Theory of functions 2

Sheet 5

Various exercises

4th July 2012

Exercise 1 Let $K \subset \mathbb{R}^n$ be a closed set and let $f : \mathbb{R}^n \to \mathbb{R}$ be the function

$$f(x) = \operatorname{dist}(x; K), \quad x \in \mathbb{R}^n.$$

Prove that $|Df(x)| = 1$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n \setminus K$.

Exercise 2 Let $A \subset \mathbb{R}^n$ be a bounded open set with boundary of class C^2. Prove that the function $\varphi : [0, \infty) \to \mathbb{R}$

$$\varphi(t) = \mathcal{H}^{n-1}(\{x \in \mathbb{R}^n : \operatorname{dist}(x; \partial A) = t\}), \quad t \geq 0,$$

is continuous at $t = 0$. Hint: $f(x) = \operatorname{dist}(x; \partial A)$, integrate $\Delta f(x)$, where Δ is the Laplace operator.

Exercise 3 Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter and let $x \in \partial^* E$ be a point in the reduced boundary. Prove that:

$$\lim_{r \to 0} \frac{\|\partial E\|(B_r(x))}{r^{n-1}} = \alpha(n-1),$$

where $\alpha(n-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.