Direct Method of the Calculus of Variations

Let X be a set and let $F: X \longrightarrow (-\infty, \infty]$ be a function, $F \not\equiv \infty$. We would like to see whether there exists $x_0 \in X$ such that

$$F(x_0) = \min \{ F(x) \; ; \; x \in X \}.$$

Existence

Fix a topology τ on X. $F: X \longrightarrow (-\infty, \infty]$ is lower-semicontinuous if for any $t \in \mathbb{R}$

$$\{ x \in X : F(x) > t \} \in \tau \quad (\text{is open}).$$

If τ is induced by a metric d on X, this is equivalent with

$$F(x) \leq \liminf_{\substack{y \to x \\ \text{in } (X,d)}} F(y) \quad \text{for any } x \in X.$$

Theorem If (X, τ) is compact and $F: X \longrightarrow (-\infty, \infty]$ is lower semicontinuous, then there is $x_0 \in X$ such $F(x_0) = \min \{ F(x) \; ; x \in X \}$.

Proof.

There is a real sequence $(t_n)_{n \in \mathbb{N}}$ such that:

(1) $t_{n+1} < t_n$ for all $n \in \mathbb{N}$;

(2) $\lim_{n \to \infty} t_n = \inf \{ F(x) \in \mathbb{R} ; x \in X \} \in [-\infty, \infty)$.

Let $A_n = \{ x \in X : F(x) > t_n \} \in \tau$. Notice that $A_{n+1} \supset A_n$.
By contradiction, assume that $F(x_0) \neq \inf \{ F(x) : x \in X \}$ for all $x_0 \in X$.
Then we have $X = \bigcup_{n=1}^{\infty} A_n$.

By compactness there is $N \in \mathbb{N}$ such that

$$X = \bigcup_{n=1}^{N} A_n = A_N.$$

This is a contradiction because there are points $x \in X$ with $F(x) \leq t_N$.

$\square$

Comments:

(1) (X,τ) is compact if τ contains few open sets.

(2) F is lower semicontinuous if τ contains many open sets.

So (1) and (2) are in competition.

Problem of Regularity

To make easier the problem of finding a point of minimum one can try to enlarge the space.

Let X be a set and let $F: X \longrightarrow (-\infty, \infty]$.

Let $\hat{X}$ be such that $X \subset \hat{X}$ and let $\hat{F}: \hat{X} \longrightarrow (-\infty, \infty]$ be such that $\hat{F} = F$ on X.

Let τ be a topology on $\hat{X}$ that makes $\hat{F}$ lower semicontinuous and $\hat{X}$ compact.

Then $\hat{F}$ has a point of minimum $\hat{x}_0 \in \hat{X}$.

Comment: If $\hat{x}_0 \in X$ then F attains the minimum on X. Showing that $\hat{x}_0 \in X$ is the "problem of regularity" in the Calculus of Variations.

Uniqueness

Assume that X has a linear structure and that $F: X \longrightarrow (-\infty, \infty]$ is strictly convex:

$$F(tx_1 + (1-t)x_0) < t\,F(x_1) + (1-t)\,F(x_0), \qquad x_0, x_1 \in X \quad x_0 \neq x_1 \quad 0<t<1 .$$

If F attains the minimum, then there is only one minimizer.

Necessary Conditions

Assume that X is a Banach space. If F is differentiable at a minimum point $x_0 \in X$ then

$$0 = dF(x_0)h = \lim_{t \to 0} \frac{F(x_0 + th) - F(x_0)}{t}$$

for all $h \in X$.

When X is a functional (Sobolev, BV) space, this leads typically to partial differential equations (Variational Approach).

Example 1 (Obstacle Problem)

Let $0 < r < R < \infty$. Let $X = \left\{ u \in Lip(\overline{B_R}) \; ; \; \begin{matrix} u(x) = 0 & \text{if } |x| = R \\ u(x) \geq 1 & \text{if } |x| \leq r \end{matrix} \right\}$.

Let $F : X \longrightarrow [0,\infty)$ be

$$F(u) = \int_{B_R} |\nabla u(x)|^2 \, dx,$$

$$B_R = \{ x \in \mathbb{R}^n \; ; \; |x| < R \}, \quad n \geq 1.$$

We study the problem

$$\min \{ F(u) \; ; \; u \in X \}.$$

Existence. Let $H_0^1(B_R) = \{ u \in H^1(B_R) \; ; \; u(x) = 0 \text{ for } |x| = R \text{ in the trace sense} \}$

where $u \in H^1(B_R) \iff u \in L^2(B_R)$ and $\nabla u \in L^2(B_R; \mathbb{R}^n)$, weak gradient.

Let $\widehat{X} = \{ u \in H_0^1(B_R) \; ; \; u(x) \geq 1 \text{ for } \mathcal{L}^n\text{-a.e. } x \in B_r \}$

$\widehat{F} : \widehat{X} \longrightarrow [0,\infty)$ is the natural extension.

We fix on $\hat{X}$ the weak topology of $H_0^1(B_R)$:

$$u_K \xrightarrow[K\to\infty]{\text{weakly } H_0^1(B_R)} u \quad \overset{\text{(Def.)}}{\Longleftrightarrow} \quad \int_{B_R} \nabla u_K \cdot \nabla v \, dx \xrightarrow[K\to\infty]{} \int_{B_R} \nabla u \cdot \nabla v \, dx$$

$$\forall v \in H_0^1(B_R)$$

$\hat{F}$ is lowersemicontinuous:

$$F(u) = \int_{B_R} |\nabla u|^2 dx = \sup_{\int_{B_R} |\nabla v|^2 \le 1} \left(\int_{B_R} \nabla u \cdot \nabla v \, dx \right)^2 =$$

$$= \sup_{\int_{B_R} |\nabla v|^2 \le 1} \lim_{K\to\infty} \left(\int_{B_R} \nabla u_K \cdot \nabla v \, dx \right)^2$$

Cauchy-Schwarz

$$\le \liminf_{K\to\infty} \int_{B_R} |\nabla u_K|^2 dx.$$

$\hat{X}$ itself is not compact (in the weak topology).

