DIPARTIMENTO MATEMATICA

Department of Pure and Applied Mathematics
Bachelor's Degree in Mathematics

Maximal Function and Lebesgue Differentiation Theorem

Supervisor:
Prof. Monti Roberto

Candidate:
Bosio Niccolò

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Quote:
Any good idea can be stated in fifty words or less.
(Stanislaw M. Ulam, "Adventures of a Matematician")

Contents

Introduction

This thesis develops around Lebesgue's Differentiation Theorem, progressively trying to generalize its statement.
The first chapter will focus on $\mathbb{R}^n$ with the lebesgue measure, here the 'Maximal function' will be introduced and its properties will be studied.
The differentiation theorem, in fact, will descend as a corollary from the theorem of the Maximal function.

Subsequently we will treat Vitali's cover lemma, necessary for previous proofs, and the last part of the first chapter will end with some application of the theorem, in particular the Marcinkiewicz integral will be introduced and some properties of this function will be studied.

At this point it is natural to wonder if the $\mathbb{R}^n$ environment is necessary to prove the previous theorems; in the second chapter we will try to answer this question by showing that in some metric spaces equipped with regular Borel measures, it is possible to show the same theorems as the first chapter using a cover lemma very similar to vitali's one.

Finally, in the last chapter, the study of the differentiation theorem in $\mathbb{R}^n$ with measures other than that of Lebesgue will be deepened. We will show that it is possible to generalize the previous concepts but not for free, in fact it is necessary to narrow yourself to metric balls and not to general families of regular sets as in the first two cases.
The latter section will close with a study of the cover lemma used to show the third variant of the theorem: the Besicovitch cover lemma.

Chapter 1

The study of Lebesgue Differentiation Theorem on $\mathbb{R}^n$ with Lebesgue measure

In this chapter we are going to study the properties of the maximal function and the applications on Differentiation Theorem. We will analyze the behaviour of this function in $\mathbb{R}^n$ with the Lebesgue measure.
Let's start with some definitions:

Definition 1.1. We define the maximal function as:

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_B |f(y)|\, dy. \tag{1.1}$$

It is to be noticed that nothing excludes the possibility that $M(f)(x)$ is infinite for any given x.

Definition 1.2. Let be $\lambda(\alpha) = m(x \mid |g(x)| > \alpha)$. This is called the distribution function of g with g a measurable function.

Obviously we are interested in studying the properties of this function when is defined on a non bounded set. This function describes the relative largeness of the function g: it is very useful for many applications and for the proof of the principal theorem of this chapter.

Observation 1.3. It's useful to notice that

$$\int_{\mathbb{R}^n} |g(y)|^p\, dy = p \int_0^\infty \alpha^{p-1}\, \lambda(\alpha)\, d\alpha. \tag{1.2}$$

We will use this equality during the proof of Theorem 1.5.

Proof.

$$\int_{\mathbb{R}^n} |g(y)|^p\, dy = \int_{\mathbb{R}^n} \left(\int_0^{|g(y)|^p} 1\, dt \right) dy = \int_{\mathbb{R}^n} \int_0^{\infty} \chi_{[0,|g(y)^p|]}(t)\, dt\, dy =$$
$$= \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\{y \in \mathbb{R}^n s.a. |g(y)|^p > t\}}\, dy\, dt = \int_0^{\infty} \lambda(t^{\frac{1}{p}}) dt.$$

Now we use this replacement: $t^{\frac{1}{p}} = \alpha, dt = p\alpha^{p-1} d\alpha^{\cdot}$
The result is:

$$p \int_0^{\infty} \alpha^{p-1}\, \lambda(\alpha)\, d\alpha.$$

□

Observation 1.4. If $g \in L^{\infty}$ we can claim that $\|g\|_{\infty} = \inf\{\alpha \,|\, \lambda(\alpha) = 0\}$ and

$$\int_{\mathbb{R}^n} |g(y)|\, dy \geq \alpha\lambda(\alpha). \tag{1.3}$$

Proof. Te first equality comes from the definition of $\|g\|_{\infty}$. Let's prove the 1.3 inequality:

$$\int_{\mathbb{R}^n} |g(y)|\, dy \;\geq\; \int_{\{x \,|\, g(x) > \alpha\}} |g(y)| dy \;\geq\; \int_{\{x \,|\, g(x) > \alpha\}} \alpha\, dy = \alpha\, \lambda(\alpha)$$

□

After these small observations we can introduce the first important theorem of this chapter:

Theorem 1.5. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and A a constant depending on p and n*
a) If $f \in \mathbb{L}^p(\mathbb{R}^n); 1 \leq p \leq \infty$ we have that:

$$M(f) < \infty \; a.e.$$

b) If $f \in \mathbb{L}^1(\mathbb{R}^n)$ we have that forall $\alpha > 0$:

$$m\{y \,|\, M(f)(x) > \alpha\} \leq \frac{A}{\alpha} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

c) If $f \in \mathbb{L}^p(\mathbb{R}^n);\; 1 < p \leq \infty$ we have that:

$$M(f) \in \mathbb{L}^p(\mathbb{R}^n) \text{ and } \|M(f)\|_p \leq A\, \|f\|_p.$$

Observation 1.6. In c), when p=1 and $f \neq 0$, we can't claim that $M(f) \in \mathbb{L}^1(\mathbb{R}^n)$:
We can observe that

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B_r(x)} \int_B |f(y)|\, dy \geq \frac{C}{|x|^n}; \;\; \forall\, |x| \geq 1$$

where C is a constant depending on $\|f\|_1$ and $|x|^n$ comes from the measure of the n-dimensional ball. Now if we try to computing $\|M(f)\|_1$:

$$\|M(f)\|_1 \geq \int_{\mathbb{R}^n} \frac{C}{|x|^n}\, d^n x$$

and passing by the n-dimensional spherical coordinates we found:

$$\int_0^\infty \frac{K^*}{r^n} r^{n-1}\, dr = +\infty$$

so we have to ask stronger conditions to f than the integrability to be $M(f)$ integrable.
(*)=K is a constant that comes from the change of variables rule and C, the prior constant.

