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Solution of Plateau's problem via Hausdorff measures

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Introduction

Plateau's problem, also known as minimal surfaces problem, is an important open question of Calculus of Variation which consists in the research of the minimal area surface "spanned" by a fixed constraint. This problem has been faced with a large number of different approaches during the last three centuries, starting from parametrization, then using a variational approach, continuosly evolving in more complex techniques such as the integral currents based method and varifold theory, both arisen with the development of the Geometric Measure Theory.

In this thesis, following the work done by C. Dellis, F. Ghilardin e F. Maggi in [DLGM], we present a direct approach to Plateau's problem based on Hausdorff measure and Preiss' theorem, in the particuolar case of codimension 1; in this setting the presented tecnhique works more effectively than the integral currents method, permitting a larger number of constraints and allowing to find also surfaces which are not smooth away from the fixed boundary.

More precisely: fixed a set $H \subset \mathbb{R}^{n+1}$, we consider $\mathcal{P}(H)$ class of sets K closed in $\mathbb{R}^{n+1} \setminus H$, which formalize the idea of "H-spanned set", then we can state the Plateau's problem as the research of sets that realizes the following minimum

$$m_0 = \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\},$$

where $\mathcal{H}^n$ denote the Hausdorff n-dimensional measure,starting from a minimizing sequence K_j of elements of $\mathcal{P}(H)$ such that $\mathcal{H}^n(K_j) \to m_0$ for $j \to +\infty$.

We introduce two important concept,which are well defined in the above presented context: for every ball $B(x,r) = \left\{y \in \mathbb{R}^{n+1} : |x-y| < r\right\} \subset \mathbb{R}^{n+1} \setminus H$ we define:

- the *cone competitor* for $K \in \mathcal{P}(H)$ in $B(x,r)$ as the set

$$(K \setminus B(x,r)) \cup \{\lambda x + (1-\lambda)z : z \in K \cap \partial B(x,r), \lambda \in [0,1]\}\,;$$

- a *cup competitor* for $K \in \mathcal{P}(H)$ in $B(x,r)$ as the set

$$(K \setminus B(x,r)) \cup (\partial B(x,r) \setminus A),$$

  where A is a connected component of $\partial B(x,r) \setminus K$.

We say that $K \in \mathcal{P}(H)$ enjoys the good comparison property in $B(x,r)$ if it holds

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{P}(H) \text{tale che } J \setminus \mathrm{cl}(B(x,r)) = K \setminus \mathrm{cl}(B(x,r))\} \leq \mathcal{H}^n(L),$$

where L is any the cone competitor or a cup competitors for K in $B(x,r)$,and we call $\mathcal{P}(H)$ a good class if , for all $K \in \mathcal{P}(H)$ and for all $x \in K$, the set K enjoys the good comparison property in $B(x,r)$ for almost every $r \in (0, \mathrm{dist}(x,H))$. We have now all the tools needed for state the main theorem of the thesis:

Theorem. Let $H \subset \mathbb{R}^{n+1}$ and $\mathcal{P}(H)$ a good class. Suppose exits finite $m_0 = \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}$ and let K_j a minimizing sequence of sets $\mathcal{H}^n$-rectifiables. Then,

- up to subsequence, the measures $\mu_j = \mathcal{H}^n \llcorner K_j$ converge weakly* in $\mathbb{R}^{n+1} \setminus H$ to a measure $\mu = \theta \mathcal{H}^n \llcorner K$, where $K = \mathrm{spt}(\mu) \setminus H$ is $\mathcal{H}^n$-rectifiable and $\theta(x) \geq 1$,
- it holds the lower semicontinuity $\liminf_{j \in \mathbb{N}} \mathcal{H}^n(K_j) \geq \mathcal{H}^n(K)$,
- for every $x \in K$ the quantity $r^{-n}\mu(B(x,r))$ is monotone increasing and the density verifies

$$\theta(x) = \lim_{r \searrow 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1,$$

  where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

The proof of the theorem above uses the concepts of cup and cone competitors to prove respectively the structure of the limit measure $\mu = \theta \mathcal{H}^n \llcorner K$ and existence and finetness of such measure, which will implies the rectifiability of K via an important theorem due to Preiss. In the proof we find also the main limitation of this approach: while using an isoperimetric inequality on the sphere, we introduce the codimension 1 constraint. Although there exists similar inequality for greater codimensions, harder to handle, this generalization will implies the loss of cup competitors on which we based our proof.

Even though the theorem does not implies in general the existence of a minimum K in the class $\mathcal{P}(H)$, the theorem can be more precise in some suitable settings: in the thesis we present two different application which permit to refind, with a direct application of the above described method, results due to J. Harrison e H. Pugh in the first case and to G. David in the second case. In particular, J. Harrison e H. Pugh show the existence of minimal surfaces spanned by closed and compact submanifolds, while G. David proved the existence of sliding minimizers, namely sets that are minimal above particular Lipschitz deformation of a chosen set with fixed boundary. Despite the differences between the settings, the strategy will be the same: we select and define a suitable class $\mathcal{P}(H)$ which encodes a particular definition of boundary, we show that $\mathcal{P}(H)$ is a good class and applying the main theorem we conclude the existence of $m_0 = \min\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}$, giving also some charaterization for the set K.

Chapter 1

Existence of sets minimizing the Hausdorff measure in good classes

In this chapter we display the approach to Plateau's problem as presented in [DLGM]. This approach, albeit limited to the case of codimension one, allows to avoid difficulties due to lack of semicontinuity or compactness trouble related to the choosen convegence, which, in a general framework, are quite delicate to face.

1.1 Cone and cups competitors and statement of the main theorem

Consider $H \subset \mathbb{R}^{n+1}$. Let $\mathcal{P}(H)$ a class of closed subset K of $\mathbb{R}^{n+1} \setminus H$ which encodes a certain notion of "K bounds H". We state the Plateau's problem, namely the research of the minimal surface "generated" by H, as the research of the minimum

$$m_0 = \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}, \tag{PP}$$

and a minimizing sequence $(K_j)_{j\in\mathbb{N}} \subset \mathcal{P}(H)$ such that $K_j \to K$ as $j \to \infty$. We introduce now two concepts that will be fondamental for our approach:

Definition 1.1.1. (**Cone competitor**)
Let $H \subset \mathbb{R}^{n+1}$ be a closed set. Given $K \subset \mathbb{R}^{n+1} \setminus H$ and $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\} \subset \mathbb{R}^{n+1} \setminus H$, we define as **cone competitor** for K in $B(x,r)$ the set

$$(K \setminus B(x,r)) \cup \{\lambda x + (1-\lambda)z : z \in K \cap \partial B(x,r), \lambda \in [0,1]\}\,. \tag{1.1}$$

Definition 1.1.2. (**Cup competitor**)
Let $H \subset \mathbb{R}^{n+1}$ be a closed set. Given $K \subset \mathbb{R}^{n+1} \setminus H$ and $B(x,r) =$

$\{y \in \mathbb{R}^n : |x-y| < r\} \subset \mathbb{R}^{n+1} \setminus H$, we define as **cup competitor** for K in $B(x,r)$ the set

$$(K \setminus B(x,r)) \cup (\partial B(x,r) \setminus A), \tag{1.2}$$

where A is a connected componets of $\partial B(x,r) \setminus K$.

Definition 1.1.3. (**Good comparison property**)
Let $H \subset \mathbb{R}^{n+1}$ be a closed set. Given $K \subset \mathbb{R}^{n+1} \setminus H$ element of $\mathcal{P}(H)$, we say that K has the **good comparison property** in $B(x,r)$ if

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{P}(H) \text{ with } J \backslash \mathrm{cl}(B(x,r)) = J \backslash \mathrm{cl}(B(x,r))\} \leq \mathcal{H}^n(L) \tag{1.3}$$

whenever L is the cone competitor or a cup competitor for K in $B(x,r)$.

Definition 1.1.4. (**Good class**)
Let $H \subset \mathbb{R}^{n+1}$ be a closed set, $\mathcal{P}(H)$ class defined as above. We say that $\mathcal{P}(H)$ is a **good class** if for every $K \in \mathcal{P}(H)$, for every $x \in K$ and foe almost every $r \in (0, \mathrm{dist}(x,H))$, the set K has the good comparison property in $B(x,r)$.

Now we state the main result:

Theorem 1.1.1. (Main Theorem)
Let $H \subset \mathbb{R}^{n+1}$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau's problem is finite and let $(K_j)_{j \in \mathbb{N}} \subset \mathcal{P}(H)$ be a minimizing sequence of $\mathcal{H}^n$-rectficable sets. Then:

(*i*) up to subsequences, the measures $\mu_j = \mathcal{H}^n \llcorner K_j$ converge weakly* in $\mathbb{R}^{n+1} \backslash H$ to a measure $\mu = \theta \mathcal{H}^n \llcorner K$, with $K = \mathrm{spt}(\mu) \backslash H$ $\mathcal{H}^n$-rectifiable set and $\theta(x) \geq 1$ for all $x \in K$,

(*ii*) it holds that $\liminf_{j \in \mathbb{N}} \mathcal{H}^n(K_j) \geq \mathcal{H}^n(K)$,

(*iii*) for every $x \in K$ the quantity $r^{-n}\mu(B(x,r))$ is monotone increasing and

$$\theta(x) = \lim_{r \searrow 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1,$$

where $\omega_n = \mathcal{L}^n(B(0,1))$.

1.2 Proof of the main theorem

Before starting, we briefly recall some standard notations:

$$\sigma_n = \mathcal{H}^n\left(\partial B_{\mathbb{R}^{n+1}}(0,1)\right) = \mathcal{H}^n\left(\left\{z \in \mathbb{R}^{n+1} : |z| = 1\right\}\right), \tag{1.4}$$

$$\omega_{n+1} = \mathcal{H}^{n+1}\left(B_{\mathbb{R}^{n+1}}(0,1)\right) = \mathcal{H}^{n+1}\left(\left\{z \in \mathbb{R}^{n+1} : |z| \leq 1\right\}\right) = \frac{\sigma_n}{n+1}. \tag{1.5}$$

Now we can state:

Lemma 1.2.1. (**Isoperimetry on the sphere**)
Let $J \subset \partial B(x,r)$ be a compact set and $\{A\}_{h\in\mathbb{N}}$ the family of all connected components of $\partial B(x,r) \setminus J$, ordered so that $\mathcal{H}^n(A_h) \geq \mathcal{H}^n(A_{h+1})$, then

$$\mathcal{H}^n(\partial B(x,r) \setminus A_0) \leq C(n)\mathcal{H}^{n-1}(J)^{\frac{n}{n-1}}. \tag{1.6}$$

Moreover, for every $\eta > 0$ there exists $\delta > 0$ such that

$$\min\{\mathcal{H}^n(A_0), \mathcal{H}^n(A_1)\} = \mathcal{H}^n(A_1) \geq \left(\frac{\sigma_n}{2} - \delta\right) r^n \tag{1.7}$$

implies

$$\mathcal{H}^{n-1}(J) \geq (\sigma_{n-1} - \eta)\, r^{n-1}. \tag{1.8}$$

The inequality (1.6) holds also replacing $\partial B(x,r)$ with ∂Q for any cube $Q \subset \mathbb{R}^{n+1}$ or with any spherical cap $\partial B(x,r) \cap \{y \in \mathbb{R}^{n+1} : (y-x)\cdot\nu > \varepsilon r\}$, where $\nu \in B_{\mathbb{R}^{n+1}}(0,1)$ and $\varepsilon \in]0,1[$.

