Example 2 (Exercise)

Let $X = \{u \in Lip(\overline{B_R}) : u(x) = 0$ for $|x| = R$ and $u(x) \geq 1$ if $|x| \leq r\}$.

Let $F, G : X \longrightarrow [0,\infty)$ be

$$F(u) = \int_{B_R} |\nabla u(x)|\, dx ,$$

$$G(u) = \int_{B_R} \sqrt{1 + |\nabla u(x)|^2}\, dx .$$

- Are there minimizers of F, G in X?
- Are there minimizers of F, G in $W^{1,1}(B_R)$ or in $BV(B_R)$?

Weak convergence and weak compactness for Radon measures

Theorem 1 Let μ, μ_K with $K \in \mathbb{N}$ be Radon (outer) measures on $\mathbb{R}^n$. The following statements are equivalent:

(1) $\lim\limits_{K\to\infty} \int_{\mathbb{R}^n} f(x)\, d\mu_K = \int_{\mathbb{R}^n} f(x)\, d\mu$ for all $f \in C_c(\mathbb{R}^n)$;

(2) • $\limsup\limits_{k\to\infty} \mu_k(K) \leq \mu(K)$ for any $K \subset \mathbb{R}^n$ compact, AND
• $\liminf\limits_{K\to\infty} \mu_K(A) \geq \mu(A)$ for any $A \subset \mathbb{R}^n$ open;

(3) $\lim\limits_{K\to\infty} \mu_K(B) = \mu(B)$ for any bounded Borel set $B \subset \mathbb{R}^n$ such that $\mu(\partial B) = 0$.

Proof. (1) $\Rightarrow$ (2) Let $K \subset \mathbb{R}^n$ be compact and let $A \subset \mathbb{R}^n$ be open such that $K \subset A$. There exists $f \in C_c(\mathbb{R}^n)$ such that $0 \leq f \leq 1$, $f = 1$ on K and $f = 0$ on $\mathbb{R}^n \setminus K$.

Then

$$\limsup_{k\to\infty} \mu_k(K) \leq \lim_{k\to\infty} (\sup) \int_{\mathbb{R}^n} f(x)\, d\mu_k = \int_{\mathbb{R}^n} f(x)\, d\mu \leq$$

$$\leq \mu(A).$$

Using $\mu(K) = \inf_{\substack{K \subset A \\ \text{open}}} \mu(A)$, we obtain the first claim.

The proof of the second claim is similar.

(2) $\Rightarrow$ (3) Let $B \subset \mathbb{R}^n$ be bounded and Borel such that $\mu(\partial B) = 0$.

Then

$$\mu(B) = \mu(\operatorname{int}(B)) \leq \liminf_{k\to\infty} \mu_k(\operatorname{int}(B)) \leq$$

$$\leq \limsup_{k\to\infty} \mu_k(\underbrace{\overline{B}}_{\text{compact}}) \leq \mu(\overline{B}) = \mu(B),$$

It follows: $\liminf = \limsup = \lim$.

(3) $\Rightarrow$ (1). Let $f \in C_c(\mathbb{R}^n)$. We may assume $f \geq 0$.

Let $R > 0$ be such that $\mu(\partial B_R(0)) = 0$ and $\operatorname{spt}(f) \subset B_R(0)$.

Fix $\varepsilon > 0$ and let $0 = t_0 < t_1 < \dots < t_N = 2\|f\|_\infty$ be such that

- $t_i - t_{i-1} < \varepsilon \quad \forall\, i = 1, \dots, N$;
- $\mu(f^{-1}(\{t_i\})) = 0, \quad i \geq 1$.

The sets $B_i = f^{-1}((t_{i-1}, t_i])$ are Borel and bounded, $i = 1, \dots, N$.
Moreover $\mu(\partial B_i) = 0$ for $i \geq 2$.

Now we have:

$$\sum_{i=2}^{N} t_{i-1}\,\mu(B_i) \leq \int_{\mathbb{R}^n} f(x)\,d\mu = \int_{\bigcup_{i=1}^{N} B_i} f(x)\,d\mu =$$

$$= \sum_{i=1}^{N} \int_{B_i} f(x)\,d\mu \leq \sum_{i=1}^{N} t_i\,\mu(B_i)$$

$$\leq \varepsilon\,\mu(B_R) + \sum_{i=2}^{N} t_i\,\mu(B_i)$$

In the same way we get

$$\sum_{i=2}^{N} t_{i-1}\,\mu_K(B_i) \leq \int_{\mathbb{R}^n} f(x)\,d\mu_K \leq \varepsilon\,\mu_K(B_R) + \sum_{i=2}^{N} t_i\,\mu_K(B_i).$$