Observation 1.7. The result obtained in b) is the best possible estimator: if we consider $f = \delta_0$ we have:

$$M(f)(x) \geq \frac{1}{m(B_{|x|}(x))} \int_B \delta_0(y)\, dy = \frac{1}{m(B)} = \frac{1}{C\,|x|^n}.$$

In this case

$$\lambda(\alpha) = m\{x \,|\, \frac{1}{|x|^n} > C\,\alpha\} = m(B_{\frac{1}{C\,\alpha}})(0) = Vol(B_1(0))\,\frac{1}{C^n \alpha^n} = \frac{1}{C^{n-1}\alpha^n}.$$

Now we put n=1 and we have concluded.

Now we can prove the Theorem 1.5.

Proof. b) We define $E_\alpha = \{x \,|\, M(f)(x) > \alpha\}$ so forall $x \in E_\alpha$ exists $B_r(x)$ such that:

$$\int_B |f(y)|\, dy \geq \int_B \alpha\, dx \geq \alpha\, m(B_r(x)).$$

This implies:

$$m(B) \leq \frac{\|f\|_1}{\alpha}.$$

Thanks to the Vitali's 5 covering lemma we can extract some disjointed balls such that

$$\sum_{k=0}^{\infty} mB_k \geq Cm(E_\alpha).$$

So

$$\|f\|_1 \geq \int_{\cup B_k} |f(y)|\, dy \geq \alpha \sum_{k=0}^{\infty} m(B_k) \geq \alpha\, C\, m(E_\alpha)$$

Let $A = \frac{1}{C}$ then we find

$$\frac{\|f\|_1\, A}{\alpha} \geq m(E_\alpha) = \lambda(\alpha).$$

Now let's prove a) and c):
If $f \in \mathbb{L}^\infty$ then

$$\sup_{r>0} \frac{1}{m(B_r(x))} \int_B |f(y)|\, dy \leq \|f\|_\infty \frac{m(B)}{m(B)} = \|f\|_\infty$$

and so

$$\|M(f)\|_\infty = \|f\|_\infty.$$

Now we consider $p \in (1; \infty)$:
Let be

$$f_1(x) = \begin{cases} f(x)\ if\ |f(x)| > \frac{\alpha}{2} \\ 0 \text{ otherwise} \end{cases}$$

It's obvious that

$$|f(x)| \leq |f_1(x)| + \frac{\alpha}{2};\ \ \forall\, x \in \mathbb{R}$$

and

$$M(f)(x) \leq M(f_1 + \alpha)(x) = M(f)(x) + \alpha.$$

That implies:

$$\{x \,|\, M(f)(x) > \alpha\} \subset \{x \,|\, M(f_1)(x) + \frac{\alpha}{2} > \alpha\} = \{x \,|\, M(f_1)(x) > \frac{\alpha}{2}\}$$

Now we can use point b):

$$m(E_\alpha) = m\{x \,|\, M(f)(x) > \alpha\} \leq \frac{2\, A\, \|f_1\|_\infty}{\alpha}$$

which mean

$$m(E_\alpha) \leq \frac{2\, A}{\alpha} \int_{\{x\,|\,|f(x)|>\frac{\alpha}{2}\}} |f(x)|\, dx.$$

Let $g = M(f)$ and λ the distribution function of g:

$$\int_{\mathbb{R}^n} |g(x)|^p\, dx = p \int_0^\infty \alpha^{p-1}\, \lambda(\alpha)\, d\alpha$$

(Observation 1.3).
So

$$\|g\|_p^p = p\int_0^\infty \alpha^{p-1}\, m(E_\alpha)\, d\alpha \le p\int_0^\infty \alpha^{p-1}\left(\frac{2\,A}{\alpha}\int_{|f|>\frac{\alpha}{2}} |f(x)|\, dx\right) d\alpha =$$

$$= p\int_0^\infty \alpha^{p-1}\left(\frac{2\,A}{\alpha}\int_{\mathbb{R}^n} |f(x)|\chi_{|f|>\frac{\alpha}{2}}(x)\, dx\right) d\alpha = *$$

$$= p\,2\,A\int_{\mathbb{R}^n} |f(x)|\left(\int_{|f|>\frac{\alpha}{2}} \alpha^{p-2}\, d\alpha\right) dx = p\,2\,A\int_{\mathbb{R}^n} |f(x)|\frac{(2|f(x)|)^{p-1}}{p-1}\, dx =$$

$$= \frac{2^p\,A\,p}{p-1}\int_{\mathbb{R}^n} |f(x)|^p\, dx = A_p\,\|f\|p.$$

*=In this passage we used Tonelli-Fubini.
So we managed to prove that:

$$\|M(f)\|_p^p \le A\,\|f\|_p^p.$$

□

There is a very important corollary of this theorem called the Lebesgue differentiation Theorem:

Corollary 1.8. *If* $f \in L^p(\mathbb{R}^n),\ 1 \le p \le \infty$; *or if* f *is locally integrable then:*

$$\lim_{r\to 0} \frac{1}{m(B_r)(x)}\int_B f(y)dy = f(x) \tag{1.4}$$

Proof. Let define

$$f_r(x) = \frac{1}{m(B_r(x))}\int_B f(y)\, dy;\ r > 0.$$

It's easy to prove that:

$$r \to 0 \Rightarrow \|f_r - f\|_p \to 0.$$

We know that C_c^0 is dense in L^p so we can write $f = f_1 + f_2$ with $f_1 \in C_c^0$ and $\|f_2\|_p < \epsilon$, forall $\epsilon \in \mathbb{R}$.
Define $\Delta(y) = \|f(x-y) - f(x)\|_p$ and writing $\Delta(y)$ with $f = f_1 + f_2$ we find $\Delta \le \Delta_1 + \Delta_2$.
Notice that $\Delta_1 \to 0$ for $y \to 0$ as an immediate consequence of the uniform convergence (f_1 is continuous with a compact support) and $\Delta_2 \le 2\epsilon$ so