Proof. We start proving (1.6) for $J \subset \partial B(x,r)$. Since $\partial A_h \subset J$ and $\mathcal{H}^{n-1}(J)$ is finite, without loss of generality, from Proposition 3.62 in [AFP] it follows that each A_h has finite perimeter and $\partial^* A_h \subset J$, and since $\partial^* A_h \subset \partial A_h$ for each $h \in \mathbb{N}$, we have $\sum_{h\in\mathbb{N}} \mathcal{H}^{n-1}\llcorner\partial^* A_h \leq 2\mathcal{H}^{n-1}\llcorner J$. Taken $A \subset \partial B(x,r)$ a set of finite perimeter, then by the Isoperimetric Inequality follows

$$\min\{\mathcal{H}^n(A), \mathcal{H}^n(\partial B(x,r) \setminus A)\} \leq C(n)\mathcal{H}^{n-1}(\partial^* A)^{\frac{n}{n-1}}, \tag{1.9}$$

then for $h \geq 1$ we have

$$\mathcal{H}^n(A_h) \leq C(n)\mathcal{H}^{n-1}(\partial^* A_h)^{\frac{n}{n-1}}, \tag{1.10}$$

so, adding up over $h \neq 0$, remebering the superadditivity of $x \mapsto x^{\frac{n}{n-1}}$, we find

$$\mathcal{H}^n(\partial B(x,r) \setminus A_0) \leq C(n)\left(\sum_{h\geq 1} \mathcal{H}^{n-1}(\partial^* A_h)\right)^{\frac{n}{n-1}} \leq C(n)\mathcal{H}^{n-1}(J)^{\frac{n}{n-1}}.$$

Simple adaptations of same argouments will prove similar results for cubes and spherical caps boundaries.
Now we prove (1.8) using a compactenss argument, arguing by contraddiction: after assuming that it fails on $\eta > 0$, we find a sequence of sets $(J_k)_{k\in\mathbb{N}}$ each one violating the statement for $\delta = \frac{1}{k}$. Let A_0^k and A_1^k be the connected components, then by compactness of finite perimeter set given in Theorem 12.26 in [Mag] we have convergence to the sets A_0^∞ and A_1^∞, with

$$\mathcal{H}^n(A_0^\infty) = \mathcal{H}^n(A_1^\infty) = \frac{\sigma_k}{2} r^n, \tag{1.11}$$

$$\mathcal{H}^n(A_0^\infty \cap A_1^\infty) = 0, \tag{1.12}$$

$$\max\{\mathcal{H}^{n-1}(\partial^* A_0^\infty), \mathcal{H}^{n-1}(\partial^* A_1^\infty)\} \leq (\sigma_{n-1} - \eta) r^{n-1}. \tag{1.13}$$

By (1.11) and (1.12) we have $\partial^* A_0^\infty = \partial^* A_1^\infty$, but then (1.13) contraditcs the sharp isoperimetric inequality on the sphere presented in Theorem 10.21 in [BZ]. $\square$

Now we state and prove Theorem 1.1.1:

Theorem. (Main Theorem)
Let $H \subset \mathbb{R}^{n+1}$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau's problem is finite and let $(K_j)_{j\in\mathbb{N}} \subset \mathcal{P}(H)$ be a minimizing sequence of $\mathcal{H}^n$-rectficable sets. Then:

(i) up to subsequences, the measures $\mu_j = \mathcal{H}^n \llcorner K_j$ converge weakly* in $\mathbb{R}^{n+1} \setminus H$ to a measure $\mu = \theta \mathcal{H}^n \llcorner K$, with $K = \mathrm{spt}(\mu) \setminus H$ $\mathcal{H}^n$-rectifiable set and $\theta(x) \geq 1$ for all $x \in K$,

(ii) it holds that $\liminf_{j\in\mathbb{N}} \mathcal{H}^n(K_j) \geq \mathcal{H}^n(K)$,

(iii) for every $x \in K$ the quantity $r^{-n}\mu(B(x,r))$ is monotone increasing and

$$\theta(x) = \lim_{r\searrow 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1,$$

where $\omega_n = \mathcal{L}^n(B(0,1))$.

Proof. Let $\mu_j = \mathcal{H}^n \llcorner K_j$: then by Theorem 1.41 in [EG], up to extracting subsequences, we have a Radon measure μ on $\mathbb{R}^{n+1} \setminus H$ such that $\mu_j \stackrel{*}{\rightharpoonup} \mu$, and we set $K = \mathrm{spt}(\mu) \setminus H$. Now we procede in 4 steps:
Step 1: We show existence of $\theta_0 = \theta_0(n) > 0$ such that $\mu(B(x,r)) \geq \theta_0 \omega_n r^n$, for all $x \in \mathrm{spt}(\mu)$, for all $r < d_x = dist(x, H)$. We start defining $f(r) = \mu(B(x,r))$ and $f_j(r) = \mathcal{H}^n(K_j \cap B(x,r))$, so

$$f_j(r) - f_j(s) \geq \int_s^r \mathcal{H}^{n-1}(K_j \cap \partial B(x,t))\,\mathrm{d}t \qquad \text{for } 0 < s < r < d_x, \tag{1.14}$$

by the coarea formula for rectifiable sets (Section 3.2.22 in [F]). Since f_j is not decreasing on $(0, d_x)$, defined as Df_j the distributional derivative of f_j and as f_j' the pointwise derivative of f_j, by properties of real functions it follows that $Df_j \geq f_j' \mathcal{L}^1$ with

$$f_j'(r) \geq \mathcal{H}^{n-1}(K_j \cap \partial B(x,r)) \qquad \text{for a.e. } r \in (0, d_x). \tag{1.15}$$

By Fatou's lemma with $g(t) = \liminf_{j\in\mathbb{N}} f_j'(t)$, then

$$f(s) - f(r) = \mu(B(x,r) \setminus B(x,s)) \geq \int_s^r g(t)\,\mathrm{d}t \tag{1.16}$$

if $\mu(\partial B(x,r)) = \mu(\partial B(x,s)) = 0$ and this implies $Df \geq g\mathcal{L}^1$. On the other hand, since f is differentiable a.e., taking $s \nearrow r$, we have $f' \geq g$ in $\mathcal{L}^1$-a.e., and since f is non decreasing it follows that $Df \geq f'\mathcal{L}^1$. Taken A_j a connected component of $\partial B(x,r) \setminus K_j$ of maximal $\mathcal{H}^n$ measure, we call K_j'' the corrisponing cup competitor of K_j in $B(x,r)$, and since $\mathcal{P}(H)$ is a good class we find

$$f_j(r) \leq \mathcal{H}^n(\partial B(x,r) \setminus A_j) + \varepsilon_j \leq C(n)(\mathcal{H}^{n-1}(\partial B(x,r) \cap K_j))^{\frac{n}{n-1}} + \varepsilon_j, \quad (1.17)$$

with $\varepsilon_j \to 0$, that is we are assumuming that $\mathcal{H}^n(K_j) \leq \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\} + \varepsilon_j$. Now we take $j \to \infty$ and we have

$$f(r) \leq C(n)g(r)^{\frac{n}{n-1}} \leq C(n)f'(r)^{\frac{n}{n-1}} \qquad \text{for a.e. } r < d_x, \quad (1.18)$$

from which

$$f(r)^{\frac{n-1}{n}} \leq C(n)f'(r) \qquad \text{for a.e. } r < d_x, \quad (1.19)$$

which implies

$$1 \leq C(n)\frac{f'(r)}{f(r)^{\frac{n-1}{n}}} = C(n)(f(r)^{\frac{1}{n}})' \qquad \text{for a.e. } r < d_x. \quad (1.20)$$

Since $Df^{\frac{1}{n}}$ is nonnegative, we deduce that $r < C(n)\left(f(r)^{\frac{1}{n}} - f(0)^{\frac{1}{n}}\right)$, so $\mu(B(x,r)) \geq \theta_0\omega_n r^n$ for some value of θ_0. By Theorem 9.6 in [Mat] we have $\mu \geq \theta_0\mathcal{H}^n \llcorner K$ on subsets of $\mathbb{R}^{n+1} \setminus H$.

Step 2: Now we fix $x \in \mathrm{spt}(\mu) \setminus H$, and we want to prove that $r \to \frac{f(r)}{r^n} = \frac{\mu(B(x,r))}{r^n}$ is nondecreasing for $(0, d_x)$. Rewrinting formula 1.17 using the cone competitor in $B(x,r)$, namely K', we find

$$f_j(r) \leq \mathcal{H}^n(B(x,r) \cap K_j') + \varepsilon_j \leq \frac{r}{n}\mathcal{H}^{n-1}(\partial B(x,r) \cap K_j) + \varepsilon_j \leq \frac{r}{n}f_j'(r) + \varepsilon_j, \quad (1.21)$$

which give us

$$f(r) \leq \frac{r}{n}g(r) \leq \frac{r}{n}f'(r) \qquad \text{for a.e. } r < d_x. \quad (1.22)$$

Reasoning as at the end of previous step, we found that the positivity of $D\log(f)$ prove the monotonicity. Now using these results we can write θ, the n-dimensional density of μ:

$$\theta(x) = \theta^n(\mu, x) = \lim_{r \to 0^+} \frac{f(r)}{\omega_n r^n}, \quad (1.23)$$

which exists, finite and positive μ-a.e. with $\theta(x) \geq \theta_0$. By Theorem 1.1 in [DL], namely Preiss' Theorem, we deduce that $\mu = \theta\mathcal{H}^n \llcorner \tilde{K}$ for some $\mathcal{H}^n$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since $K = \mathrm{spt}(\mu)$, it must be that $\mathcal{H}^n(\tilde{K} \setminus K) = 0$; on the other side we have $\mathcal{H}^n(K \setminus \tilde{K}) = 0$

from the conclusion of the first step. It follows that K is a $\mathcal{H}^n$-rectifiable set and $\mu = \theta \mathcal{H}^n \llcorner K$.

Step 3: We prove that $\theta(x) \geq 1$ for every $x \in K$ such that the approximate tangent space to K exists. Fix any $x \in K \setminus H$ and suppose, up to change of coordinates, that $T = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ is the approximate tanget space to K in x, in particular we have

$$\mathcal{H}^n \llcorner \frac{K - x}{r} \overset{*}{\rightharpoonup} \mathcal{H}^n \llcorner T \qquad \text{as } r \to 0^+. \tag{1.24}$$

By the density lower bound proved in the first step, for every $\varepsilon > 0$ there is $\rho > 0$ such that

$$K \cap B(x, \rho) \subset x + \{y \in \mathbb{R}^{n+1} : |y_{n+1}| < \varepsilon r\} \qquad \text{for every } r < \rho. \tag{1.25}$$

Now we choose r sufficient small that

$$\mu \left(B(x, 2r) \setminus \left(x + \left\{ y \in \mathbb{R}^{n+1} : |y_{n+1}| < \frac{\varepsilon r}{2} \right\} \right) \right) < \theta_0 \frac{\varepsilon^n r^n}{2^n}, \tag{1.26}$$

so $K \cap (x + \{y \in \mathbb{R}^{n+1} : |y_{n+1}| < \varepsilon r\}) \cap B(x, r)$ must be empty, infact taken $\tilde{y}$ in this set, we will have

$$\begin{aligned} \mu \left(B(x, 2r) \setminus \left(x + \left\{ y \in \mathbb{R}^{n+1} : |y_{n+1}| < \frac{\varepsilon r}{2} \right\} \right) \right) \geq \\ \geq \mu \left(B \left(\tilde{y}, \frac{\varepsilon r}{2} \right) \right) \geq \theta_0 \frac{\varepsilon^n r^n}{2^n}, \end{aligned} \tag{1.27}$$

and we have a contradiction. Now we set $c(\varepsilon) = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$, and we have

$$K \cap B(x, \rho) \subset x + \left\{ (y', y_{n+1}) \in \mathbb{R}^{n+1} : |y_{n+1}| < c(\varepsilon)|y'| \right\}. \tag{1.28}$$

Now we choose ρ satisfying $\mathcal{H}^n(K \cap \partial B(x, \rho)) = 0$, and by the coarea formula we find

$$\begin{aligned} 0 = \lim_{j \to \infty} \mu_j \left(\mathrm{cl}(B(x, \rho)) \cap \left(x + \left\{ (y', y_{n+1}) \in \mathbb{R}^{n+1} : |y_{n+1}| < c(\varepsilon)|y'| \right\} \right) \right) \geq \\ \geq \int_0^\rho \liminf_{j \to \infty} \mathcal{H}^{n-1} \left(K_j \cap \partial B(x, r) \cap \left(x + \left\{ (y', y_{n+1}) : |y_{n+1}| < c(\varepsilon)|y'| \right\} \right) \right) \, \mathrm{d}r. \end{aligned} \tag{1.29}$$