Take the difference:

$$-\varepsilon\mu(B_R) + \sum_{i=2}^{N} t_{i-1}\,\mu_K(B_i) - \sum_{i=2}^{N} t_i\,\mu_K(B_i) \leq \int_{\mathbb{R}^n} f(x)\,d\mu_K - \int_{\mathbb{R}^n} f(x)\,d\mu \leq \varepsilon\,\mu_K(B_R) + \sum_{i=2}^{N} t_i\,\mu_K(B_i) - \sum_{i=2}^{N} t_{i-1}\,\mu(B_i)$$

Letting $K \to \infty$ we find

$$\limsup_{K \to \infty} \left| \int_{\mathbb{R}^n} f(x)\,d\mu_K - \int_{\mathbb{R}^n} f(x)\,d\mu \right| \leq \varepsilon\,\mu(B_R) + \sum_{i=2}^{N} (t_i - t_{i-1})\,\mu(B_i)$$

$$\leq \varepsilon\,\mu(B_R) + \varepsilon \sum_{i=2}^{N} \mu(B_i) \leq$$

$$\leq 2\varepsilon\,\mu(B_R).$$

Letting $\varepsilon \downarrow 0$ we prove that $\limsup = \lim = 0$.

$\square$

<u>Definition</u> We say that a sequence $(\mu_K)_{K\in\mathbb{N}}$ of (outer) Radon measures on $\mathbb{R}^n$ converges <u>weakly</u> to a Radon measure μ on $\mathbb{R}^n$ if one of the three statements of Theorem 1 holds. We write

$$\mu_K \xrightarrow[K\to\infty]{} \mu .$$

<u>Comment</u>: It would be better to speak of $*$-weak convergence and to write

$$\mu_K \overset{*}{\underset{K\to\infty}{\longrightarrow}} \mu .$$

<u>Theorem 2</u> Let $(\mu_K)_{K\in\mathbb{N}}$ be a sequence of (outer) Radon measures on $\mathbb{R}^n$ satisfying

$$\sup_{K\in\mathbb{N}} \mu_K(K) < \infty \quad \text{for each compact set } K\subset\mathbb{R}^n.$$

Then there exist a subsequence $(\mu_{K_j})_{j\in\mathbb{N}}$ and a Radon (outer) measure μ on $\mathbb{R}^n$ such that

$$\mu_{K_j} \xrightarrow[j\to\infty]{} \mu .$$

Proof. Assume first that

$$M := \sup_{K\in\mathbb{N}} \mu_K(\mathbb{R}^n) < \infty .$$

Recall that $C_c(\mathbb{R}^n)$ with the sup-norm is <u>separable</u>. Then there exists a sequence $(f_K)_{K\in\mathbb{N}}$, $f_K\in C_c(\mathbb{R}^n)$ that is dense in $C_c(\mathbb{R}^n)$ with the sup-norm.

The real sequence $\left(\int_{\mathbb{R}^n} f_1(x)\, d\mu_j \right)_{j\in\mathbb{N}}$ is bonded by $\|f_1\|_\infty M$.

There exists $\alpha_1 \in \mathbb{R}$ and $(\mu_j^1)_{j\in\mathbb{N}}$ sub sequence of $(\mu_j)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_1(x)\, d\mu_j^1 = \alpha_1 .$$

The real sequence $\left(\int_{\mathbb{R}^n} f_2(x)\, d\mu_j^1 \right)_{j\in\mathbb{N}}$ is bonded.

There exists $\alpha_2 \in \mathbb{R}$ and $(\mu_j^2)_{j\in\mathbb{N}}$ subsequence of $(\mu_j^1)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_2(x)\, d\mu_j^2 = \alpha_2 .$$

By induction there exist $\alpha_K \in \mathbb{R}$ and $(\mu_j^K)_{j\in\mathbb{N}}$ sub-sequence of $(\mu_j^{K-1})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_K(x)\, d\mu_j^K = \alpha_K , \qquad K \in \mathbb{N}.$$

Diagonal argument : Let $\nu_j = \mu_j^j$, $j \in \mathbb{N}$.

Then we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f_K(x)\, d\nu_j = \alpha_K \qquad \forall K \in \mathbb{N}.$$

Now we construct a linear and bounded operator $T : C_c(\mathbb{R}^n) \to \mathbb{R}$.
For each $k \in \mathbb{N}$ let

$$T(f_k) = \alpha_k \in \mathbb{R} .$$

For $f \in C_c(\mathbb{R}^n)$ let $g_j \in \{ f_k : k \in \mathbb{N} \}$, with $j \in \mathbb{N}$, be such that $g_j \underset{j \to \infty}{\rightrightarrows} f$ (we are using the density).