$\Delta(y) \to 0 \; for \; y \to 0.$
Let be

$$\phi_r(y) = \frac{1}{m(B_r(x))}\,\chi_B(y)$$

and notice that

$$\int_{\mathbb{R}^n} \phi_r(y)\,dy = 1.$$

Now we have

$$f_r(x-y) = \int_B f(x-y)\,\frac{1}{m(B)}dy = \int_{\mathbb{R}^n} f(x-y)\frac{\chi_B(y)}{m(B)}\,dy =$$
$$= \int_{\mathbb{R}^n} f(x-y)\phi_r(y)\,dy = (f*\phi_r)(x).$$

Clearly

$$f*\phi_r - f = \int_{\mathbb{R}^n} [f(x-y) - f(x)]\phi_r(y)\,dy,$$

because

$$\int_{\mathbb{R}^n} \phi_r(x)\,dx = 1.$$

So

$$\|f_r - f\|_p = \|f*\phi_r - f\|_p \leq \int_{\mathbb{R}^n} \Delta(y)|\phi_r(y)|\,dy =$$
$$= \int_{\mathbb{R}^n} \Delta(ry)|\phi_1(y)|\,dy \to 0 \; for \; r \to 0$$

where in the first inequality we used Minkowsky and in the last limit we used the Lebesgue dominated convergence theorem.
What remains to be seen is that $\lim_{r\to 0} f_r(x)$ exists almost everywhere: for this purpose we denote forall $g \in L^1$ and $x \in \mathbb{R}^n$,

$$\Omega g(x) = |\limsup_{r\to 0} g_r(x) - \liminf_{r\to 0} g_r(x)|.$$

(We reduce the consideration to the case p=1).
If g is continuous with compact support, then $g_r \rightrightarrows g$ so $\Omega g = 0$.
If $g \in L^1$ we can use b) of Theorem 1.5:

$$m\{x|\, 2M(g)(x) > \epsilon\} \leq \frac{2\,A}{\epsilon}\|g\|_1.$$

Clearly $\Omega g \leq 2M(g)$, thus

$$m\{x|\, \Omega g(x) > \epsilon\} \leq \frac{2\,A}{\epsilon}\|g\|_1.$$

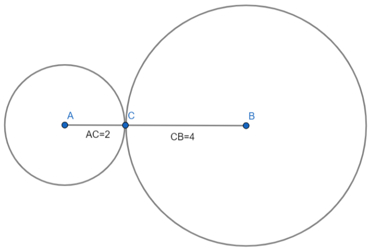

Figure 1.1: this is the limit case where $\operatorname{diam}(B_i) = \frac{1}{2}\operatorname{diam}(B_j)$ and they are tangent

Finally we can write g as $g_1 + g_2$ with $g_1 \in C_c^1$ and g_2 such that $\|g_2\|_1 < \epsilon^2$ so

$$m\{x|\,\Omega g(x) > \epsilon\} \leq \frac{2\,A}{\epsilon}\epsilon^2 \to 0 \ \ for\ \epsilon \to 0.$$

$\square$

Now we must study the Vitali's 5 covering Lemma that we used for the proof of the Theorem 1.5 and we will explain why the constant A of the 1.5 Theorem is $2(\frac{5^n\,p}{p-1})^{\frac{1}{p}}$.

Lemma 1.9. *Let E be a measurable subset of $\mathbb{R}^n$ which is covered by the union of a family of balls $\{B_i\}$, of bounded diameter. Then from this family we can select a disjointed subsequence, $B_1, ..., B_n, ...$ (finite or infinite) such that:*
$\sum_n m(B_n) \geq Cm(E)$.

Proof. Let's take $B_1; ...; B_k$ such that $\operatorname{diam}(B_{k+1}) \geq \frac{1}{2}\{\sup(d(B_j) \mid B_j$ is disjointed with $B_1; ...; B_k)\}$.We call $\{1; ...; k\} = \mathcal{K}$.
This sequence could be finite or infinite: if it's finite then it's impossible to find B_{k+1} disjointed with $B_1; ...; B_k$ so we have to show that if we define $B_i^* = 5B_i$ forall $i \in \{1; ...; k\}$ we find $\bigcup_{i\leq k} B_i^* \supset E$.
Let's show that forall $j \in \mathbb{N}$ $B_j \subset \cup_{i\leq k} B_i^*$:
We know that forall B_j exists $i \leq k$ such that $\operatorname{diam}(B_i) \geq \frac{1}{2}\operatorname{diam}(B_j)$ and $B_i \cap B_j \neq \emptyset$, otherwise we would have taken B_j in $\mathcal{K}$.
So $5B_i \supset B_j$.
The same proof works for an infinite set of index $\mathcal{K}$ with $\sum_k m(B_k) < \infty$; if it's equal to ∞ the proof is trivial.

$\square$

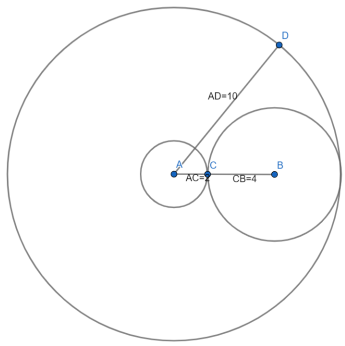

Figure 1.2: we multiplied the radius of the small ball by 5.

Now a legitimate question is: what if we try to apply the theorem 1.5 with a different family of sets? We will see that it works if the family has a properties:

Definition 1.10. A family of sets $\mathcal{F}$ is regular if forall $S \in \mathcal{F}$ exists $B_r(0)$ such that $S \subset B_r(0)$ and $m(S) \geq C\, m(B)$.

