Theory of functions 2

Sheet 2

Various exercises

Within 2 April 2012

Exercise 1 For any $t \geq 0$ and for any $A \subset \mathbb{R}^n$ define the *spherical* Hausdorff measure of A of dimension t as

$$\mathcal{S}^t(A) = \sup_{\delta>0} \inf \Big\{ \sum_{j=1}^{\infty} \alpha(t) \Big(\frac{\operatorname{diam}(B_j)}{2}\Big)^t : B_j \subset \mathbb{R}^n \text{ balls}, A \subset \bigcup_{j=1}^{\infty} B_j, \operatorname{diam}(B_j) < \delta \Big\}.$$

Prove that $\mathcal{S}^t(A) = 0$ if and only if $\mathcal{H}^t(A) = 0$

Exercise 2 Let $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$, $n \geq 1$. Using the splitting $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ along with Fubini-Tonelli theorem and polar coordinates in $\mathbb{R}^2$, prove that

$$\alpha(n) = 2\pi\alpha(n-2) \int_0^1 r(1-r^2)^{(n-2)/2} dr.$$

By induction, compute an explicit formula for $\alpha(2n)$ and $\alpha(2n+1)$.

Exercise 3 Let $0 < \lambda < 1/2$ be a fixed parameter, and for $x \in [0,1]$ define $T_1(x) = \lambda x$, $T_2(x) = 1 - \lambda + \lambda x$. Let $T : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ be the mapping $T(E) = T_1(E) \cup T_2(E)$. By induction, define $E_0 = [0,1]$ and $E_{n+1} = T(E_n)$, for any $n \in \mathbb{N}$. Compute the Hausdorff dimension and measure of the set

$$K = \bigcap_{n=0}^{\infty} E_n.$$

Exercise 4 Let $0 \leq \delta < 1$ be a number. Construct a function $\varphi : [0,1] \to \mathbb{R}$ such that: 1) φ is strictly increasing; 2) $|\varphi(t) - \varphi(s)| \leq |t - s|$ for all $t, s \in [0,1]$; 3) There holds

$$\mathcal{L}^1\big(\{t \in [0,1] : \text{there exists } \varphi'(t) \text{ and } \varphi'(t) = 0\}\big) \geq \delta.$$