However,
$$\{ u \in \hat{X} \; : \; \hat{F}(u) \le M \}$$
is weakly compact for any $M < \infty$. Indeed, the sets
$$\{ u \in H_0^1(B_R) \; : \; \int_{B_R} |\nabla u|^2 dx \le M \}$$
are compact. Moreover

$$\left. \begin{array}{l} u_K \xrightarrow{\text{weakly } H_0^1(B_R)} u \\ \\ u_K \ge 1 \text{ a.e. on } B_r \end{array} \right\} \Longrightarrow u \ge 1 \text{ on } B_r.$$

To prove this implication we need the compact embedding

$$H_0^1(B_R) \underset{\text{compact}}{\subset\subset} L^2(B_R).$$

The details are left as an exercise.

<u>Conclusion</u>: $\hat{F}$ attains the minimum on $\hat{X}$.

Moreover: $\hat{X}$ is convex and $\hat{F} : \hat{X} \longrightarrow [0,\infty)$ is <u>strictly</u> convex;

Proof: the function $\xi \longmapsto |\xi|^2$ from $\mathbb{R}^n$ to $[0,\infty)$ is strictly convex. Let $u_0, u_1 \in H_0^1(B_R)$ be such that $u_0 \not\equiv u_1$ it follows that (Exercise)

$$\nabla u_0 \not\equiv \nabla u_1 \quad \text{on a set of positive measure.}$$

Then for $t \in (0,1)$ we have:

$$\int_{B_R} |\nabla(t u_1 + (1-t) u_0)|^2 dx < \int_{B_R} \{ t |\nabla u_1|^2 + (1-t) |\nabla u_0|^2 \} dx.$$

<u>Conclusion</u>: $\hat{F}$ has only one minimizer on $\hat{X}$.

Let u be the minimizer.

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an orthogonal transformation.
The function $v(x) = u(T(x))$, $x \in B_R$, is in $\hat{X}$.
Moreover we have

$$\int_{B_R} |\nabla v(x)|^2 dx = \int_{B_R} |\nabla u(x)|^2 dx.$$

It follows that $u = u \circ T$ for any $T \in O(n)$.

Then $u(X) = \varphi(|X|)$ for some $\varphi : [r, R] \longrightarrow \mathbb{R}$ such that $\varphi(r) = 1$ and $\varphi(R)$. The function φ is at least continuous (a posteriori: $\varphi \in C^\infty([r, R])$).

<u>Necessary condition</u> Let $v \in C_c^\infty(r < |X| < R)$.

Then for any $\varepsilon \in \mathbb{R}$: $u + \varepsilon v \in \hat{X}$.

It follows that $\forall \varepsilon \in \mathbb{R}$:

$$\theta(0) := \int_{B_R} |\nabla u|^2 dx \leq \int_{B_R} |\nabla(u+\varepsilon v)|^2 dx = \theta(\varepsilon).$$

Clearly $\theta \in C^\infty(\mathbb{R})$. It must be $\theta'(0) = 0$.

Now

$$\theta'(\varepsilon) = \frac{d}{d\varepsilon} \int_{B_R} \left\{ |\nabla u|^2 + 2\varepsilon \langle \nabla u, \nabla v \rangle + \varepsilon^2 |\nabla v|^2 \right\} dx$$

$$= 2 \int_{B_R} \left\{ \langle \nabla u, \nabla v \rangle + \varepsilon |\nabla v|^2 \right\} dx$$

We find the necessary condition

$$0 = \theta'(0) = 2 \int_{B_R} \langle \nabla u, \nabla v \rangle \, dx \qquad \forall v \in C_c^\infty(r < |X| < R)$$

Weakly harmonic functions are harmonic in the classical sense:

$$\int_{B_R} \langle \nabla u, \nabla v \rangle dx = 0 \;\; \forall v \in C_c^\infty(r < |X| < R) \implies u \in C^\infty(\{r < |X| < R\})$$

and $\Delta u = 0$ in $\{r < |X| < R\}$.

We know that $u(X) = \varphi(|X|)$. After some computation (write Δ in spherical coordinates) we obtain:

$$\Delta u = 0 \iff \varphi''(t) + \frac{n-1}{t} \varphi'(t) = 0 \qquad t \in [r, R].$$

We also have the conditions $\varphi(r) = 1$ and $\varphi(R) = 0$.
We integrate the differential equation:

$$\varphi'' + \frac{n-1}{t}\varphi' = 0 \iff t^{n-1}\varphi'' + (n-1)\, t^{n-2}\varphi' = 0$$
$$\iff (t^{n-1}\varphi')' = 0$$
$$\iff t^{n-1}\varphi' = c_0 = \text{constant}$$
$$\iff \varphi' = c_0\, t^{1-n}$$

Assume now $n \geq 3$;

$$\varphi' = c_0 t^{1-n} \iff \varphi = C_1 t^{2-n} + C_2 \,, \quad C_1, C_2 \in \mathbb{R}$$

Using $\varphi(r) = 1$ and $\varphi(R) = 0$ we determine C_1 and C_2.
The solution (the minimizer) is

$$u(x) = \begin{cases} C_1 |x|^{2-n} + C_2 & r \leq |x| \leq R, \\ 1 & |x| \leq r. \end{cases}$$

Clearly $u \in Lip(B_R)$