Definition 1.11. We can define a different maximal function:

$$M_{\mathcal{F}}(f(x)) = \sup_{S \in \mathcal{F}} \frac{1}{m(s)} \int_S |f(x-y)\, dy. \tag{1.5}$$

Observation 1.12. $M_{\mathcal{F}}(f(x)) \leq C^{-1} M(f(x))$ so $M_{\mathcal{F}}$ satisfies the same conclusions of Theorem 1.5.
In particular, if f is locally integrable we have:

$$\lim_{S \in \mathcal{F};\, m(S) \to 0} \frac{1}{m(S)} \int_S f(x-y)\, dy = f(x) \tag{1.6}$$

Proof. We know that exists $B_r(0)$ such that $B \supset S$ and $C\, m(B) \leq m(S)$ so

$$\sup_{S \in \mathcal{F}} \frac{1}{m(S)} \int_S |f(x-y)|\, dy \leq \sup_{S \in \mathcal{F}} \frac{1}{C\, m(B)} \int_S |f(x-y)|\, dy \leq$$
$$\leq \frac{1}{C\, m(B)} \int_B |f(y)|\, dy \leq C^{-1} M(f(x)).$$

□

Observation 1.13. The equation 1.4 is true almost everywhere but the exeptional set where 1.4 is not valid depends on $\mathcal{F}$. Our goal is to find the exeptional set depending on f; we consider the relation:

$$\lim_{r \to 0} \frac{1}{m(B_r(0))} \int_B |f(y) - c|\, dy = |f(x) - c| \tag{1.7}$$

It's valid almost everywhere except on a set E_c such that $m(E_c) = 0$.
Now forall $c \in \mathbb{Q}$ be $E = \bigcup_{c \in \mathbb{Q}} E_c$: obviously $m(E) = 0$ and forall $x \notin E$ the relation 1.5 works.

Definition 1.14. An element $x \in E$ is called "point of density" of E if:

$$\lim_{r \to 0} \frac{m(E \cap B_r(0))}{m(B_r(0))} = 1 \tag{1.8}$$

Observation 1.15. If we apply (1.5) to x "point of density" of E with $f = \chi_E$ we find:

$$\lim_{r \to 0} \frac{1}{m(B_r(0))} \int_B \chi_E(y)\, dy = \chi_E(x) = 1 = \lim_{r \to 0} \frac{m(E \cap B_r(0))}{m(B_r(0))} \tag{1.9}$$

Proposizione 1.16. *For almost every $x \in E$ the limit (1.5) holds then almost every $x \in E$ is a point of density.*

From now we will consider $E = \overline{E}$ and it's not restrictive because m is a regular measure.
$\delta(x; F)$ will be the distance from x to F closed set. Obviously
$\delta(x; F) = 0 \iff x \in F$;
$\forall x \in F,\ \delta(x + y; F) \leq |y|$;
$\forall \epsilon > 0,\ \exists \eta \,|\, |y| < \eta$ such that $\delta(x + y; F) < \epsilon |y|$.

Proposizione 1.17. *Let F a closed set. For almost every $x \in F$*

$$\delta(x + y; F) = o(|y|).$$

Proof. Let $x \in F$ be point of density.
Obviously $B(x + y; \epsilon\, |y|) \subset B(x; |y| + \epsilon\, |y|)$.
We claim that exists $z \in F$ such that $z \in B(x + y; \epsilon |y|)$ if $|y|$ is 'small'. Otherwise:

$$\frac{m(F \cap B(x; |y| + \epsilon\, |y|))}{m(B(x; |y| + \epsilon\, |y|))} \leq \frac{m(B(x; |y| + \epsilon\, |y|)) - m(B(x + y; \epsilon\, |y|))}{m(B(x; |y| + \epsilon\, |y|))} \leq 1 - \left(\frac{\epsilon}{1 + \epsilon} \right)^n \neq 1$$

so x is not a point of density. $\perp$
Thus exists $z \in F$ such that $z \in B(x + y; \epsilon\, |y|)$ forall ϵ small, therefore:

$$\delta(x + y; F) \leq \delta(x + y; z) \leq \epsilon\, |y|.$$

$\square$

The last thing that we present in this chapter is the Marcinkiewicz integral:

$$I(x) = \int_{\{|y| \leq 1\}} \frac{\delta(x + y; F)}{|y|^{n+1}}\, dy \tag{1.10}$$

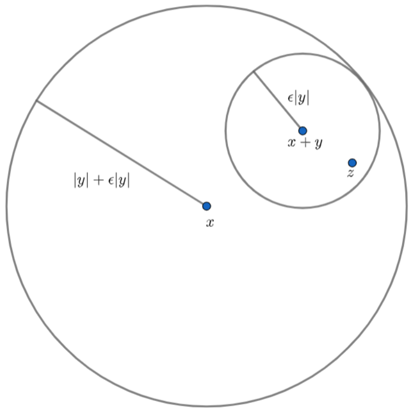

Figure 1.3

Theorem 1.18. *Let $I(x)$ be the Marcinkiewicz integral and F a closed set in $\mathbb{R}^n$.*
a) If $x \in F^c$ then $I(x) = \infty$.
b) For almost every $x \in F$: $I(x) < \infty$

Proof. a) Obviously exists $c > 0$ such that $\delta(x+y) > c$; this implies

$$I(x) > c \int_{\mathbb{R}^n} \frac{1}{|y|^{n+1}}\, dy = \infty$$

b) this result is a simple consequence of the following lemma. □

Lemma 1.19. *Let F be a closed set whose complement has finite measure. Let*

$$I_*(x) = \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}}\, dy. \tag{1.11}$$

Then $I_*(x) < \infty$ for almost every $x \in F$. Moreover:

$$\int_F I_*(x)\, dx \leq c\, m(F^c). \tag{1.12}$$

Proof. Clearly is sufficient to prove (1.10) since the integrand is positive:

$$\int_F I_*(x)\, dx = \int_F \int_{\mathbb{R}^n} \frac{\delta(x+y)}{|y|^{n+1}}\, dy\, dx = \int_F \int_{\mathbb{R}^n} \frac{\delta(y)}{|x-y|^{n+1}}\, dy\, dx =$$
$$= \int_F \int_{F^c} \frac{\delta(y)}{|x-y|^{n+1}}\, dy\, dx = \int_{F^c} \delta(y) (\int_F \frac{dx}{|x-y|^{n+1}})\, dy.$$

