It follows that $\underline{D}_\mu \nu = \overline{D}_\mu \nu < \infty$ μ-a.e.

<u>CLAIM</u> The function $x \longmapsto D_\mu \nu(x)$ is μ-measurable.

We first show that for any fixed $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\limsup_{y \to x} \mu(B(y,r)) \leq \mu(B(x,r)).$$

That is:

$x \longmapsto \mu(B(x,r))$ is upper semicontinuous ($\Longrightarrow$ Borel measurable)

Let $y_k \xrightarrow[k\to\infty]{} x$ be any sequence. Let

$$f_k(y) = \chi_{B(y_k,r)}(y),$$

$$f(y) = \chi_{B(x,r)}(y), \quad y \in \mathbb{R}^n.$$

Because $B(x,r)$ is closed, we have

$$\limsup_{k \to \infty} f_k(y) \leq f(y) \qquad \forall y \in \mathbb{R}^n.$$

It follows that

$$\liminf_{k \to \infty} (1 - f_k(y)) \geq 1 - f(y) \qquad \forall y \in \mathbb{R}^n,$$

By Fatou Lemma:

$$\int_{B(x,2r)} (1-f)\, d\mu \leq \int_{B(x,2r)} \liminf_{k\to\infty} (1-f_k)\, d\mu \overset{\text{Fatou}}{\leq}$$

$$\leq \liminf_{k \to \infty} \int_{B(x,2r)} (1 - f_k)\, d\mu$$

that is

$$\mu(B(x,2r)) - \mu(B(x,r)) \leq \liminf_{k\to\infty} \left(\mu(B(x,2r)) - \mu(B(y_k,r)) \right).$$

As $\mu(B(x,2r)) < \infty$ this is equivalent with

$$\mu(B(x,r)) \geq \limsup_{k\to\infty} \mu(B(y_k,r)).$$

We are ready to prove the claim: For $r>0$ let

$$f_r(x) = \begin{cases} \dfrac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ \infty & \text{if } \mu(B_r(x)) = 0. \end{cases}$$

The function $x \mapsto f_r(x)$ is μ-measurable.
Then also

$$D_\mu \nu = \lim_{r \downarrow 0} f_r = \lim_{k\to\infty} f_{1/k}$$

is μ-measurable.

$\square$

Proof of Theorem 2. We can assume: $\mu(\mathbb{R}^n) < \infty$, $\nu(\mathbb{R}^n) < \infty$.
Let $A \subset \mathbb{R}^n$ be μ-measurable. There is $B \in \mathcal{B}(\mathbb{R}^n)$ such that $A \subset B$ and $\mu(B \setminus A) = 0 \overset{\nu \ll \mu}{\Longrightarrow} \nu(B\setminus A) = 0$

$\Rightarrow$ A ν-measurable.

Let

$$I = \{ x \in \mathbb{R}^n ; \ D_\mu \nu(x) = \infty \} \quad \mu\text{-meas}$$

$$Z = \{ x \in \mathbb{R}^n ; \ D_\mu \nu(x) = 0 \} \quad \mu\text{-meas}$$

From the Lemma we know that $\mu(I)=0 \Rightarrow \nu(I)=0$
Moreover, we have

$$\nu(Z) \leq \alpha\mu(Z) \quad \forall \alpha>0 \quad \Rightarrow \quad \nu(Z)=0.$$

It follows that

$$\nu(I)=0 = \int_I D_\mu \nu \, d\mu$$
$$\nu(Z)=0 = \int_Z D_\mu \nu \, d\mu .$$

Now let $A \subset \mathbb{R}^n$ be measurable. For $m \in \mathbb{Z}$ let

$$A_m = \{ x \in A ; \, t^m \leq D_\mu \nu < t^{m+1} \}$$

where $t>1$ is fixed. Then

$$\nu \left(A \setminus \bigcup_{m\in\mathbb{Z}} A_m \right) = 0 .$$

It follows that

$$\nu(A) = \nu\left(\bigcup_{m\in\mathbb{Z}} A_m\right) = \sum_{m\in\mathbb{Z}} \nu(A_m) \leq$$

$$\leq \sum_{m\in\mathbb{Z}} t^{m+1} \mu(A_m) \leq \sum_{m\in\mathbb{Z}} t \int_{A_m} D_\mu \nu \, d\mu$$

$$= t \int_{\bigcup_{m\in\mathbb{Z}} A_m} D_\mu \nu \, d\mu = t \int_A D_\mu \nu \, d\mu .$$

With a similar proof:

$$\nu(A) \geq \frac{1}{t} \int_A D_\mu \nu \, d\mu,$$

Letting $t \downarrow 1$ we obtain

$$\nu(A) = \int_A D_\mu \nu \, d\mu.$$

<u>Definition</u> We say that two Radon measures μ and ν on $\mathbb{R}^n$ are mutually singular (or "orthogonal") if there exists $B \subset \mathbb{R}^n$ Borel set such that

$$\mu(\mathbb{R}^n \setminus B) = 0 \quad \text{and} \quad \nu(B) = 0.$$

In this case, we write $\mu \perp \nu$.

<u>Theorem 3 (Decomposition)</u>

Let μ, ν be Radon measures on $\mathbb{R}^n$,

(1) There are Radon measures ν_{ac} and ν_s on $\mathbb{R}^n$ such that:

$$\nu = \nu_{ac} + \nu_s, \quad \nu_{ac} << \mu, \quad \text{and} \quad \nu_s \perp \mu.$$

(2) Moreover, we have $D_\mu \nu = D_\mu \nu_{ac}$, $D_\mu \nu_s = 0$ and

$$\nu(A) = \int_A D_\mu \nu_{ac} \, d\mu + \nu_s(A)$$

for any Borel set $A \subset \mathbb{R}^n$.

Proof. We assume $\mu(\mathbb{R}^n) < \infty$, $\nu(\mathbb{R}^n) < \infty$.

Let

$$\mathcal{E} = \{ A \subset \mathbb{R}^n : A \text{ Borel set}, \ \mu(\mathbb{R}^n \setminus A) = 0 \}.$$

For any $k \in \mathbb{N}$ there exists $B_k \in \mathcal{E}$ such that

$$\nu(B_k) \leq \inf_{A \in \mathcal{E}} \nu(A) + \frac{1}{k}.$$

The set

$$B = \bigcap_{k=1}^{\infty} B_k$$

is a Borel set and

$$\mu(\mathbb{R}^n \setminus B) = \mu\left(\bigcup_{k=1}^{\infty} \mathbb{R}^n \setminus B_k \right) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus B_k) = 0,$$

We therefore have $B \in \mathcal{E}$.

Let us define

$$\nu_{ac} = \nu \llcorner B \qquad \text{Radon measures.}$$
$$\nu_s = \nu \llcorner (\mathbb{R}^n \setminus B)$$

We claim that $\nu_{ac} \ll \mu$. Let $A \subset \mathbb{R}^n$ be a Borel set such that $\mu(A) = 0$. Then $B \cap (\mathbb{R}^n \setminus A) = B \setminus A \in \mathcal{E}$

$$\nu(B) = \min_{A \in \mathcal{E}} \nu(A) \Rightarrow \nu(B) \leq \nu(B \setminus A).$$
$$\nu(B) = \nu(B \setminus A) + \nu(A \cap B)$$

We deduce that

$$\nu_{ac}(A) = \nu(A \cap B) = 0.$$