Finally, we have

$$\left.\begin{array}{l} \nu_s(B) = \nu(B \cap (\mathbb{R}^n \setminus B)) = 0 \\ \mu(B) = 0 \end{array}\right\} \Rightarrow \quad \nu_s \perp \mu .$$

□

The Lebesgue-Besicovitch differentiation theorem is a corollary of the differentiation theorem for Radon measures.

Theorem Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n; \mu)$. Then for μ-a.e. $x \in \mathbb{R}^n$ we have:

$$\lim_{r \downarrow 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) = f(x).$$

The details of the proof are omitted. See [EG] p. 43.

Decomposition of the derivative of BV functions

Let $U \subset \mathbb{R}^n$ be an open set and let $f \in BV_{loc}(U)$. Then there is a Radon measure μ on U and $\sigma : U \to \mathbb{R}^n$ μ-measurable such that

$$\int_U f \operatorname{div} \varphi \, dx = - \int_U \varphi \cdot \sigma \, d\mu$$

for any $\varphi \in C^1_c(U; \mathbb{R}^n)$. Equivalently, we have

$$\int_U f \frac{\partial \varphi}{\partial x_i} \, dx = - \int_U \varphi \, \sigma^i \, d\mu$$

for any $\varphi \in C^1_c(U)$. Here, $\sigma = (\sigma^1, \dots, \sigma^n)$.

Notation We let

$$\| Df \| = \mu \qquad \text{Radon measure}$$

$$[Df] = \sigma \mu \qquad \text{Vector valued Radon measure}$$

We also let

$$\mu^i = \sigma^i \mu \qquad i = 1, \dots, n.$$

This is a signed Radon measure.

By the decomposition theorem there are signed Radon measures μ^i_{ac} and μ^i_s in U such that

$$\mu^i = \mu^i_{ac} + \mu^i_s \, , \qquad \mu^i_{ac} \ll \mathcal{L}^n \text{ and } \mu^i_s \perp \mathcal{L}^n$$

By the differentiation theorem for Radon measures there are functions $g_i \in L^1_{loc}(U)$ such that

$$\mu^i_{ac} = g_i \, \mathcal{L}^n, \qquad i = 1, \dots, n.$$

We let

$$\frac{\partial f}{\partial x_i} := g_i$$

$$Df = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$[Df]_{ac} = Df \, \mathcal{L}^n = (\mu^1_{ac}, \dots, \mu^n_{ac})$$

$$[Df]_s = (\mu^1_s, \dots, \mu^n_s)$$

Example 1 Let $f \in W^{1,1}_{loc}(U)$ with $U \subset \mathbb{R}^n$ open set.

Then for any open set V with $\underset{compact}{\overline{V}} \subset U$ we have

$$\int_U f \operatorname{div} \varphi \, dx = -\int_U Df \cdot \varphi \, dx \leq \int_V |Df| \, dx < \infty$$

for $\varphi \in C^1_c(V; \mathbb{R}^n)$ with $|\varphi| \leq 1$, where $Df \in L^1_{loc}(U; \mathbb{R}^n)$ is the weak gradient of f.

It follows that $f \in BV_{loc}(U)$ and moreover

$$\|Df\| = |Df| \, \mathcal{L}^n, \quad \text{and}$$

$$\sigma = \begin{cases} \dfrac{Df}{|Df|} & \text{if } |Df| \neq 0, \\ 0 & \text{if } |Df| = 0. \end{cases}$$

In fact, for any $\varphi \in C^1_c(U; \mathbb{R}^n)$:

$$\int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \overbrace{\frac{Df}{|Df|}}^{=\sigma} |Df| \, dx .$$

This shows that

$$W^{1,1}_{loc}(U) \subset BV_{loc}(U) \qquad (\text{also without "} loc \text{"})$$

and similarly

$$W^{1,p}_{loc}(U) \subset BV_{loc}(U) \qquad 1 \leq p \leq \infty .$$

In particular, for $f \in BV_{loc}(U)$ we have

$$f \in W^{1,1}_{loc}(U) \iff \mu^1_s = \dots = \mu^n_s = 0$$

Example 2 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then $f \in L^1_{loc}(\mathbb{R})$. For any $\varphi \in C^1_c(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = \int_0^\infty \varphi'(x)\,dx = \underset{\overset{\parallel}{0}}{\varphi(\infty)} - \varphi(0)$$

and so

$$\sup_{\substack{\varphi \in C^1_c(\mathbb{R}) \\ \|\varphi\|_\infty \leq 1}} \int_{\mathbb{R}} f(x)\varphi'(x)\,dx = 1 < \infty.$$

Then $f \in BV_{loc}(\mathbb{R})$.