Let's consider the integral on F:
$y \in F^c$ implies $|x-y| \geq \delta(y)$ thus:

$$\int_F I_*(x)\, dx \leq \int_{|x| \geq \delta(y)} \frac{dx}{|x|^{n+1}} = \int_{\delta(y)}^{\infty} \frac{c\, r^{n-1}}{r^{n+1}} dr = \frac{c}{\delta(y)}.$$

So

$$\int_F I_*(x)\,dx \leq \int_{F^c} c\,\delta(y)\,\delta(y)^{-1}\,dy = c\,m(F^c).$$

$\square$

Now we can finish the proof of the theorem 1.18:
$F_m = F \cup B_m(0)^c$.
F_m is closed and $(F \cup B_m^c)^c \subset B_m$.
We can apply the lemma 1.19 to F_m with δ_m the distance from F_m.
Forall F exists $m > 0$ such that $\delta(x+y) = \delta_m(x+y)$ if $|y| \leq 1$ and $x \in B_{m-2}$.
For the lemma 1.19: $I(x) < \infty$ for almost every $x \in B_{m-2} \cap F$.

Chapter 2

The study of Lebesgue Differentiation Theorem on doubling metric spaces

In this chapter we will redefine the structures as in the previous chapter but generalizing the definition of Maximal Function on some particular metric spaces: doubling metric spaces. Let's start defining the properties of the measure that we need for this chapter:

Definition 2.1. Let μ be a outer measure on (X, d) metric space.
It's called a Borel regular measure if:
1) For every B Borel set and for every $A \subset X$ we have: (Borel condition)

$$\mu(A) = \mu(A \cap B) + \mu(A \cap B^c).$$

2) For every set $A \subset X$ there exists an open set $B \supset A$ such that: (regular condition from above)

$$\mu(A) = \mu(B).$$

3) For every set $A \subset X$ there exists a compact set $K \subset A$ such that: (regular condition from below)

$$\mu(A) = \mu(K).$$

Definition 2.2. Let (X, d, μ) be a metric measure space, it is said to be doubling if there exists a constant $C > 0$ such that:

$$0 < \mu(B_{2r}(x)) \leq C\,\mu(B_r(x))\;\forall\, x \in X,\; r \in \mathbb{R}. \tag{2.1}$$

Looking at the properties of the Lebesgue measure that we used for the Vitali's covering lemma we notice the importance of the previous property of m:
we said that if we multiply the diameter of the selected balls for 5 we found

that every balls is contained in this new set and this implied the thesis: $\sum_n m(B_n) \geq Cm(E)$.
It's looks like very trivial that $m(B_{5r}(x)) = C\, m(B_r(x))$ but if we change measure and we chose μ as a non positive measure we could even find $\mu(B_{5r}(x)) < \mu(B_r(x))$. So if finding C such that $0 < \mu(B_{2r}(x)) \leq C\, \mu(B_r(x))$ is impossible, the proof of the Vitali's covering theorem falls apart.

Now we can define the maximal function in (X, μ) doubling space with μ regular and Borel measure.

Definition 2.3. Let $f \in L^p(X)$, we define the maximal function of f:

$$M(f)(x) = \sup_{\delta>0} \frac{1}{\mu(B_\delta(x)} \int_{B_\delta(x)} |f|\, d\mu. \tag{2.2}$$

From now we will use the following notation:

$$\sup_{\delta>0} \frac{1}{\mu(B_\delta(x)} \int_{B_\delta(x)} |f|\, d\mu = \sup_{\delta>0} \fint_B |f|\, d\mu.$$

Observation 2.4. If $X = \mathbb{R}^n$ and $M(f)$ is lower semi-continuous than $M(f)$ is measurable but this claim is not true with a general metric space.
For this reason we define the following "decentralized" maximal function:

$$\tilde{M}(f)(x) = \sup_{B\,|x\in B} \fint_B |f|\, d\mu. \tag{2.3}$$

We observe that $\tilde{M}(f) \geq M(f)$ and $\tilde{M}$ is lower semi-continuous, in fact $\{x|\tilde{M}(f)(x) > \alpha\}$ is open forall $\alpha \in \mathbb{R}$.

Proof. Let's prove that $\{x|\tilde{M}(f)(x) > \alpha\}$ is open:
if $x \in \{\tilde{M}(f) > \alpha\}$ then $\tilde{M}(f)(x) > \alpha$ so we know that exists B such that $\fint_B |f|\, d\mu > \alpha$ now clearly if $y \in B$ than $\tilde{M}(f)(y) > \alpha$ thus $B \subset \{\tilde{M}(f) > \alpha\}$. ☐

Now we are ready to state the Maximal Function Theorem:

Theorem 2.5. *Let (X, d) be a doubling metric spaces and μ a regular Borel measure on X.*
If $p > 1$ and $f \in L^p(X)$ then $\tilde{M}(f) \in L^p(X)$ and exists a constant C_p depending on p such that:

$$\|\tilde{M}(f)\|_p \leq C_p(X)\|f\|_p. \tag{2.4}$$

If $p = 1$ then:

$$\mu(\{\tilde{M}(f) > \alpha\}) \leq \frac{C_p}{\alpha}\|f\|_1.$$

Before proving this theorem we need a new covering lemma, similar to the Vitali's one but for compact sets.

Lemma 2.6. *Let (X, d, μ) be a doubling metric space. Let $K \subset X$ a compact set and let $\{B_i \,|\, i \in I\}$ a finite cover of X. Then there exists a subcover $\{\bar{B}_j \,|\, j \in J \subset I\}$ such that:*
1) $\bar{B}_j \cap \bar{B}_i = \emptyset$ for $i \neq j$
2) $K \subset \bigcup 3\bar{B}_j$

Proof. Let be $\bar{B}_1 \in \{B_i\}_{i\in I}$ such that $\mathrm{diam}(\bar{B}_i) = \max\{\mathrm{diam}(B_i)|i \in I\}$.
Let be $\bar{B}_2$ such that $\bar{B}_1 \cap \bar{B}_2 = \emptyset$ and $\mathrm{diam}(\bar{B}_2) = \max\{\mathrm{diam}(B_i)|i \in I \neq 1\}$. (Notice that exists because the cover is finite.) With this method we define a subset of I.
Now it's easy to see that if we consider $A = \bigcup_{i=1}^{p} \bar{B}_i$ every $B_j, j \in I$ is contained in A.