We now define

$$\partial B_\varepsilon^+(x, r) = \{y \in \partial B(x, r) : y_{n+1} > x_{n+1} + \varepsilon r\}, \tag{1.30}$$

$$\partial B_\varepsilon^-(x, r) = \{y \in \partial B(x, r) : y_{n+1} < x_{n+1} - \varepsilon r\}, \tag{1.31}$$

so we obtain

$$\liminf_{j \to \infty} \mathcal{H}^{n-1}(K_j \cap \partial B_\varepsilon^\pm(x, r)) = 0, \qquad \text{for a.e. } r < \rho. \tag{1.32}$$

Taken $r < \rho$ such that the above condition is satisfied, $f'(r)$ exists, it holds that $f'(r) \geq g(r)$ and each K_j has the good comparison property in $B(x,r)$. Using the Lemma 1.2.1, defined A_j^+ as the connected component of $\partial B_\varepsilon^+(x,r) \setminus K_j$ with the largest $\mathcal{H}^n$-measure, then

$$\mathcal{H}^n(\partial B_\varepsilon^+(x,r) \setminus A_j^+) \leq C(n)\mathcal{H}^{n-1}(K_j \cap \partial B_\varepsilon^+(x,r))^{\frac{n}{n-1}}, \tag{1.33}$$

and thus by (1.32)

$$\lim_{j\to\infty} \mathcal{H}^n(A_j^+) = \mathcal{H}^n(\partial B_\varepsilon^+(x,r)), \tag{1.34}$$

and similarly $\mathcal{H}^n(A_j^+) \to \mathcal{H}^n(\partial B_\varepsilon^+(x,r))$ if A_j^- is the largest connected component of $\partial B_\varepsilon^-(x,r)\setminus K_j$. Now we prove that, for j sufficiently large, A_j^+ and A_j^- cannot belong to the same connected component of $\partial B(x,r)\setminus K_j$: infact, if it is not true we can compare with the cup competitor of K_j in $B(x,r)$ defined by the connected component of $\partial B(x,r) \setminus K_j$ containing $A_j^+ \cup A_j^-$, that lead to

$$\begin{aligned} \mu(B(x,r)) &\geq \liminf_{j\to\infty} \mathcal{H}^n(B(x,r) \cap K_j) \leq \\ &\leq \liminf_{j\to\infty} \mathcal{H}^n(\partial B(x,r) \cap (A_j^+ \cup A_j^-)) \leq \\ &\leq \mathcal{H}^n(\partial B(x,r) \cap \{|y_{n+1} - x_{n+1}| < \varepsilon r\}) \leq C\varepsilon r^n, \end{aligned} \tag{1.35}$$

that contradicts the density lower bound of first step. Now we can fix η and choose ε as in Lemma 1.2.1 and for j large enough we find

$$(\sigma_{n-1} - \eta)r^{n-1} \leq \liminf_{j\to\infty} \mathcal{H}^{n-1}(K_j \cap \partial B(x,r)) \leq f'(r). \tag{1.36}$$

So $f'(r) \geq (\sigma_{n-1} - \eta)r^{n-1}$ for a.e. $r < \rho$, that leads to $f(r) \geq (\sigma_{n-1} - \eta)\frac{r^n}{n}$ for $r < \rho$, and one obtain $\theta(x) \geq \frac{\sigma_{n-1}-\eta}{n\omega_n}$. For $\eta \to 0$ we conclude $\theta(x) \geq 1$.
Step 4: We prove the upper semicontinuity of θ and conclude that $\theta(x) = \lim_{r\searrow 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1$. Following Corollary 17.8 in [Si], we fix two parameters $0 < \rho < r < d_x$, and choose x, $y \in K$ such that $|x - y| < \varepsilon$, so we have

$$\frac{\mu(B(y,\rho))}{\rho^n} \leq \frac{\mu(B(y,r))}{r^n} + \frac{c}{r^{n-1}} \leq \frac{\mu(B(x,r+\varepsilon))}{r^n} + \frac{c}{r^{n-1}}. \tag{1.37}$$

Taking $\rho \searrow 0$ and dividing by ω_n we find

$$\theta(y) \leq \frac{\mu(B(x,r+\varepsilon))}{\omega_n(r+\varepsilon)^n}\Big(1 + \frac{\varepsilon}{r}\Big)^n + \frac{c}{r^{n-1}}. \tag{1.38}$$

Fixing $\delta > 0$ so that $r < \delta$ and tking ε small enough we find

$$\frac{\mu(B(x,r+\varepsilon))}{\omega_n(r+\varepsilon)^n}\Big(1 + \frac{\varepsilon}{r}\Big)^n < \theta(x) + \delta, \tag{1.39}$$

and thus

$$\theta(y) \leq \theta(x) + \frac{c}{r^{n-1}}, \tag{1.40}$$

so we conclude

$$\limsup_{y \to x} \theta(y) \leq \theta(x). \tag{1.41}$$

□

Chapter 2

Existence of minimal sets spanned by closed compact submanifolds

In this chapter we are able to prove, using the previously given scheme, a result obtained by Harrison and Pugh. Our approach allows both a more general setting and a more immediate way: the original result was infact proved only for Hausdorff spherical measures, and then upgraded to Hausdorff measure (we refer to [DLGM]) for the complete list of papers), while we find the result directly for Hausdorff measure, exploiting theory of finite perimeter sets and minimal partitions.

2.1 The class $\mathcal{F}(H, \mathcal{C})$

Let $H \subset \mathbb{R}^{n+1}$, with $n \geq 2$, be a closed compact submanifold of dimension $n-1$. Then we define:

Definition 2.1.1. (Spanning set)
In the setting presented above, we say that $K \subset \mathbb{R}^{n+1} \setminus H$ **spans** H if it intersect any embedded smooth closed curve γ in $\mathbb{R}^{n+1} \setminus H$ with linking number 1 around H.

For an arbitrary closed set H we will consider the family

$$\mathcal{C}_H = \{\gamma : S^1 \to \mathbb{R}^{n+1} \setminus H : \gamma \text{ is a smooth embedding of } S^1 \text{ into } \mathbb{R}^{n+1}\}.$$

So we can define:

Definition 2.1.2. (Closure by homotpy)
We will say that $\mathcal{C} \subset \mathcal{C}_H$ is **closed by homotopy** (with respect to H) if $\gamma \in \mathcal{C}$ implies that $\tilde{\gamma} \in \mathcal{C}$ for every $\tilde{\gamma} \in \pi_1(\mathbb{R}^{n+1} \setminus H)$ with $[\gamma] = [\tilde{\gamma}]$.

Definition 2.1.3. ($\mathcal{C}$-spanning set)
Given $\mathcal{C} \subset \mathcal{C}_H$ closed by homotopy, we say that K closed subset of $\mathbb{R}^{n+1} \setminus H$ is a **$\mathcal{C}$-spanning set** if $K \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$. We will denote with $\mathcal{F}(H, \mathcal{C})$ the family of $\mathcal{C}$-spanning set of H.

Given these definitions, we will prove:

Theorem 2.1.1. Let $n \geq 2$, $H \subset \mathbb{R}^{n+1}$ closed subset and $\mathcal{C} \subset \mathcal{C}_H$ closed by homotopy with respect to H. Assume that $m_0 = \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}$ is finite, where we put $\mathcal{P}(H) = \mathcal{F}(H, \mathcal{C})$. Then we have:

(i) $\mathcal{F}(H, \mathcal{C})$ is a good class,

(ii) there exists a minimizing sequence $(K_j)_{j \in \mathbb{N}} \subset \mathcal{F}(H, \mathcal{C})$ of $\mathcal{H}^n$-rectfiable sets, and the limit set K is a minimizer for the Plateau's problem and $K \in \mathcal{F}(H, \mathcal{C})$,

(iii) the set K is such that $\mathcal{H}^n(K) \leq \mathcal{H}^n(\varphi(K))$ for every $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ Lipschitz with $\varphi|_{\mathbb{R}^{n+1} \setminus B(x,r)} = \mathrm{Id}$ and $\varphi(B(x,r)) \subset B(x,r)$ for $x \in \mathbb{R}^{n+1} \setminus H$ and $r < \mathrm{dist}(x, H)$.

2.2 Application of the Main Theorem

We start stating and proving a geometric lemma:

Lemma 2.2.1. Given $K \in \mathcal{F}(H, \mathcal{C})$, $B(x,r) \subset\subset \mathbb{R}^{n+1} \setminus H$ and $\gamma \in \mathcal{C}$, then:

- either $(K \setminus B(x,r)) \cap \gamma \neq \emptyset$,

- or there exists σ, connected component of $\gamma \cap \mathrm{cl}(B(x,r))$, which is homeomorphic to an interval and whose end points belong to distinct connected components of $\mathrm{cl}(B(x,r) \cap K$.

Same conclusion holds for $Q \subset\subset \mathbb{R}^{n+1} \setminus H$ open cube istead of $B(x,r)$.

Proof. We split the proof in two steps: in the first step we assume that γ and $\partial B(x,r)$ intersect trasversally, and in second step we remove the auxiliary hypothesis. One can archive the same result on open cubes adapting the argument below. Assuming $(K \setminus B(x,r)) \cap \gamma = \emptyset$ we have:
Step 1: Since γ and $\partial B(x,r)$ intersect trasversally we can find a finite number of disjointed closed circular arcs $J_i \subset S^1$ in the form $J_i = [a_i, b_i]$, such that

$$\gamma \cap B(x,r) = \bigcup_i \gamma((a_i, b_i)) \tag{2.1}$$

and

$$\gamma \cap \partial B(x,r) = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}. \tag{2.2}$$

Now we argue by contradiction, assuming that for every index i there exists a connected component A_i such that both $\gamma(a_i)$ and $\gamma(b_i)$ belong to A_i. Since A_i is connected for each index i we can find a smooth embedding $\tau_i : J_i \to A_i$ with

$$\tau_i(a_i) = \gamma(a_i) \tag{2.3}$$

and

$$\tau_i(b_i) = \gamma(b_i), \tag{2.4}$$

moreover we can suppose that

$$\tau_i(J_i) \cap \tau_k(J_k) = \emptyset \quad \text{ for } i \neq k. \tag{2.5}$$

Now we can define

$$\bar{\gamma} = \begin{cases} \gamma & \text{on } S^1 \setminus \bigcup_i J_i, \\ \tau_i & \text{on } J_i, \end{cases} \tag{2.6}$$

and we notice that $[\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$. Also we have $\bar{\gamma} \cap (K \setminus \mathrm{cl}(B(x,r))) = \gamma \cap (K \setminus \mathrm{cl}(B(x,r))) = \emptyset$ and $\bar{\gamma} \cap K \cap \mathrm{cl}(B(x,r)) = \emptyset$ by construction, which imply $\bar{\gamma} \cap K = \emptyset$. We now choose $\tilde{\gamma} \in \mathcal{C}_H$ with $[\tilde{\gamma}] = [\bar{\gamma}] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$ which is uniformely close to $\bar{\gamma}$, obtaining $\tilde{\gamma} \cap K = \emptyset$ that contradicts $K \in \mathcal{F}(H, \mathcal{C})$.

