 0 .$$

For the proof see [EG] p. 35

<u>Lemma</u> Let μ, ν be (outer) Radon measures on $\mathbb{R}^n$. Let $0<\alpha<\infty$, Then:

(i) $A \subset \{x\in\mathbb{R}^n : \underline{D}_\mu \nu(x) \le \alpha\} \implies \nu(A) \le \alpha\mu(A)$;

(ii) $A \subset \{x\in\mathbb{R}^n : \overline{D}_\mu \nu(x) \ge \alpha\} \implies \nu(A) \ge \alpha\mu(A)$.

Proof. Without loss of generality, we assume $\mu(\mathbb{R}^n)<\infty$ and $\nu(\mathbb{R}^n)<\infty$.
Let $\varepsilon>0$ and let $U\subset\mathbb{R}^n$ be an open set such that $A\subset U$.
For $x\in A$ we have

$$\liminf_{r\downarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \le \alpha \implies \exists\, r_k>0,\ r_k\downarrow 0 \text{ as } k\to\infty$$
$$\text{and } \nu(B(x,r_k)) \le (\alpha+\varepsilon)\mu(B(x,r_k))$$

Consider

$$\mathcal{F} = \{B : B \text{ closed ball centered on } A,\ B\subset U,\ \nu(B)\le(\alpha+\varepsilon)\mu(B)\}$$

Then for any $x\in A$:
$$\inf\{r>0 : B(x,r)\in\mathcal{F}\} = 0 .$$

Then there exists $\mathcal{G} \subset \mathcal{F}$ countable and disjoint such that

$$\nu\Big(A \setminus \bigcup_{B\in\mathcal{G}} B\Big) = 0 .$$

It follows that:

$$\nu(A) \le \underbrace{\nu\Big(A\setminus \bigcup_{B\in\mathcal{G}} B\Big)}_{\substack{\| \\ 0}} + \nu\Big(\bigcup_{B\in\mathcal{G}} B\Big) = \sum_{B\in\mathcal{G}} \nu(B) \le$$

$$\le \sum_{B\in\mathcal{G}} (\alpha+\varepsilon)\,\mu(B) = (\alpha+\varepsilon)\,\mu\Big(\bigcup_{B\in\mathcal{G}} B\Big) \le$$

$$\le (\alpha+\varepsilon)\,\mu(U).$$

Because $\mu(A) = \inf\limits_{\substack{A\subset U \\ \text{open}}} \mu(U)$, it follows that

$$\nu(A) \le (\alpha+\varepsilon)\,\mu(A) .$$

Letting $\varepsilon \downarrow 0$ we get $\nu(A) \le \alpha\,\mu(A)$.

This proves (i). The proof of (ii) is analogous.

Proof of Theorem 1 W.l.g. we can assume that $\mu(\mathbb{R}^n) < \infty$ and $\nu(\mathbb{R}^n) < \infty$.

Claim: $D_\mu \nu$ exists and is finite μ-a.e.

Let $I = \{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) = \infty\}$. Then for any $\alpha > 0$:

$$\{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\} \supset I \qquad \forall \alpha > 0$$

$$\Downarrow$$

$$\mu(I) \leq \frac{1}{\alpha} \nu(I) \qquad \forall \alpha > 0$$

$$\Downarrow$$

$$\mu(I) = 0.$$

For $0 < \alpha < \beta < \infty$ let

$$R_{\alpha,\beta} = \{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) \leq \alpha < \beta \leq \overline{D}_\mu \nu(x)\}$$

From the Lemma it follows:

$$\left.\begin{array}{l} \nu(R_{\alpha\beta}) \leq \alpha\, \mu(R_{\alpha\beta}) \\ \nu(R_{\alpha\beta}) \geq \beta\, \mu(R_{\alpha\beta}) \end{array}\right\} \Rightarrow \mu(R_{\alpha,\beta}) = 0$$

because $\alpha < \beta$.

Then we have:

$$\{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x)\} = \bigcup_{\substack{0 < \alpha < \beta < \infty \\ \alpha, \beta \in \mathbb{Q}}} R(\alpha, \beta)$$

countable union

$$\Downarrow$$

$$\mu\{\underline{D}_\mu \nu < \overline{D}_\mu \nu\} = 0$$