□

Now we can prove Theorem 2.5:

Proof. 1) Let's start with $p = 1$. We have to prove that if $f \in L^1(X)$:

$$\mu(\{\tilde{M}(f) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1.$$

Let K be a subset of $\{\tilde{M}(f) > \alpha\}$, thus:

$$\|f\|_1 = \int_X |f|\, d\mu \geq \int_K |f|\, d\mu \geq \mu(K)\, \alpha\, C.$$

The last inequality comes from lemma 2.6 and the regularity of μ:
if $x \in K$ then $\tilde{M}(f)(x) > \alpha$ and so exists B_x such that:

$$\frac{1}{\mu(B_x)} \int_{B_x} |f|\, d\mu > \alpha \implies \mu(B_x) < \frac{1}{\alpha} \int_{B_x} |f|\, d\mu.$$

So if we take $\{B_i|i \in I\}$ a cover of K and we extract $\{\bar{B}_j|j \in J \subset I\}$ with the properties of lemma 2.6:

$$\mu(K) \leq \sum_{j\in J} \mu(3\bar{B}_j) \leq 3\sum_{j\in J} \mu(\bar{B}_j) \leq \frac{3}{\alpha}\sum_{j\in J}\int_{\bar{B}_j} |f|\, d\mu = \frac{3}{\alpha}\int_{\cup\bar{B}_j} |f|\, d\mu \leq \frac{C}{\alpha}\|f\|_1. \tag{2.5}$$

Thanks to the regularity of μ we can conclude.
2) Now let's consider $p > 1$:

We define $f_1 : X \longrightarrow \mathbb{R}$ such that: $f_1 = \begin{cases} f(x) \;\; if \; |f(x)| > \alpha \\ 0 \;\text{ otherwise.} \end{cases}$

Clearly:

$$|f| \leq |f_1| + \alpha; \;\; \forall x \in X.$$

Thanks to the monotony of $\tilde{M}$ we can claim that

$$\tilde{M}(f) \leq \tilde{M}(f_1) + \tilde{M}(\alpha) = \tilde{M}(f_1) + \alpha$$

This implies:

$$\{\tilde{M}(f) > 2\alpha\} = \{\tilde{M}(f) - \alpha > \alpha\} \subset \{\tilde{M}(f_1) > \alpha\},$$

so

$$\mu\{\tilde{M}(f) > 2\alpha\} \leq \mu\{\tilde{M}(f_1) > \alpha\} \leq (*)\frac{C}{\alpha}\int_X |f_1|\, d\mu = \frac{C}{\alpha}\int_{|f|>\alpha} |f|\, d\mu$$

(*)= this equality comes from the series of inequalities 2.5.
Now we can prove 2.4:

$$\begin{aligned}
\|\tilde{M}(f)\|_p^p &= \int_X |\tilde{M}(f)|^p\, d\mu \\
&= p\int_0^\infty \alpha^{p-1}\, \mu\{x|\, \tilde{M}(f)(x) > \alpha\}\, d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\left(\frac{C}{2\alpha}\int_{|f|>\alpha} |f|\, d\mu\right) d\alpha \\
&= \frac{p\,C}{2}\int_0^\infty \alpha^{p-2}\left(\int_X |f|\chi_{|f|>\alpha}\, d\mu\right) d\alpha \\
&= \frac{p\,C}{2}\int_X |f|\left(\int_{\alpha<|f|} \alpha^{p-2}\, d\alpha\right) d\mu \\
&= \frac{p\,C}{2(p-1)}\|f\|_p^p.
\end{aligned}$$

$\square$

Chapter 3

The study of Lebesgue Differentiation Theorem on $\mathbb{R}^n$ with Radon measures

In this third chapter we are going to study the same theorem but with a different type of measures: the Radon measures. Let's start with a definition:

Definition 3.1. A Radon measure defined on the σ-algebra of Borel sets is a regular measure that is finite on all of the compact sets of X.

Our goal is to study the differentiation of Radon measures and to do that we must define some functions:

Definition 3.2. Let μ, ν be Radon measures on $\mathbb{R}^n$.
For each point $x \in \mathbb{R}^n$, define:

$$\overline{D}_\mu \nu(x) \begin{cases} \limsup_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}; & \mu(B) > 0 \, \forall r > 0 \\ +\infty \ \ otherwise \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}; & \mu(B) > 0 \forall r > 0 \\ +\infty \ \ otherwise \end{cases}$$

Definition 3.3. If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < \infty$ we say that ν is differentiable with respect to μ at x and we write:

$$\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) = D_\mu \nu(x)$$

We will call $D_\mu \nu$ the density of ν with respect to μ.

We want to study two things about the density function:
1) when exists,
2) when ν can be recovered by integrating $D_\mu \nu$. To reach this goal we must start with an important lemma:

Lemma 3.4. *Fix* $0 < \alpha < \infty$:

$$A \subset \{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha\} \implies \nu(A) \leq \alpha\mu(A) \tag{3.1}$$

$$B \subset \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) \geq \alpha\} \implies \nu(A) \geq \alpha\mu(A) \tag{3.2}$$

Proof. We can assume $\mu(\mathbb{R}^n)$ and $\nu(\mathbb{R}^n)$ finite, since we could otherwise consider these two measures restricted to compact subsets of $\mathbb{R}^n$.
Fix $\epsilon > 0$ and U open subset of $\{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha\}$ such as $A \subset U$.
Let's define $\mathcal{F} = \{B \mid B = B(a,r), a \in A, B \subset U, \nu(B) \leq (a+\epsilon)\mu(B)\}$.
We notice that $\inf\{r \mid B(a,r) \in \mathcal{F}\} = 0 \ \forall a \in A$ so we can use a corollary of the Besicovitch covering theorem that assures us that exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\nu(A - \bigcup_{B \in \mathcal{G}} B) = 0$$

Then:

$$\nu(A) \leq \sum_{B \in \mathcal{G}} \nu(B) \leq (\alpha+\epsilon) \sum_{B \in \mathcal{G}} \mu(B) \leq (\alpha+\epsilon)\mu(U).$$

We can conclude thanks to the regularity of ν and μ.