Step 2: Consider now a generic ball $B(x,r) \subset\subset \mathbb{R}^{n+1} \setminus H$. Since γ is a smooth embedding, applying Theorem 1.3 in [H], namely Morse-Sard's theorem, to $|\gamma|$ we find that γ and $\partial B(x,s)$ intersect trasversally for almost every $s > 0$. Chosen ε small enough we have that for every $s \in (r - \varepsilon, r)$ we can costruct a diffeomorphism $f_s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ definded by

$$f_s = \mathrm{Id} \quad \text{ on } \mathbb{R}^{n+1} \setminus B(x, r + 2\varepsilon), \tag{2.7}$$

$$f_s = x + \frac{r}{s}(y - x) \text{ for } y \in B(x, r + \varepsilon), \tag{2.8}$$

and it holds that

$$f_s \to \mathrm{Id} \text{ on } \mathbb{R}^{n+1} \text{ for } s \to r^-. \tag{2.9}$$

Now $f_s \circ \gamma$ is in $\mathcal{C}$ and intersect trasversally $\partial B(x,s)$ and we notice that $\mathrm{dist}(\gamma, K \cap B(x,r)) > 0$, by the trasversality of intersection, so one have

$$(f_s \circ \gamma) \cap K \setminus B(x,r) = \emptyset, \tag{2.10}$$

and we can apply the previous step and conclude. $\square$

Now we recall and prove:

Theorem. Let $n \geq 2$, $H \subset \mathbb{R}^{n+1}$ closed subset and $\mathcal{C} \subset \mathcal{C}_H$ closed by homotopy with respect to H. Assume that $m_0 = \inf\{\mathcal{H}^n(K) : K \in \mathcal{P}(H)\}$ is finite, where we put $\mathcal{P}(H) = \mathcal{F}(H, \mathcal{C})$. Then we have:

(*i*) $\mathcal{F}(H, \mathcal{C})$ is a good class, (*ii*) there exists a minimizing sequence $(K_j)_{j\in\mathbb{N}} \subset \mathcal{F}(H,\mathcal{C})$ of $\mathcal{H}^n$-rectfiable sets, and the limit set K is a minimizer for the Plateau's problem and $K \in \mathcal{F}(H,\mathcal{C})$,

(*iii*) the set K is such that $\mathcal{H}^n(K) \leq \mathcal{H}^n(\varphi(K))$ for every $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ Lipschitz with $\varphi|_{\mathbb{R}^{n+1}\setminus B(x,r)} = \mathrm{Id}$ and $\varphi(B(x,r)) \subset B(x,r)$ for $x \in \mathbb{R}^{n+1} \setminus H$ and $r < \mathrm{dist}(x,H)$.

Proof. For sake of clarity let's divide the proof into some steps.
Step 1: In order to prove that $\mathcal{F}(H,\mathcal{C})$ is a good class, we show that for every $V \in \mathcal{F}(H,\mathcal{C})$, for $x \in V$ and for almost ever $r \in (0, \mathrm{dist}(x,H))$, called V' and V'' the cone competitor and the cup competitor respectively for V in $B(x,r)$, we have $V', V'' \in \mathcal{F}(H,\mathcal{C})$. We start from the cup competitor V'': fixing $\gamma \in \mathcal{C}$ and assuming that $\gamma \cap (V \setminus B(x,r)) = \emptyset$, by Lemma 2.2.1 we have an arc in $\mathrm{cl}(B(x,r))$ which is homeomorphic to $[0,1]$. We denote by $\sigma : [0,1] \to \mathrm{cl}(B(x,r))$ a parametrization of this arc, and we know that $\sigma(0)$ and $\sigma(1)$ belong to different connected components of $\mathrm{cl}(B(x,r) \setminus V$. This imply that one of the endpoints of σ belongs to $V'' \cap \gamma \cap \partial B(x,r)$, so $V'' \in \mathcal{F}(H,\mathcal{C})$.
Now we consider the cone competitor V' and we show that $V' \cap \gamma \cap \partial B(x,r) \neq \emptyset$. There are two cases: if $x \in \sigma$, then $\sigma \cap V' \neq \emptyset$, since x is the vertex of the cone, and we have nothing else left to prove. If instead $x \notin \sigma$ then we project radially σ on $\partial B(x,r)$ via a projection $\pi : \sigma \to \partial B(x,r)$. Now by connectedness $\pi \circ \sigma$ must intersect $V' \cap \partial B(x,r)$, which is by definition of cone competitor equal to $V \cap \partial B(x,r)$. Consider a point z in such intersection: for such z it is true that $\emptyset \neq \pi^{-1}(z) \cap \sigma([0,1]) \subset V'$, since $\pi^{-1}(z) = \lambda z$ for some $\lambda \in (0,1)$. So $V' \in \mathcal{F}(H,\mathcal{C})$, and we have shown point (*i*).
Step 2: Now, assuming that the minimizing sequence in (*ii*) is given, we prove that $K \in \mathcal{F}(H,\mathcal{C})$. Suppose for contradiction that exists $\gamma \in \mathcal{C}$ that does not intersect K. Since both K and γ are compact, there exists $\varepsilon > 0$ such that the tubular neighborhood $U_\varepsilon(\gamma) = \bigcup_{x\in\gamma} B(x,\varepsilon)$ does not intersect K and is contained in $\mathbb{R}^{n+1} \setminus H$. Noticing $\mu(U_\varepsilon(\gamma)) = 0$, since $\mu = \theta\mathcal{H}^n \llcorner K$, we can write

$$\lim_{j\to\infty} \mathcal{H}^n(K_j \cap U_\varepsilon(\gamma)) = 0. \tag{2.11}$$

If ε is small enough, we consider a diffeomorphism $\Phi : S^1 \times D_\varepsilon \to U_\varepsilon(\gamma)$, where $D_\varepsilon = \{y \in \mathbb{R}^n : |y| < \varepsilon\}$, with $\Phi|_{S^1\times\{0\}} = \gamma$, and denote with γ_y the parallel curve $\Phi|_{S^1\times\{y\}}$. So $\gamma_y \in [\gamma] \in \pi^1(\mathbb{R}^{n+1} \setminus H)$ for every $y \in D_\varepsilon$. It holds $K_j \cap \gamma_y \neq \emptyset$ for every $y \in D_\varepsilon$ and every $j \in \mathbb{N}$, since $K_j \in \mathcal{F}(H,\mathcal{C})$. Called $\pi_2 : S^1 \times D_\varepsilon \to D_\varepsilon$ the projection on the second component, it is true that $\pi = \pi_2 \circ \Phi^{-1} : U_\varepsilon(\gamma) \to D_\varepsilon$ is a Lipschitz map. The coarea formula for rectifiable sets give us

$$\mathcal{H}^n(K_j \cap U_\varepsilon(\gamma)) \geq \frac{\omega_n \varepsilon^n}{(\mathrm{Lip}\ \pi)^n} > 0. \tag{2.12}$$

Given that (2.11) and (2.12) are in contradiction, it must be that $K \in \mathcal{F}(H,\mathcal{C})$.

Step 3: In order to prove point *(iii)* now it suffices to show that $\varphi(K) \in \mathcal{F}(H,\mathcal{C})$. We fix $\gamma \in \mathcal{C}$ and assume that $\gamma \cap (K \setminus B(x,\rho)) = \emptyset$ for some $\rho \in (r, \mathrm{dist}(x,H))$. By Lemma 2.2.1 there exists two different connected component A and A' of $B(x,\rho) \setminus K$ and a connected component of $\gamma \cap \mathrm{cl}(B(x,\rho))$ with the endpoints $p \in \mathrm{cl}(A) \cap \partial B(x,\rho)$ and $q \in \mathrm{cl}(A') \cap \partial B(x,\rho)$. We complete the proof by showing that $p = \varphi(p)$ and $q = \varphi(q)$ are in the closure of distinct connected components of $B(x,\rho) \setminus \varphi(K)$. Arguing by contradiction we suppose that p and q are in the closure of Ω, connected component of $B(x,\rho) \setminus \varphi(K)$. We call $h = \varphi|_{\mathrm{cl}(A)}$ and observe that the topological degree of h is defined on Ω. Since $\varphi = \mathrm{Id}$ in a neighborhood of $\partial B(x,\rho)$, it holds that $\deg(h,p') = 1$ for every p' sufficiently close to p, and since the topological degree is locally constant and Ω is a connected set, it follows that $\deg(h,x) = 1$ for every $x \in \Omega$. So $\varphi^{-1}(y) \cap A \neq \emptyset$ for every $y \in \Omega$, and for $y = q'$, where $q' \in \Omega$ is sufficiently close to q, we find $w \in \varphi^{-1}(q')$. Since $|q'| > r$, then $w = q'$, because $\varphi|_{\mathbb{R}^{n+1} \setminus B(x,r)} = \mathrm{Id}$, and so $q' \in A$. Since we chose $\rho > r$ for every $q' \in B(x,\rho)$, where q' is sufficiently close to q, we have $q' \in A$. A is connected so we can connect $p', q' \in A$, and since p' and q' are sufficiently close to p and q respectively we can connect p and q in A. This implies a contradiction, since $A \neq A'$. Then $\varphi(K) \in \mathcal{F}(H,\mathcal{C})$.

Step 4: In order to conclude the prove, we need to show that given $K \in \mathcal{F}(H,\mathcal{C})$ with $\mathcal{H}^n(K)$ finite there exists $K' \in \mathcal{F}(H,\mathcal{C})$ rectifiable and such that $\mathcal{H}^n(K') < \mathcal{H}^n(K)$. We split the prove of this fact in three steps. By Theorem 2.10.25 in [F], we have

$$\int_{\mathbb{R}^n}^{*} \mathcal{H}^n(K \cap \{x_1 = t\})\,\mathrm{d}t \leq \frac{\omega_1 \omega_n}{\omega_{n+1}} \mathcal{H}^{n+1}(K) = 0, \tag{2.13}$$

thus

$$\mathcal{L}^1(\{t \in \mathbb{R}^n : \mathcal{H}^n(K \cap \{x_1 = t\}) > 0\}) = 0. \tag{2.14}$$

So in particular

$$\mathcal{L}^1\left(\bigcup_{j\in\mathbb{N}} \left\{ t \in (0,1) : \mathcal{H}^n\left(K \cap \bigcup_{h\in\mathbb{Z}} \left\{ x_1 = t + \frac{h}{2^j} \right\}\right) > 0 \right\}\right) = 0, \tag{2.15}$$

and we can find $x_1^0 \in (0,1)$ such that

$$\mathcal{H}^n\left(K \cap \left\{ x_1 = x_1^0 + \frac{h}{2^j} \right\}\right) = 0 \quad \begin{array}{l} \text{for every } j \in \mathbb{N}, \\ \text{for every } h \in \mathbb{Z}, \end{array}. \tag{2.16}$$

We can argue similarly for each coordinate, and obtain a point $x_0 \in \mathbb{R}^{n+1}$ such that

$$\mathcal{H}^n\left(K \cap \left\{ x_m = x_m^0 + \frac{h}{2^j} \right\}\right) = 0 \quad \begin{array}{l} \text{for every } m \in \{1,\dots,n+1\}, \\ \text{for every } j \in \mathbb{N}, \\ \text{for every } h \in \mathbb{Z}. \end{array} \tag{2.17}$$

In this way we find a grid $\mathcal{Q}$ of open dyadic cubes such that $\mathcal{H}^n(K \cap \partial Q) = 0$ for every $Q \in \mathcal{Q}$. Let $\mathcal{W}$ be a Whitney covering of $\mathbb{R}^{n+1} \setminus H$ obtained from $\mathcal{Q}$ as in Theorem 3 of [St], more precisely we find a family of closed cubes Q_k with faces parallel to coordinated axis such that:

(i) $\mathbb{R}^{n+1} \setminus H = \bigcup_{k=1}^{+\infty} Q_k$,

(ii) $\mathrm{Int}(Q_k) \cap \mathrm{Int}(Q_j) = \emptyset$ for $k \neq j$,

(iii) there exists c_1, $c_2 > 0$ constants such that

$$c_1 \operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, H) \leq c_2 \operatorname{diam}(Q_k). \tag{2.18}$$

We choose $\mathcal{W}$ such that if Q' is the concentric cube with twice the side of $Q \in \mathcal{W}$, then $Q' \cap H = \emptyset$.
Step 5: Now for every $Q \in \mathcal{W}$ we define a suitable replacement K_Q of K such that

(i) $K_Q \cap \mathrm{cl}(Q)$ is $\mathcal{H}^n$-rectifiable,

(ii) $\mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \mathcal{H}^n(K \cap \mathrm{cl}(Q))$,

(iii) $K_Q \setminus \mathrm{cl}(Q) = K \setminus \mathrm{cl}(Q)$.