Let $\mu = \delta_0$, the Dirac measure in 0. Equivalently, let $\mu = \mathcal{H}^0 \llcorner \{0\}$. Then

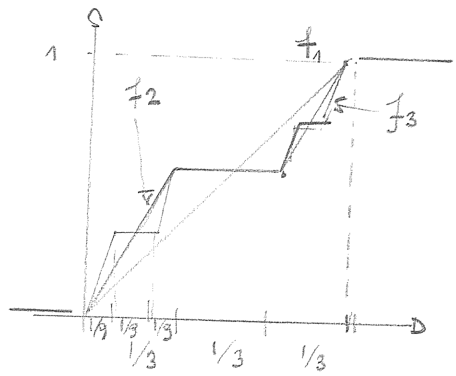
$$\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = -\int_{\mathbb{R}} \varphi(x)\,d\mu(x).$$

The distributional derivative of f is μ.

Notice that $\mu \perp \mathcal{L}^1$, that is $\mu_{ac} = 0$!

Example 3 Let $f : [0,1] \longrightarrow [0,1]$ be the Vitali function

$$f(x) = \lim_{\substack{k \to \infty \\ \text{uniform}}} f_k(x)$$



We extend $f = 0$ for $x \leq 0$ and $f = 1$ for $x \geq 1$.

Then $f \in L^1_{loc}(\mathbb{R}) \cap C(\mathbb{R})$. Clearly, we have $f \uparrow$.

It is known that

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}.$$

Let $\varphi \in C^1_c(\mathbb{R})$. Then if $\|\varphi\|_\infty \leq 1$

$$\int_{\mathbb{R}} f(x)\, \varphi'(x)\, dx = \lim_{k \to \infty} \int_{\mathbb{R}} f_k(x)\, \varphi'(x)\, dx =$$

f_k Lipschitz

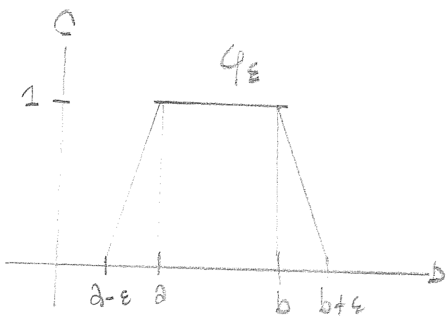
$$= - \lim_{k \to \infty} \int_{\mathbb{R}} f_k'(x)\, \varphi(x)\, dx$$

$$\leq \lim_{k \to \infty} \int_0^1 f_k'(x)\, dx = \lim_{k \to \infty} \left(f_k(1) - f_k(0) \right) = 1$$

It follows that $f \in BV_{loc}(\mathbb{R})$. There is a Radon measure μ on $\mathbb{R}$ such that $\forall \varphi \in C^1_c(\mathbb{R})$ We take $\sigma \equiv 1$

$$\int_{\mathbb{R}} f(x)\, \varphi'(x)\, dx = - \int_{\mathbb{R}} \varphi(x)\, d\mu .$$

Fix $a < b$ and $\varepsilon > 0$, Let $\varphi_\varepsilon \in Lip(\mathbb{R})$ be the function in the picture



Then we have:

$$-\int_{\mathbb{R}} \varphi_\varepsilon \, d\mu = \int_{\mathbb{R}} f(x) \, \varphi_\varepsilon'(x) \, dx = \int_{a-\varepsilon}^{a} f(x) \frac{1}{\varepsilon} dx + \int_{b}^{b+\varepsilon} f(x) \left(-\frac{1}{\varepsilon}\right) dx$$

letting $\varepsilon \downarrow 0$ we obtain

$$\mu([a,b]) = f(b) - f(a).$$

We differentiate μ w.r.t. $\mathcal{L}^1$. For $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$ we have

$$D_{\mathcal{L}^1} \mu(x) = \lim_{r \downarrow 0} \frac{\mu([x-r, x+r])}{\mathcal{L}^1([x-r, x+r])} = \lim_{r \downarrow 0} \frac{f(x+r) - f(x-r)}{2r} =$$

$$= \lim_{r \downarrow 0} \left\{ \frac{1}{2} \frac{f(x+r) - f(x)}{r} + \frac{1}{2} \frac{f(x) - f(x-r)}{r} \right\} = 0.$$

This means that $\mu_{ac} = 0$.

Conclusion: $f \in BV_{loc}(\mathbb{R})$ and $\underset{\text{measure}}{f'} \perp \mathcal{L}^1$.