□

Now we can enunciate a theorem that answer to the first question we did after the definition 3.3 about the density function:

Theorem 3.5. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Than $D_\mu \nu$ exists and is finite μ-almost everywhere. Furthermore, $D_\mu \nu$ is measurable.*

Proof. As before we can assume $\mu(\mathbb{R}^n)$ and $\nu(\mathbb{R}^n) < \infty$.
1)$D_\mu \nu$ exists and is finite μ a.e.:
Let $I = \{x \mid \overline{D}_\mu \nu(x) = \infty\}$, and for all $0 < a < b$ let $R(a,b) = \{x \mid \underline{D}_\mu \nu(x) < a < b < \overline{D}_\mu \nu(x) < \infty\}$.
Now we can apply the lemma 3.4 observing that for each $\alpha > 0$, $I \subset \{x \mid \overline{D}_\mu \nu(x) \geq \alpha\}$. So we can claim that $\mu(I) \leq \frac{1}{\alpha}\nu(I)$.
Sending $\alpha \to \infty$ we conclude that $\mu(I) = 0$.
Now we must show that $D_\mu \nu$ exists μ a.e.:
Using lemma 3.4 we can claim the following inequalities: $\nu(R(a,b)) \leq a\mu(R(a,b))$ and $\nu(R(a,b)) \geq b\mu(R(a,b))$ but $b > a \implies \mu(R(a,b)) = 0$.
Now we can write:

$$\{x \mid \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x) < \infty\} = \bigcup_{0<a<b \,\mid\, a,b \in \mathbb{Q}} R(a,b)$$

Obviously a countable union of zero-measure sets is a zero-measure set so $D_\mu \nu$ exists μ a.e.

2)We will show that $x \to \nu(B(x,r))$ and $x \to \mu(B(x,r))$ are upper semicontinuous and thus Borel measurable:
Let's prove

$$\limsup_{y \to x} \mu(B(y,r)) \leq \mu(B(x,r)).$$

Choose a sequence $\{y_k\} \subset \mathbb{R}^n$ s.a. $y_k \to x$. Set $f_k = \chi_{B(y_k,r)}$ and $f = \chi_{B(x,r)}$.
Then

$$\limsup_{k \to \infty} f_k(z) \leq f(z) \;\; \forall z \in \mathbb{R}^n \tag{3.3}$$

because if we fix $z \in B(x,r) \implies f(z) = 1$ and we can conclude, otherwise if we take $z \in B(y_k,r) - B(x,r) \implies \limsup f_k(z) = 0$.
From the 3.3 equality we have

$$\liminf_{k \to \infty} (1 - f_k) \geq (1 - f).$$

Thus by Fatou's Lemma:

$$\int_{B(x,2r)} (1-f)\, d\mu \leq \int_{B(x,2r)} \liminf_{k \to \infty} (1-f_k)\, d\mu \leq \liminf_{k \to \infty} \int_{B(x,2r)} (1-f_k)\, d\mu.$$

So

$$\mu(B(x,2r)) - \mu(B(x,r)) \leq \liminf_{k \to \infty} (\mu(B(x,2r)) - \mu(B(y_k,r)))$$
$$\implies \mu(B(x,r)) \geq \limsup_{k \to \infty} \mu(B(y_k,r)).$$

Now we claim that for every $r > 0$:

$$f_r(x) = \begin{cases} \frac{\nu(B(x,r))}{\mu(B(x,r))} & if\ \mu(B) > 0 \\ +\infty \quad if\ \mu(B) = 0 \end{cases}$$

is μ-measurable because is quotient of measurable functions. But

$$D_\mu \nu = \lim_{r \to 0} f_r = \lim_{k \to \infty} f_{\frac{1}{k}} \quad \mu - a.e.$$

and so $D_\mu \nu$ is μ-measurable. $\square$

This part was indispensable to define the absolutely continuity and the mutually singularity of measures, definitions that are mandatory to enunciate the differentiation theorem for Radon measures. In the following pages we are going to define such things and at the end we will be ready for the Lebesgue differentiation theorem.

Definition 3.6. The measure ν is absolutely continuous with respect to μ provided $\mu(A) = 0 \implies \nu(A) = 0 \;\forall\, A \subset \mathbb{R}^n$.
Written:

$$\nu \ll \mu.$$

Definition 3.7. The measures ν and μ are mutually singular if there exists a Borel subset B such that:

$$\mu(\mathbb{R}^n - B) = \nu(B) = 0.$$

Written:

$$\nu \perp \mu.$$

Theorem 3.8. *Let ν and μ be Radon measures on $\mathbb{R}^n$, with $\nu \ll \mu$. Then*

$$\nu(A) = \int_A D_\mu \nu \, d\mu \tag{3.4}$$

for all μ-measurable sets $A \subset \mathbb{R}^n$.

Proof. Let A be μ-measurable. Then there exists a Borel set B with $A \subset B$, and $\mu(B - A) = 0$. Thus $\nu(B - A) = 0$ and so A is ν-measurable. This prove that each μ-measurable set is also ν-measurable.
Let's define:

$$Z = \{x \in \mathbb{R}^n \mid D_\mu \nu(x) = 0\}$$

and

$$I = \{x \in \mathbb{R}^n \mid D_\mu \nu(x) = \infty\};$$

Thanks to the theorem 3.5 I and Z are μ-(thus ν) measurable sets and by the same theorem $\mu(I) = \nu(I) = 0$. Also lemma 3.4 implies $\nu(Z) \le \alpha\mu(Z)$ for all $\alpha > 0$; thus $\nu(Z) = 0$. So

$$\nu(Z) = 0 = \int_Z D_\mu \nu \, d\mu$$

and

$$\nu(I) = 0 = \int_I D_\mu \nu \, d\mu.$$

This works because we are integrating a measurable and well defined function (the set of point where $D_\mu \nu$ is infinite has measure equal to zero) on a zero-measure set hence the integral is obviously zero.