Fixed $Q \in \mathcal{W}$, denote by $(F_i)_{i\in\mathbb{N}}$ the connected components of $Q' \setminus K$ and consider the partition problem in finite perimeter sets

$$\inf \left\{ \mathcal{H}^n \left(Q' \cap \bigcup_i \partial^* E_i \right) \middle| \begin{array}{l} (E_i)_{i\in\mathbb{N}} \text{ partition modulo } \mathcal{H}^{n+1} \text{ of } Q' \\ \text{with } E_i \setminus Q = F_i \setminus Q \end{array} \right\}. \tag{2.19}$$

Since F_i is open with $\partial F_i \subset K$ and $\mathcal{H}^n(K) < \infty$, the infimum above is finite and there exists, by Theorem 4.19 and Remark 4.20 in [AFP], a minimizing partition $(E_i)_{i\in\mathbb{N}}$. In particular, for $\mathcal{H}^n$-almost every point $x \in E_i$ for some i either x has density 0 or 1, or x is a point with density $\frac{1}{2}$. We remember that all points $x \in \partial^* E_i$ for some i have density $\frac{1}{2}$. Define now, for $Q \in \mathcal{W}$, the replacement

$$K_Q = (K \setminus Q) \cup \left(\mathrm{cl}(Q) \cap \mathrm{cl} \left(\bigcup_{i\in\mathbb{N}} \partial^* E_i \right) \right). \tag{2.20}$$

By Lemma 30.2 in [Mag] we have

$$\mathcal{H}^n \left(Q \cap \left(K_Q \setminus \bigcup_{i\in\mathbb{N}} \partial^* E_i \right) \right) = 0, \tag{2.21}$$

so that $\mathrm{cl}(Q) \cap K_Q$ is $\mathcal{H}^n$-rectifiable.
In order to prove

$$\mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \mathcal{H}^n(K \cap \mathrm{cl}(Q)) \tag{2.22}$$

if suffices to show that

$$\mathcal{H}^n\left(\mathrm{cl}(Q) \cap \left(K_Q \setminus \bigcup_{i\in\mathbb{N}} \partial^* E_i\right)\right) = 0. \tag{2.23}$$

Since

$$\mathcal{H}^n\left(Q \cap \left(K_Q \setminus \bigcup_{i\in\mathbb{N}} \partial^* E_i\right)\right) = 0 \tag{2.24}$$

and

$$\mathcal{H}^n(K \cap \partial Q) = 0, \tag{2.25}$$

we need to show

$$\begin{aligned}\mathcal{H}^n\left(\partial Q \cap \left((K_Q \setminus K) \setminus \bigcup_{i\in\mathbb{N}} \partial^* E_i\right)\right) &= \\ = \mathcal{H}^n\left(\partial Q \cap \left(\mathrm{cl}\left(\bigcup_{i\in\mathbb{N}} \partial^* E_i\right) \setminus \bigcup_{i\in\mathbb{N}} \partial^* E_i\right)\right) &= 0, \end{aligned} \tag{2.26}$$

and by Corollary 6.5 in [Mag] we just need to find $c_0 > 0$ such that

$$\mathcal{H}^n\left(B(x,r) \cap \bigcup_{i\in\mathbb{N}} \partial^* E_i\right) \geq c_0 r^n \tag{2.27}$$

for every $x \in \partial Q \cap (K_Q \setminus K)$ and every $r < r_x = \mathrm{dist}(x, K \setminus Q)$. To do so, let i_0 be such that $x \in F_{i_0}$, and, for $r < r_x$, let $G_i = E_i \setminus B(x,r)$ if $i \neq i_0$, and we set $G_{i_0} = E_{i_0} \cup B(x,r)$. Since $\{G_i\}_i$ is a valid partition in (2.19) we have

$$\begin{aligned} f(r) = \mathcal{H}^n\left(\mathrm{cl}(B(x,r)) \cap \bigcup_{i\in\mathbb{N}} \partial^* E_i\right) \leq \mathcal{H}^n\left(\mathrm{cl}(B(x,r)) \cap \bigcup_{i\in\mathbb{N}} \partial^* G_i\right) &= \\ = \mathcal{H}^n\left(\partial(B(x,r)) \cap \bigcup_{i\in\mathbb{N}} \partial^* G_i\right)&. \end{aligned} \tag{2.28}$$

Now we denote with $E_i^{(\tau)}$ the set of points $x \in E_i$ of density τ, namely

$$\lim_{r\to 0} \frac{\mathcal{H}^{n+1}(B(x,r) \cap E_i)}{\omega_{n+1} r^{n+1}} = \tau \quad \text{for every } x \in E_i^{(\tau)}. \tag{2.29}$$

For almost every $r < r_x$ one has

$$\mathcal{H}^n(\partial B(x,r) \cap (E_{i_0}^{(0)} \triangle \partial^* G_{i_0})) = 0 \tag{2.30}$$

and for $i \neq i_0$

$$\mathcal{H}^n \left(\partial B(x,r) \cap \left(E_{i_0}^{(0)} \triangle \bigcup_{i \neq i_0} E_i^{(1)} \right) \right) = 0 \tag{2.31}$$

and

$$\mathcal{H}^n(\partial B(x,r) \cap (E_i^{(1)} \triangle \partial^* G_i)) = 0. \tag{2.32}$$

So we find $f(r) \leq \mathcal{H}^n(\partial B(x,r) \cap E_{i_0}^{(0)})$ for almost every $r < r_x$. By Theorem 18.11 and Remark 18.14 from [Mag], the set $\partial B(x,r) \cap E_{i_0}^{(0)}$ has finite perimeter in $\partial B(x,r)$ and

$$\mathcal{H}^{n-1} \left(\partial^*_{\partial B(x,r)} \left(\partial B(x,r) \cap E_{i_0}^{(0)} \right) \triangle \left(\partial B(x,r) \cap \partial^* E_{i_0} \right) \right) = 0, \tag{2.33}$$

since $\mathcal{H}^n(\partial B(x,r) \setminus E_{i_0}^{(0)}) \geq \frac{\mathcal{H}^n(\partial B(x,r))}{2}$ by convexity of Q, the isoperimetric inequality on $\partial B(x,r)$ leads to

$$f(r) \leq C(n) \mathcal{H}^{n-1}(\partial^* E_{i_0} \cap \partial B(x,r))^{\frac{n}{n-1}} \leq C(n) f'(r) \tag{2.34}$$

for almost every $r < r_x$. Arguing like in Step 1 of Theorem 1.1.1, we complete the proof of 2.27 and conclude this step.

Step 6: In order to conclude the proof, we set

$$K' = \bigcup_{Q \in \mathcal{W}} K_Q \cap \mathrm{cl}(Q), \tag{2.35}$$

which is $\mathcal{H}^n$-rectifiable by Step 2, with

$$\mathcal{H}^n(K') \leq \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K_Q \cap \mathrm{cl}(Q)) \leq \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K \cap \mathrm{cl}(Q)) = \sum_{Q \in \mathcal{W}} \mathcal{H}^n(K \cap Q), \tag{2.36}$$

that leads to $\mathcal{H}^n(K') \leq \mathcal{H}^n(K)$. Now we show that $K' \in \mathcal{F}(H, \mathcal{C})$. Consider $\gamma \in \mathcal{C}$, with $\gamma \cap K \cap \mathrm{cl}(Q) \neq \emptyset$ for some $Q \in \mathcal{W}$, since $K \cap \partial Q \subset K_Q \cap \partial Q \subset K' \cap \partial Q$, we can assume $\gamma \cap K \cap Q \neq \emptyset$ directly. So by Lemma 2.2.1 we find a connected component σ of $\gamma \cap \mathrm{cl}(Q)$ with endpoints $p \in F_i \cap \partial Q$ and $q \in F_j \cap \partial Q$, with F_i and F_j distinct connected components of $\mathrm{cl}(Q) \setminus K$. If p or q are in K_Q, we conclude, otherwise if $p \in E_i$ and $q \in E_j$ we have, by connectdness of σ, that $\sigma \cap K_Q \cap \mathrm{cl}(Q) \neq \emptyset$, and we conclude the proof. $\square$

Chapter 3

Existence of sliding minimizer

We present in this chapter another application of the method illustrated in Chapter 1, which is based on the concept of "sliding minimizer" by David. We will consider a class of Lipschitz deformations of a fixed set spanned by the given boundary and exploiting the Allard's regularity theorem from the varifold theory we will be able to conclude our proof.

3.1 Existence of minimal sets in $\mathcal{A}(H, K_0)$

Let $H \subset \mathbb{R}^{n+1}$ be closed set and $K_0 \subset \mathbb{R}^{n+1} \setminus H$ be relative closed. In setting we introduce a suitable class of functions and the concept of "sliding minimizer":

Definition 3.1.1. (**Family of functions** $\Sigma(H)$)
Given $H \subset \mathbb{R}^{n+1}$ closed we define

$$\Sigma(H) = \left\{ \varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \text{ Lipschitz } \middle| \begin{array}{l} \text{exists } \Phi : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \\ \text{with } \Phi(1, \cdot) = \varphi, \Phi(0, \cdot) = \mathrm{Id} \\ \text{and } \Phi(t, H) \subset H \text{ for every } t \in [0,1] \end{array} \right\}.$$

Definition 3.1.2. (**Class of sets** $\mathcal{A}(H, K_0)$)
Given $H \subset \mathbb{R}^{n+1}$ closed and $K_0 \subset \mathbb{R}^{n+1} \setminus H$ relative closed we define

$$\mathcal{A}(H, K_0) = \{K : K = \varphi(K_0) \text{ for } \varphi \in \Sigma(H)\}.$$

Definition 3.1.3. (**Sliding minimizer**)
Given $H \subset \mathbb{R}^{n+1}$ closed and $K_0 \subset \mathbb{R}^{n+1} \setminus H$ relative closed we say that K_0 is a **sliding minimizer** if

$$\mathcal{H}^n(K_0) = \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\}.$$

Now we state and prove:

Theorem 3.1.1. Given $H \subset \mathbb{R}^{n+1}$ closed and $K_0 \subset \mathbb{R}^{n+1} \backslash H$ relative closed, $\mathcal{A}(H, K_0)$ is a good class. Moreover, if we assume

(*i*) K_0 bounded and $\mathcal{H}^n$-rectifiable with $\mathcal{H}^n(K_0) < \infty$,

(*ii*)$\mathcal{H}^n(H) = 0$ and for every $\eta > 0$ exists $\delta > 0$ and $\pi \in \Sigma(H)$ such that

$$\mathrm{Lip}(\pi) \leq 1 + \eta \tag{3.1}$$

and

$$\pi(U_\delta(H)) \subset H, \tag{3.2}$$

where $U_\delta(E) = \bigcup_{x \in E} \{y \in \mathbb{R}^{n+1} : \mathrm{dist}(y, E) \leq \delta\}$ for $E \subset \mathbb{R}^{n+1}$,

then the minimizing sequence $\{K_j\}_{j \in \mathbb{N}}$ correspoding the Plateau's problem with $\mathcal{P}(H) = \mathcal{A}(H, K_0)$ and the limit set K satisfies

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} = \mathcal{H}^n(K) = \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K)\},$$

namely, K is a sliding minimizer.