We have

$$\left| T(g_k) - T(g_h) \right| = \left| \lim_{j \to \infty} \left\{ \int_{\mathbb{R}^n} g_k \, d\nu_j - \int_{\mathbb{R}^n} g_h \, d\nu_j \right\} \right|$$

$$\leq \| g_k - g_h \|_\infty \sup_{j \in \mathbb{N}} \nu_j(\mathbb{R}^n) \leq$$

$$\leq M \| g_k - g_h \|_\infty ,$$

Then $(T(g_k))_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$,
We can define

$$T(f) = \lim_{k \to \infty} T(g_k) ,$$

Exercise: $T : C_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is linear and bounded

(*) $$|T(f)| \leq M \|f\|_\infty \quad \forall f \in C_c(\mathbb{R}^n).$$

By Riesz theorem there is a Radon measure μ (Exercise: we can choose $6 \equiv 1$) such that

$$T(f) = \int_{\mathbb{R}^n} f(x) \, d\mu .$$

Moreover, (*) implies that $\mu(\mathbb{R}^n) \leq M$.

We claim that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f(x)\, d\nu_j = \int_{\mathbb{R}^n} f(x)\, d\mu \qquad \forall f \in C_c(\mathbb{R}^n).$$

Fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $\| f_k - f \|_\infty \leq \varepsilon$.

Then

$$\int_{\mathbb{R}^n} f\, d\nu_j - \int_{\mathbb{R}^n} f\, d\mu = \int_{\mathbb{R}^n} (f - f_k)\, d\nu_j + \int_{\mathbb{R}^n} f_k\, d\nu_j + \int_{\mathbb{R}^n} (f_k - f)\, d\mu - \int_{\mathbb{R}^n} f_k\, d\mu$$

For $j \geq \bar{j}$ we have

$$\left| \int_{\mathbb{R}^n} f_k\, d\nu_j - \int_{\mathbb{R}^n} f_k\, d\mu \right| < \varepsilon$$

It follows that for $j \geq \bar{j}$:

$$\left| \int_{\mathbb{R}^n} f\, d\nu_j - \int_{\mathbb{R}^n} f\, d\mu \right| \leq \varepsilon\, \nu_j(\mathbb{R}^n) + \varepsilon\, \mu(\mathbb{R}^n) + \varepsilon \leq$$
$$\leq \varepsilon\, (2M + 1).$$

This ends the proof when $\sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}^n) < \infty$.

For the general case: Consider

$$\mu_k^l = \mu_k \llcorner B_l(0), \qquad k \in \mathbb{N},\ l \in \mathbb{N}$$

and use a diagonal argument.

$\square$

<u>Corollary (Weak compactness in L^p)</u> Let $A \subset \mathbb{R}^n$ be an open set. Let $1 < p < \infty$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $L^p(A)$ such that

$$\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(A)} < \infty .$$

Then there exists a sub-sequence $(f_{k_j})_{j \in \mathbb{N}}$ and a function $f \in L^p(A)$ such that

$$f_{k_j} \xrightarrow[j \to \infty]{} f \quad \text{weakly in } L^p(A).$$

This means:

$$\lim_{j \to \infty} \int_A f_{k_j}(x)\, g(x)\, dx = \int_A f(x)\, g(x)\, dx$$

for all $g \in L^q(A)$ with $1/p + 1/q = 1$.

Proof. Idea:
- Let $\mu_k = f_k \mathcal{L}^n$
- Then $\mu_{k_j} \to \mu$ (there is μ R. measure)
- $\mu \ll \mathcal{L}^n$. Here, we need $p > 1$.
- Then $\mu = f \mathcal{L}^n$. Prove: $f \in L^p$ and $f_{k_j} \to f$.

<u>Example</u> $f_k : \mathbb{R} \to \mathbb{R}$, $f_k(x) = \begin{cases} k/2 & \text{if } |x| < \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$

Then $\|f_k\|_{L^1(\mathbb{R})} = 1$ for all $k \in \mathbb{N}$.

For any $\varphi \in C_c(\mathbb{R})$ we have

$$\lim_{k \to \infty} \int_{\mathbb{R}} f_k\, \varphi\, dx = \lim_{k \to \infty} \frac{k}{2} \int_{-1/k}^{1/k} \varphi(x)\, dx = \varphi(0)$$