Now fix $1 < t < \infty$ and define for each integer m

$$A_m = A \cap \{x \in \mathbb{R}^n \mid t^m \le D_\mu \nu(x) < t^{m+1}\}.$$

Then A_m is μ-(and ν) measurable (as it's intersection on measurable sets) and:

$$A - \bigcup_{m=-\infty}^{\infty} A_m \subset Z \cup I \cup \{x \mid \overline{D}_\mu \nu(x) \neq \underline{D}_\mu \nu(x)\}.$$

This implies:

$$\mu(A - \bigcup_{m=-\infty}^{\infty} A_m) = \nu(A - \bigcup_{m=-\infty}^{\infty} A_m) = 0$$

and consequently:

$$\nu(A) = \sum_{m=-\infty}^{\infty} \nu(A_m) \leq \sum_m t^{m+1}\mu(A_m) \quad (\textit{thanks to the lemma } 3.4)\ \forall t \in \mathbb{R}$$

$$= t\sum_m t^m \mu(A_m) \leq t\sum_m \int_{A_m} D_\mu \nu \, d\mu = t\int_A D_\mu \nu \, d\mu$$

Similarly:

$$\nu(A) = \sum_m \nu(A_m) \geq \sum_m t^m \mu(A_m) = \frac{1}{t}\sum_m t^{m+1}\mu(A_m) \geq \frac{1}{t}\int_A D_\mu \nu \, d\mu$$

In conclusion:

$$\frac{1}{t}\int_A D_\mu \nu \, d\mu \leq \nu(A) \leq t\int_A D_\mu \nu \, d\mu$$

Sending $t \to 1$ we prove the theorem.

□

The last theorem we need to reach our goal: proving the Lebesgue differentiation theorem, is the Lebesgue differentiation theorem.

Theorem 3.9. *Let μ and ν be Radon measures on $\mathbb{R}^n$.*
Then $\nu = \nu_{ac} + \nu_s$ where ν_{ac}, ν_s are Radon measures on $\mathbb{R}^n$ and

$$\nu_{ac} \ll \mu, \quad \nu_s \perp \mu$$

Furthermore

$$D_\mu \nu = D_\mu \nu_{ac}, \quad D_\mu \nu_s = 0$$

and consequently we have:

$$\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A).$$

for each Borel set $A \subset \mathbb{R}^n$.

Proof. We assume $\nu(\mathbb{R}^n)$, $\mu(\mathbb{R}^n) < \infty$.
Define

$$\mathcal{Z} = \{A \subset \mathbb{R}^n \mid \mu(\mathbb{R}^n - A) = 0\}$$

and we chose $B_k \in \mathcal{Z}$ s.a.

$$\nu(B_k) \leq \inf_{A \in \mathcal{Z}} \nu(A) + \frac{1}{k}$$

Now we call

$$B = \bigcap_k B_k$$

so we have:

$$\mu(\mathbb{R}^n - B) \leq \sum_k \mu(\mathbb{R}^n - B_k) = 0$$

and

$$\nu(B) \leq \inf_{A \in \mathcal{Z}} \nu(A). \tag{3.5}$$

Define

$$\nu_{ac} = \nu_{|\,B}, \quad \nu_s = \nu_{|\,B^c}$$

It's easy to verify that $\nu_{ac} \ll \mu$ and $\nu_s \perp \mu$:
if we take $A \subset B$ we have $\mu(A) = 0$, and we suppose $\nu(A) > 0 \implies B - A \in \mathcal{Z}$ and $\nu(B - A) < \nu(B)$ that is a contradiction to the equation 3.5.
On the other and we can see: $\mu(\mathbb{R}^n - B) = 0$ thus $\nu_s \perp \mu$. Finally, fix $\alpha > 0$ and set

$$\mathcal{C} = \{x \in B \,|\, D_\mu \nu_s(x) \geq \alpha\}$$

According to lemma 3.4

$$\alpha\mu(\mathcal{C}) \leq \nu_s(\mathcal{C}) = 0$$

and therefore

$$D_\mu \nu_s = 0 \ a.e.$$

This implies

$$D_\mu \nu_{ac} = D_\mu \nu, \ \mu \ a.e.$$

□

Now we can start with the Lebesgue differentiation theorem for Radon measures, let's begin with some notation:

Definition 3.10. Like in the previous chapter we will denote

$$\fint_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu$$

Theorem 3.11. *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r \to 0} \fint_{B(x,r)} f \, d\mu = f(x)$$

for μ a.e. $x \in \mathbb{R}^n$.

Proof. For Borel $B \subset \mathbb{R}^n$, define

$$\nu^{\pm}(B) = \int_B f^{\pm}\, d\mu$$

and for $A \subset \mathbb{R}^n$

$$\nu^{\pm}(A) = \inf\{\nu^{\pm}(B) \mid A \subset B,\ B\, Borel\}.$$

Now $\nu^{\pm}$ are Radon measures and so we can apply the theorem 3.8.
Consequently

$$\lim_{r\to 0} \fint_{B(x,r)} f\, d\mu = \lim_{r\to 0} \fint_{B} f^{+} - f^{-}\, d\mu = \lim_{r\to 0} \frac{1}{\mu(B)}(\nu^{+}(B) - \nu^{-}(B))$$

(by theorem 3.8)

$$= D_{\mu}\nu^{+}(x) - D_{\mu}\nu^{-}(x) = f^{+}(x) - f^{-}(x) = f(x), \quad \mu - a.e.$$

$\square$

Conclusion

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