Proof. We split again the proof in some steps.
Step 1: We begin proving that if $K \in \mathcal{A}(H, K_0)$, then his cup competitor in $B(x, r) \subset \mathbb{R}^{n+1} \setminus H$ has the good comparison property for almost every r. Fixed $B(x, r) \subset\subset \mathbb{R}^{n+1} \setminus H$ such that $\mathcal{H}^n(K \cap \partial B(x, r)) = 0$ for almost every r, we rescale and translate conveniently in a way such that the ball becomes $B(0, 1) = B$. Now we consider the cup competitor of K in B given by A, connected component of $\partial B \setminus K$, which has Hausdorff measure $\mathcal{H}^n(K \setminus B) + \mathcal{H}^n(\partial B \setminus A)$. We want to show that for any $\sigma > 0$ there exists $J \in \mathcal{A}(H, K_0)$ with the two following properties:

(*i*) $J \setminus \mathrm{cl}(B) = K \setminus \mathrm{cl}(B)$,

(*i*) $\mathcal{H}^n(J) \leq \mathcal{H}^n(K \setminus B) + \mathcal{H}^n(\partial B \setminus A) + \sigma$,

which togheter imply

$$\mathcal{H}^n(J \cap \mathrm{cl}(B)) \leq \mathcal{H}^n(\partial B \setminus A) + \sigma. \tag{3.3}$$

Thus we need to find a map $\psi \in \Sigma(H)$ such that $J = \psi(K)$, and we will build a map ψ such that $\psi|_{\mathbb{R}^{n+1} \setminus B(0,1+\eta)} = \mathrm{Id}$ for some sufficiently small η. The map ψ will be build by composition of two maps ϕ_1 and ϕ_2.
To costruct ϕ_1 we fix $x_0 \in A$ and $\rho > 0$ such that $B(x_0, \rho) \cap K = \emptyset$, then ϕ_1 maps B $B(x_0, \rho)$ onto ∂B along the rays outgoing from x_0 and retract $B(x_0, \rho) \cap \mathrm{cl}(B)$ onto $\mathrm{cl}(B)$. In this way we find

$$K_1 = \phi_1(K \cap \mathrm{cl}(B)) \subset \partial B, \tag{3.4}$$

which is disjoint from $B(x_0, \rho)$.
Now we claim the existence of a Lipschitz map $\phi_2 : \partial B \to \partial B$, which effectively existence's proof is postpones, with

$$\phi_2 = \mathrm{Id} \ \text{ on } U_\varepsilon(K \cap \partial B) \text{ for some } \varepsilon > 0 \tag{3.5}$$

and

$$\mathcal{H}^n(\phi_2(K_1)) \leq \mathcal{H}^n(\partial B \setminus A) + \sigma. \tag{3.6}$$

After fixing $\varepsilon > 0$, we can find $\eta > 0$ such that $B(0, 1+\eta) \subset\subset \mathbb{R}^n \setminus H$ and

$$\frac{K \cap \partial B(0, 1+t)}{1+t} \subset U_\varepsilon(K \cap \partial B) \quad \text{ for all } t \in (0, \eta). \tag{3.7}$$

We can finally define $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\psi(x) = \begin{cases} \phi_2(\phi_1(x)) & \text{for } |x| < 1, \\ \frac{|x|-1}{\eta} x + \frac{1+\eta-|x|}{\eta} \phi_2(\phi_1(x)) & \text{for } 1 \leq |x| < 1+\eta, \\ x & \text{for } |x| \geq 1+\eta, \end{cases} \tag{3.8}$$

which is a Lipschitz map with

$$\psi = \mathrm{Id} \ \text{ on } (\mathbb{R}^{n+1} \setminus B(0, 1+\eta)) \cup \{(1+t)x : t \in (0, \eta), x \in U_\varepsilon(K \cap \partial B)\}. \tag{3.9}$$

Since

$$J \setminus \mathrm{cl}(B) = \psi(K \setminus \mathrm{cl}(B)) = K \setminus \mathrm{cl}(B) \tag{3.10}$$

and

$$J \cap \mathrm{cl}(B) = \psi(K \setminus \mathrm{cl}(B)) = \phi_2(K_1), \tag{3.11}$$

by the claim done on ϕ_2 we conclude this proof.
Now it remain to prove the existence of ϕ_2 with the suitable properties, and, up to conjugation with a stereographic projection with pole x_0, the existence problem can be reduced to the follow: given

- (i) $\Omega \subset \mathbb{R}^n$ open connected set with bounded component and $\mathcal{H}^n(\partial\Omega) = 0$,
- (ii) a ball $B(0, R) \subset \mathbb{R}^n$ such that $\partial\Omega \subset\subset B(0, R)$,
- (iii) $\sigma > 0$,

find $\varepsilon > 0$ and a Lipschitz map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that

- (a) $\phi = \mathrm{Id}$ on $U_\varepsilon(\partial\Omega) \cup (\mathbb{R}^n \setminus \Omega) \cup (\mathbb{R}^n \setminus B(0, 2R))$,
- (b) $\mathcal{H}^n(\phi(B(0, R) \cap \Omega)) < \sigma$.

We can consider a Whitney covering $\mathcal{W}$ of $B(0,2R)\cap\Omega$ constructed on a grid of dyadic cubes, as in Step 4 of Theorem 2.1.1. Given $\varepsilon>0$ we can find a "faced connected" finite subfamily $\mathcal{W}_0\subset\mathcal{W}$ such that

$$(B_R\cap\Omega)\setminus U_\varepsilon(\partial\Omega)\subset\bigcup_{Q\in\mathcal{W}_0}Q, \tag{3.12}$$

for which exists $Q_0\in\mathcal{W}_0$ such that $Q_0\setminus B(0,R)\neq\emptyset$. We now construct a Lipschitz map $f:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ such that

$$f=\mathrm{Id}\quad \text{ on } \mathbb{R}^{n+1}\setminus\bigcup_{Q\in\mathcal{W}_0}Q \tag{3.13}$$

and

$$f\left(\bigcup_{Q\in\mathcal{W}_0}Q\cap B(0,R)\right)\subset\bigcup_{Q\in\mathcal{W}_0}\partial Q. \tag{3.14}$$

In order to do this we choose a ball $U_0\subset\subset Q_0\setminus B(0,R)$, and then we define $f_0:\mathbb{R}^n\to\mathbb{R}^n$ with

$$f_0=\mathrm{Id}\quad \text{ on } \mathbb{R}\setminus Q_0, \tag{3.15}$$

$$f_0(U_0)=Q_0, \tag{3.16}$$

and

$$f_0(Q_0\setminus U_0)=\partial Q_0, \tag{3.17}$$

by projrcting $Q_0\setminus U_0$ radially from the center U_0 onto ∂Q_0, and then stretching U_0 onto Q_0 . Let now $Q_1\in\mathcal{W}_0$ with a face in common in Q_0, so that the side of Q_1 has lenght at most double of the lenght side of Q_0. If the side of Q_1 is twice of the side of Q_0, we subdivide Q_1 into 2^n subcubes and denote by $\hat{Q}_1$ the one sharing a face with Q_0, otherwise we set $\hat{Q}_1=Q_1$. Let then $x_1\in Q_0$ be the reflection of th center of $\hat{Q}_1$ with respect to the common hyperface between Q_0 and $\hat{Q}_1$. So we can find a ball $U_1\subset\subset Q_0$ and define a Lipschitz map $\hat{f}_1:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$\hat{f}_1=\mathrm{Id}\quad \text{ on } \mathbb{R}^n\setminus(Q_0\cup\hat{Q}_1), \tag{3.18}$$

$$\hat{f}_1((Q_0\cup\hat{Q}_1)\setminus U_1)\subset\partial(Q_0\cup\hat{Q}_1), \tag{3.19}$$

and

$$\hat{f}_1(U_1)=Q_0\cup\hat{Q}_1. \tag{3.20}$$

In the case $\hat{Q}_1\neq Q_1$ we exploit another radial projection onto ∂Q_1 from a small ball centered in the center of $\hat{Q}_1$. We further construct a Lipschitz map $f_1:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$f_1=\mathrm{Id}\quad \text{ on } \mathbb{R}^n\setminus(Q_0\cup Q_1), \tag{3.21}$$

$$f_1((Q_0\cup Q_1)\setminus U_1)\subset\partial Q_0\cup\partial Q_1, \tag{3.22}$$

and

$$f_1(U_1) = Q_0 \cup Q_1. \tag{3.23}$$

Thus we consider $f_2 = f_1 \circ f_0$ Lipschitz map such that

$$f_2 = \mathrm{Id} \qquad \text{on } \mathbb{R}^n \setminus (Q_0 \cup Q_1) \tag{3.24}$$

and

$$f_2((Q_0 \cup Q_1) \setminus U_0) \subset \partial Q_0 \cup \partial Q_1, \tag{3.25}$$

and by iteration we conclude.

Step 2: Now we want to prove the similar result for cone competitors. As before, we translate so $B(x,r) \subset\subset \mathbb{R}^{n+1} \setminus H$ becomes $B(0,r)$, then we assume $K \cap \partial B(0,r)$ is $\mathcal{H}^{n-1}$-rectifiable with $\mathcal{H}^{n-1}(K \cap \partial B(0,r)) < \infty$ and r is a Lebesgue point for the map $t \mapsto \mathcal{H}^{n-1}(K \cap \partial B(0,t))$. These conditions are satisfied for almost every r, so we can rescale so $B(0,r)$ turns into $B(0,1) = B$. Let's call K' the cone competitor of K in B. For $s \in (0,1)$ we set

$$\phi_s(r) = \begin{cases} 0 & r \in [0, 1-s), \\ \frac{r-(1-s)}{s} & r \in [1-s, 1], \\ r & r > 1, \end{cases} \tag{3.26}$$

and

$$\psi_s(x) = \frac{\phi_s(|x|)}{|x|} x \qquad \text{for } x \in \mathbb{R}^{n+1}. \tag{3.27}$$

Then $\psi_s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz map with

$$\psi_s = \mathrm{Id} \qquad \text{on } \mathbb{R}^{n+1} \setminus B. \tag{3.28}$$

In particular, we have $\psi_s(K) \setminus B = K \setminus B$, and thus we only need to show

$$\limsup_{s \to 0^+} \mathcal{H}^n(\psi_s(K \cap B)) \le \mathcal{H}^n(K' \cap B). \tag{3.29}$$

Since $\psi_s(K \cap B(0, 1-s)) = \{0\}$ we just have to show

$$\limsup_{s \to 0^+} \mathcal{H}^n(\psi_s(K) \cap (B \setminus B(0,1-s))) \le \frac{\mathcal{H}^{n-1}(K \cap \partial B)}{n}. \tag{3.30}$$

We denote by $\mathrm{J}^K\psi_s$ the tangential Jacobian of ψ_s with respect to K, we find by the area formula in Theorem 11.3 in [Mag] and and the coarea formula for rectifiable set

$$\begin{aligned} \mathcal{H}^n(\psi_s(K) \cap (B \setminus B(0,1-s))) &= \int_{K \cap (B \setminus B(0,1-s))} \mathrm{J}^K\psi_s \, \mathrm{d}\mathcal{H}^n = \\ &= \int_{1-s}^1 \int_{K \cap \partial B(0,t) \cap \{\nu \cdot \hat{x} < 1\}} \frac{\mathrm{J}^K\psi_s}{\sqrt{1 - (\nu \cdot \hat{x})^2}} \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t + \\ &\quad + \underbrace{\int_{K \cap (B \setminus B(0,1-s)) \cap \{\nu \cdot \hat{x} = 1\}} \mathrm{J}^K\psi_s \, \mathrm{d}\mathcal{H}^n}_{\mathfrak{J}}. \end{aligned} \tag{3.31}$$

where $\nu(x) \in S^{n+1} \cap (\mathrm{T}_x K)^\perp$ for almost every $x \in K$ and $\hat{x} = \frac{x}{|x|}$. Since $\mathrm{J}^K \psi_s \leq 1$ on $K \cap (B \setminus B(0, 1-s)) \cap \{\nu \cdot \hat{x} = 1\}$ and

$$\lim_{s \to 0} \mathcal{H}^n(K \cap (B \setminus B(0, 1-s))) = 0 \tag{3.32}$$

the term $\mathfrak{J}$ goes to 0 for $s \to 0$. Moreover, for a costant C, it holds that

$$\mathrm{J}^K \psi_s(x) \leq C + \sqrt{1 - (\nu \cdot \hat{x})^2} \phi_s'(|x|) \left(\frac{\phi_s(|x|)}{|x|} \right)^{n-1} \tag{3.33}$$

for $\mathcal{H}^n$-almost every $x \in K$, with this C that give neglible contribution as $s \to 0$. Since we have $\phi_s' = \frac{1}{s}$ on the interval $(1-s, 1)$, so we have

$$\begin{aligned} \int_{1-s}^1 \mathcal{H}^{n-1}(K \cap \partial B(0,t)) \phi_s(t) \left(\frac{\phi_s(t)}{t} \right)^{n-1} \mathrm{d}t = \\ = \frac{1}{s} \int_{1-s}^1 \mathcal{H}^{n-1}(K \cap \partial B(0,t)) \left(\frac{\phi_s(t)}{t} \right)^{n-1} \mathrm{d}t. \end{aligned} \tag{3.34}$$

For the initial choose of t as Lebesgue point for the suitable map, we have

$$\lim_{s \to 0^+} \frac{1}{s} \int_{1-s}^1 |\mathcal{H}^{n-1}(K \cap \partial B(0,t)) - \mathcal{H}^{n-1}(K \cap \partial B)| \, \mathrm{d}t = 0. \tag{3.35}$$

From the previous step we have

$$\begin{aligned} &\limsup_{s \to 0^+} \mathcal{H}^n(\psi(K \cap B)) \leq \\ &\leq \mathcal{H}^{n-1}(K \cap \partial B) \limsup_{s \to 0^+} \frac{1}{s} \int_{1-s}^1 \left(\frac{\phi_s(t)}{t} \right)^{n-1} \mathrm{d}t = \frac{\mathcal{H}^{n-1}(K \cap \partial B)}{n}. \end{aligned} \tag{3.36}$$

So $\mathcal{A}(H, K_0)$ is a good class.

