Theory of functions 2

Sheet 4

Various exercises

10 May 2012

Exercise 1 Let $K \subset \mathbb{R}^n$ be a closed set, let $x_0, x_1 \in K$ and assume there is a Lipschitz curve $\gamma : [0,1] \to K$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Prove that there exists a Lipschitz curve contained in K having minimal length and joining x_0 to x_1.

Exercise 2 Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, let $B \subset \partial A$ be a Borel set and let

$$\mathcal{A} = \big\{ E \subset A : \ E \text{ with finite perimeter in } A \text{ and } T(\chi_E) = \chi_B \big\},$$

where $T : BV(A) \to L^1(\partial A; \mathcal{H}^{n-1})$ is the trace operator. Assuming that there is $E \in \mathcal{A}$ with $\|\partial E\|(A) < \infty$, prove that the minimum

$$\min \big\{ \|\partial E\|(A) : E \in \mathcal{A} \big\}$$

is attained.

Exercise 3 Let $E \subset \mathbb{R}^n$ be a set of finite perimeter.

1) For some orthogonal mapping $T \in O(n)$ let $F = T(E)$. Show that $\|\partial F\|(\mathbb{R}^n) = \|\partial E\|(\mathbb{R}^n)$.
2) For any $\lambda > 0$ let $E_\lambda = \big\{ \lambda x \in \mathbb{R}^n : x \in E \big\}$. Prove that $\|\partial E_\lambda\|(\mathbb{R}^n) = \lambda^{n-1} \|\partial E\|(\mathbb{R}^n)$.

Exercise 4 Let $E \subset \mathbb{R}^n$ be a set with locally finite perimeter and denote by ν_E its measure theoretic inner normal. Assume that there exists a unit vector $\mathrm{v} \in \mathbb{S}^{n-1}$ such that $\nu_E(x) = \mathrm{v}$ for $\|\partial E\|$-a.e. $x \in \mathbb{R}^n$. Characterize the set E.