Step 3: Since K_0 is $\mathcal{H}^n$-rectifiable we take any minimizing sequence in $\mathcal{A}(H, K_0)$ consisting of rectifiable sets and apply Theorem 1.1.1: so find that $\mu_j = \mathcal{H}^n \llcorner K_j$ converges weakly* to $\mu = \theta \mathcal{H}^n \llcorner K$, with K is $\mathcal{H}^n$-rectifiable and $\theta \geq 1$. We assume $\varepsilon_j \searrow 0$ such that

$$\inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} \geq \mathcal{H}^n(K_j) - \varepsilon_j. \tag{3.37}$$

In this step we show that in our setting must hold $\theta \leq 1$. We reach the result arguing by contradiction: let's suppose that $\theta(x) = 1 + \sigma$ with $\sigma > 0$ for some $x \in K$ that admits approximate tangent plane. Without loss of generality, as in Step 3 of Theorem 1.1.1, let T be the approximate tangent plane at x,, and change coordinates in a way that $x = 0$ and $\mathrm{T} = \{(y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$. Since

$$\theta(x) = \lim_{r \searrow 0} \frac{\mu(B(x,r))}{\omega_n r^n} \geq 1 \tag{3.38}$$

we can find $r_0 > 0$ such that

$$K \cap B(0,r) \subset B(0,r) \cap \{x \in \mathbb{R}^{n+1} : |x_{n+1}| < \varepsilon r\} = S_{\varepsilon r} \tag{3.39}$$

and

$$1 + \sigma \leq \frac{\mu(\mathrm{cl}(B(0,r)))}{\omega_n r^n} \leq 1 + \sigma + \varepsilon\sigma \tag{3.40}$$

for all $r < r_0$. We fix one of such $r < r_0$ and find $j_0 = j_0(r) \in \mathbb{N}$ such that

$$\mathcal{H}^n(K_j \cap B(0,r)) > \left(1 + \frac{\sigma}{2}\right)\omega_n r^n \tag{3.41}$$

and

$$\mathcal{H}^n((K_j \cap B(0,r)) \setminus S_{\varepsilon r}) < \frac{\sigma}{4}\omega_n r^n \tag{3.42}$$

for all $j \geq j_0$, that combined give

$$\mathcal{H}^n(K_j \cap S_{\varepsilon r}) > \left(1 + \frac{\sigma}{4}\right)\omega_n r^n \quad \text{for all } j \geq j_0. \tag{3.43}$$

So we set

$$X_{\varepsilon r} = \{x = (x', x_{n+1}) \in S_{\varepsilon r} : |x'| < (1 - \sqrt{\varepsilon})r\}, \tag{3.44}$$

and define $f : X_{\varepsilon r} \cup (\mathbb{R}^{n+1} \setminus B(0,r)) \to \mathbb{R}^{n+1}$ with

$$f(x) = \begin{cases} (x', 0) & \text{for } x \in X_{\varepsilon r}, \\ x & \text{otherwise }, \end{cases} \tag{3.45}$$

and notice that $\mathrm{Lip}(f) \leq 1 + C\sqrt{\varepsilon}$. By Kirszbraun Theorem, for which we refer to Theorem 2.10.43 in [F], we consider a Lipschitz extension $\hat{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of f, with the same Lipschitz constant, namely $\mathrm{Lip}(\hat{f}) \leq 1 + C\sqrt{\varepsilon}$. Since $\hat{f} \in \Sigma(H)$ we have

$$\begin{aligned} &\mathcal{H}^n(K_j \cap B(0,r)) - \varepsilon_j \leq \\ &\leq \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap X_{\varepsilon r}))}_{\mathfrak{I}_1} + \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap (S_{\varepsilon r} \setminus X_{\varepsilon r})))}_{\mathfrak{I}_2} + \underbrace{\mathcal{H}^n(\hat{f}(K_j \cap (B(0,r) \setminus S_{\varepsilon r})))}_{\mathfrak{I}_3}. \end{aligned} \tag{3.46}$$

By contruction, we have

$$\mathfrak{I}_1 \leq \omega_n r^n, \tag{3.47}$$

and by the properties presented above we find

$$\mathfrak{I}_3 \leq (\mathrm{Lip}(\hat{f}))^n \mathcal{H}^n(K_j \cap (B(0,r) \setminus S_{\varepsilon r})) < (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4}\omega_n r^n. \tag{3.48}$$

We take $j \to \infty$, so

$$\left(1 + \frac{\sigma}{2}\right)\omega_n r^n \leq \omega_n r^n + \liminf_{j\to\infty} \mathfrak{I}_2 + (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4}\omega_n r^n, \tag{3.49}$$

which is

$$\left(\frac{1}{2} - \frac{(1+C\sqrt{\varepsilon})^n}{4}\right)\sigma \leq \liminf_{j\to\infty} \frac{\mathfrak{I}_2}{\omega_n r^n}. \tag{3.50}$$

From the choose of r and by the monotonicity of $s^{-n}\mu(B(0,s))$ we finally estimate

$$\begin{aligned}\limsup_{j\to\infty} \mathfrak{I}_2 &\leq (1+C\sqrt{\varepsilon})^n \mu(\mathrm{cl}(B(0,r)\setminus B(0,1-\sqrt{\varepsilon}r)) \leq \\ &\leq (1+C\sqrt{\varepsilon})^n((1+\sigma+\varepsilon\sigma)-(1+\sigma)(1-\sqrt{\varepsilon})^n)\omega_n r^n. \end{aligned} \tag{3.51}$$

Since $\sigma > 0$, for ε small enough we reach contradiction between (3.50) and (3.51). So we have $\theta(x) = 1$ for $\mathcal{H}^n$-almost every $x \in K$.

Step 4: Here we show that $\mathcal{H}^n(K_j) \to \mathcal{H}^n(K)$. We only need to exclude any concentration of mass in H or loss of mass at infinity. Let $R_0 > 0$ be such that $H \subset B(0,R_0)$ and consider the Lipschitz map

$$\varphi(x) = \min\{|x|, R_0\}\frac{x}{|x|}. \tag{3.52}$$

Since $\varphi \in \Sigma(H)$ and it follows that

$$\mathcal{H}^n(K_j) - \varepsilon_j \leq \mathcal{H}^n(\varphi(K_j)) \leq \mathcal{H}^n(K_j \cap B(0,2R_0)) + \frac{1}{2^n}\mathcal{H}^n(K_j \setminus B(0,2R_0)), \tag{3.53}$$

and so $\mathcal{H}^n(K_j \setminus B(0,2R_0)) \to 0$. Now it remains to show that there are not loss of mass at H. So we fix $\eta > 0$ and consider $\delta > 0$ and a map π as in the statement, since $\pi \in \Sigma(H)$ and $\mathcal{H}^n(\pi(U_\delta(H))) \leq \mathcal{H}^n(H) = 0$, we have

$$\begin{aligned}\mathcal{H}^n(K) &\leq \limsup_{j\to\infty} \mathcal{H}^n(K_j) \leq \limsup_{j\to\infty} \mathcal{H}^n(\pi(K_j) \leq \\ &\leq (1+\eta)^n \limsup_{j\to\infty} \mathcal{H}^n(K_j \setminus U_\delta(H)) = \\ &= (1+\eta)^n \limsup_{j\to\infty} \mathcal{H}^n\left((K_j \cap \mathrm{cl}(B(0,2R_0))) \setminus U_\delta(H)\right) \leq \\ &\leq (1+\eta)^n \mathcal{H}^n(K_j \cap \mathrm{cl}(B(0,2R_0))) \leq (1+\eta)^n \mathcal{H}^n(K). \end{aligned} \tag{3.54}$$

Since η is arbitrary, we conclude that

$$\limsup_{j\to\infty} \mathcal{H}^n(K_j) = \mathcal{H}^n(K). \tag{3.55}$$

Step 5: In order to conclude the proof we need to show that

$$\mathcal{H}^n(K) = \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H,K)\}. \tag{3.56}$$

Following the ideas in [DPH], we will show that

$$\mathcal{H}^n(K) \leq \mathcal{H}^n(\varphi(K)) \quad \text{for all diffeomorphism } \varphi \in \Sigma(H). \tag{3.57}$$

Let $G(n)$ be the Grassmannian of n-dimensional planes in $\mathbb{R}^{n+1}$, $d(\tau,\sigma)$ the geodesic distance on $G(n)$ and $\mathrm{J}^{\tau}\varphi$ the tangential Jacobian of φ with respect to $\tau \in G(n)$. Given $\varepsilon > 0$, by Lusin Theorem, for example presented as Theorem 4.1 of [Mag], we can find $\delta > 0$ and a compact $\hat{K} \subset K$ such that $\mathcal{H}^n(K \setminus \hat{K}) < \varepsilon$ and K admits approximate tangent plane $\tau(x)$ for $x \in \hat{K}$, with

$$\sup_{x\in\hat{K}} \sup_{y\in B(x,\delta)} |\nabla\varphi(x) - \nabla\varphi(y)| \leq \varepsilon, \tag{3.58}$$

$$\sup_{x\in\hat{K}} \sup_{y\in\hat{K}\cap B(x,\delta)} d(\tau(x),\tau(y)) < \varepsilon, \tag{3.59}$$

and, called $S(x,r) = \{y \in B(x,r) : |x + \tau(x) - y| \leq \varepsilon r\}$, we have $K \cap B(x,r) \subset S(x,r)$ for every $r < \delta$ and $x \in \hat{K}$. By Theorem 2.19 in [AFP], namely the Vitali-Besicovitch covering theorem, and obtain a family of finite closed ball $\{\mathrm{cl}(B_i)\}_{i\in\mathbb{N}}$, with $B_i = B(x_i, r_i) \subset\subset \mathbb{R}^{n+1}\setminus H$, $x_i \in \hat{K}$, and $r_i < \delta$ with

$$\mathcal{H}^n\left(\hat{K} \setminus \bigcup_{i\in\mathbb{N}} \mathrm{cl}(B_i)\right) = 0. \tag{3.60}$$

We can in particular choose a subfamily, which we still call $\{B_i\}_{i\in\mathbb{N}}$,

$$\mathcal{H}^n\left(\hat{K} \setminus \bigcup_{i\in\mathbb{N}} \mathrm{cl}(B_i)\right) < \varepsilon, \tag{3.61}$$

moreover slighty increasing the radii maintaing the balls disjoint and with radii less then δ, with

$$\mathcal{H}^n\left(\hat{K} \setminus \bigcup_{i\in\mathbb{N}} B_i\right) < \varepsilon. \tag{3.62}$$

Reasoning as in Step 3 we can find $j(\varepsilon) \in \mathbb{N}$ and maps $f_i : \mathrm{cl}(B_i) \to \mathrm{cl}(B_i)$ with $\mathrm{Lip}(f_i) \leq 1 + C\sqrt{\varepsilon}$ such that for a certain $X_i \subset S_i = S(x_i, \varepsilon r_i)$ it holds

$$f_i(X_i) \subset B_i \cap (x_i + \tau(x_i)), \tag{3.63}$$

and

$$\mathcal{H}^n\left(f_i((K_j \cap B_i) \setminus X_i)\right) \leq C\sqrt{\varepsilon}\omega_n r_i^n \quad \text{for all } j \geq j(\varepsilon). \tag{3.64}$$

From the estimates above and using the area formula, since by monotonicity we have $\omega_n r_i^n \le \mathcal{H}^n(K \cap B_i)$ and setting $\alpha_i = \mathcal{H}^n((K \setminus \hat{K}) \cap B_i)$ we can state

$$\begin{aligned}
\mathcal{H}^n\left(\varphi(f_i(K_j \cap X_i))\right) &= \int_{f_i(K_j \cap X_i)} \mathrm{J}^{\tau(x_i)}\varphi(x)\, \mathrm{d}\mathcal{H}^n(x) \le \\
&\le \left(\mathrm{J}^{\tau(x_i)}\varphi(x_i) + \varepsilon\right) \omega_n r_i^n \le \left(\mathrm{J}^{\tau(x_i)}\varphi(x_i) + \varepsilon\right) \mathcal{H}^n(K \cap B_i) \le \\
&\le \left(\mathrm{J}^{\tau(x_i)}\varphi(x_i) + \varepsilon\right) \left(\mathcal{H}^n(\hat{K} \cap B_i) + \alpha_i\right) \le \\
&\le \int_{\hat{K} \cap B_i} \left(\mathrm{J}^{\tau(x_i)}\varphi(x_i) + 2\varepsilon\right) \mathrm{d}\mathcal{H}^n(x) + (\mathrm{Lip}(\varphi) + \varepsilon)^n \alpha_i = \\
&= \mathcal{H}^n\left(\varphi(\hat{K} \cap B_i)\right) + 2\varepsilon \mathcal{H}^n\left(K \cap B_i\right) + (\mathrm{Lip}(\varphi) + \varepsilon)^n \alpha_i,
\end{aligned} \tag{3.65}$$

where the last equality follows by the injectivity of φ. Since every $f_i = \mathrm{Id}$ on ∂B_i and $\{\mathrm{cl}(B_i)\}_{i \in \mathbb{N}}$ is a finite disjoint family of closed balls, we can define $f : \mathbb{R}^n \to \mathbb{R}^n$ defining

$$f = \begin{cases} f_i & \text{on } \partial B_i, \\ \mathrm{Id} & \text{on } \mathbb{R}^n \setminus \bigcup_{i \in \mathbb{N}} B_i. \end{cases} \tag{3.66}$$

Thus we have $f \in \Sigma(H)$. From $\mathcal{H}^n\left(f_i((K_j \cap B_i) \setminus X_i)\right) \le C\sqrt{\varepsilon}\omega_n r_i^n$ and $\omega_n r_i^n \le \mathcal{H}^n(K \cap B_i)$, adding up over $i \in \mathbb{N}$ and putting $j \to \infty$ we find

$$\mathcal{H}^n(K_j) - \varepsilon_j \le \mathcal{H}^n(\varphi(f(K_j))) \le \mathcal{H}^n(\varphi(\hat{K})) + \rho(\varepsilon), \tag{3.67}$$

for every $j \ge j(\varepsilon)$, and for $\rho(\varepsilon) \to 0$ for $\varepsilon \to 0$, with dependence only from n, $\mathrm{Lip}(\varphi)$ and $\mathcal{H}^n(K)$. We put in order first $j \to \infty$ and then $\varepsilon \to 0$, proving the claim.

Step 6: By the previous step, the varifold of density one canonically associated to the rectifiable set K, that is the set K equipped with a density function $\theta(x) = 1$ for $x \in K$, is stationary in $\mathbb{R}^{n+1} \setminus H$. By Chapter 5 in [Si], namely the Allard's regularity theorem, there exists an $\mathcal{H}^n$-negligible closed set $S \subset K$ such that $\Gamma = K \setminus S$ is a real analytic hypersurface, and we will use this fact to show that $\mathcal{H}^n(K) \le \mathcal{H}^n(\varphi(K))$ for every $\varphi \in \Sigma(H)$, and concluding that K is a sliding minimizer. Since $\mathcal{H}^n(H \cup S) = 0$ and $\mathcal{H}^n(K) < \infty$ we have

$$\limsup_{j \to \infty} \mathcal{H}^n(K_j \cap U_\delta(H \cup S)) \le \mathcal{H}^n(K_j \cap \mathrm{cl}(U_\delta(H \cup S))) = \varrho(\delta), \tag{3.68}$$

where $\varrho(\delta) \to 0$ for $\delta \to 0$. Called $N_\varepsilon(A)$ the normal ε-neighborhood of $A \subset \Gamma$, then by compactness of $\Gamma_\delta = \Gamma \setminus U_\delta(H \cup S)$ there exists $\varepsilon < \delta$ such that the projection onto Γ defines a smooth map $p : N_{2\varepsilon}(\Gamma_\delta) \to \Gamma_\delta$. Defined the Lipschitz map

$$f_{\varepsilon,\delta} : N_\varepsilon(\Gamma_\delta) \cup U_{\frac{\delta}{2}}(H \cup S) \cup (\mathbb{R}^{n+1} \setminus U_\delta(\Gamma)) \to \mathbb{R}^{n+1}, \tag{3.69}$$

by the conditions

$$f_{\varepsilon,\delta} = \begin{cases} p & \text{on } N_\varepsilon(\Gamma_\delta), \\ \mathrm{Id} & \text{on the rest,} \end{cases} \tag{3.70}$$

and observe that it holds

$$\lim_{\varepsilon \searrow 0} \mathrm{Lip}(f_{\varepsilon,\delta}) = 1. \tag{3.71}$$

For every δ we choose $\varepsilon < \delta$ so that for $f = f_{\varepsilon,\delta}$ we have $\mathrm{Lip}(f) < 2$,and f can be extended to $\hat{f}$ on $\mathbb{R}^{n+1}$ Lipschitz map with the same Lipschitz constant $\mathrm{Lip}(f)$, and $\hat{f} \in \Sigma(H)$. So we have

$$\mathcal{H}^n(\hat{f}(K_j) \setminus \Gamma_\delta) \leq \mathrm{Lip}(\hat{f})^n \mathcal{H}^n(K_j \setminus N_\varepsilon(\Gamma_\delta)). \tag{3.72}$$

Now we observe that $\mathbb{R}^{n+1} \setminus N_\varepsilon(\Gamma_\delta) \subset\subset \mathbb{R}^{n+1} \setminus U_{\frac{\delta}{2}}(K) \cup U_{2\delta}(H \cup S)$, and so

$$\limsup_{j \to \infty} \mathcal{H}^n(K_j \setminus N_\varepsilon(\Gamma_\delta)) \leq \mathcal{H}^n(K \cap U_{2\delta}(H \cap S)) \leq \varrho(2\delta). \tag{3.73}$$

From the two formulas above we deduce

$$\limsup_{j \to \infty} \mathcal{H}^n(\hat{f}(K_j) \setminus \Gamma_\delta) \leq 2^n \varrho(2\delta), \tag{3.74}$$

and, using a diagonal argument, we can select a sequence of maps $f_i \in \Sigma(H)$ such that $\mathcal{H}^n(f_j(K_j) \setminus K) \to 0$ when $j \to \infty$. Since for every K_j there exists a map $\psi_j \in \Sigma(H)$ such that $K_j = \psi_j(K_0)$, we consider the sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subset \Sigma(H)$ with the property

$$\mathcal{H}^n(\varphi_j(K_0)) \setminus K) \to 0 \quad \text{for } j \to \infty. \tag{3.75}$$

In the end we conclude showing that, fixed $\phi \in \Sigma(H)$, we have

$$\begin{aligned} \mathcal{H}^n(\phi(K)) \geq \liminf_{j \to \infty} \mathcal{H}^n(\phi \circ \varphi_j(K_0)) \geq \\ \geq \inf\{\mathcal{H}^n(J) : J \in \mathcal{A}(H, K_0)\} = \mathcal{H}^n(K), \end{aligned} \tag{3.76}$$

so K is a sliding minimizer. □

3.2 A limit of the measure theoretic approach

We now present an application where the above strategy seems to fail: for the well known catenoid problem we are not able to find exactly the expected minimal surface. Infact, considered the set

$$H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, |x_3| = R\}, \tag{3.77}$$

namely two disjont circonferences of radius 1 distant $2R$ from each other, and fixed

$$K_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, |x_3| < R\}, \tag{3.78}$$

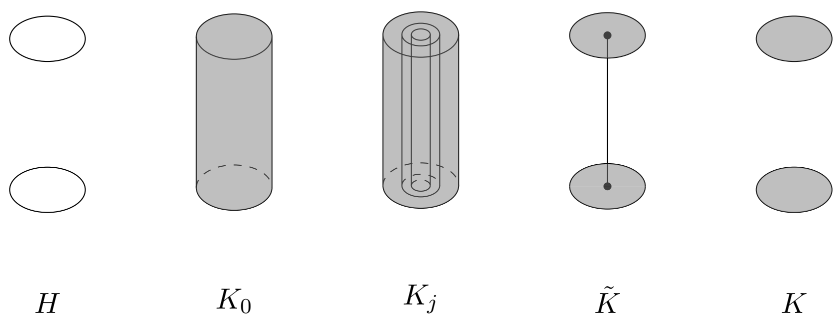
namely the total surface of the cylinder with bases the circonferences in H and choosing R large enough, we know that the minimal surfaces K generated by H is the surface obtain by the union of the two disjoint circles, namely

$$K = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, |x_3| = R\}. \tag{3.79}$$

although $K \notin \mathcal{A}(H, K_0)$, since we cannot make the lateral surface of the cylinder disappear with Lipschitz deformations. However, if we consider a minimizing sequence $\{K_j\}_{j\in\mathbb{N}} \subset \mathcal{A}(H, K_0)$, taking the contractions along the x_3-axis on the lateral surfaces we have

$$\mathcal{H}^2 \llcorner K_j = \mu_j \stackrel{*}{\rightharpoonup} \mu = \mathcal{H}^2 \llcorner \bar{K}, \tag{3.80}$$

where $\bar{K} = K \cup \{(0,0,t) \in \mathbb{R}^3 : |t| \leq R\}$, which is the circles in K with centers joint by a vertical segment, and it holds that $\bar{K} \in \mathcal{A}(H, K_0)$. This is not contraddiction: since the segment $\{(0,0,t) \in \mathbb{R}^3 : |t| \leq R\}$ is $\mathcal{H}^2$-negligible, we have $\mathcal{H}^2(K) = \mathcal{H}^2(\bar{K})$, so the expected minimal surface K is obtain, but the measure-based approach cannot archive the result directly. It is conjectured that, upon $\mathcal{H}^n$-negligible sets, every set K as in Theorem 3.1.1 is an element of $\mathcal{A}(H, K_0)$, eventually requiring some more properties on the boundary.



Bibliography

[AFP] Ambrosio L., Fusco N., Pallara D.: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 1999

[BZ] Burago Y. D., Zalgaller V. A.: *Geometric Inequalities*, Springer, 1988

[DL] De Lellis C.: *Rectifiable Sets, Densities and Tangent Measures*, European Mathematical Society, 2008

[DLGM] De Lellis C., Ghiraldin F., Maggi F.:*A direct approach to Plateau's problem*, J. Eur. Math. Soc. **19**, 2219 - 2240 (2017)

[DPH] De Pauw T., Hardt R.: *Size Minimization and Approximating Problems*, Calc. Var. Partial Differential Equations **17**, 405 - 442 (2003)

[EG] Evans L.C., Gariepy R.F.: *Measure Theory and Fine Properties of Functions. Revised Edition*, CRC Press, 2015

[F] Federer H.: *Geometric Measure Theory*, Springer, 1969

[H] Hirsch M. W.: *Differential Topology*, Springer, 1994

[Mag] Maggi F.: *Sets of Finite Perimeter and Geometric Variational Problems: an Introduction to Geometric Measure Theory*, Cambridge University Press, 2012

[Mat] Mattila P.: *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, Cambridge University Press, 1995

[Si] Simon L.: *Lectures on Geometric Measure Theory*, Australian National University, 1983

[St] Stein E. M.: *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970