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Variation formulas for H-perimeter in Heisenberg groups

Laureando:
Mattia Fogagnolo
Matricola 1084234

Relatore:
Prof. R. Monti

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Introduction

Analysis in metric spaces is an important field of today's research, in particular it appears to be fruitful in the so called CC-spaces, Carnot-Carathéodory spaces, or Carnot groups. The prototype of these spaces is the *Heisenberg group* $\mathbb{H}^n$, that is the manifold $\mathbb{C}^n \times \mathbb{R}$ endowed with the group product

$$(z,t)\cdot(\zeta,\tau) = (z+\zeta, t+\tau+2\Im\langle z,\overline{\zeta}\rangle),$$

where $z,\zeta \in \mathbb{C}^n$, $t,\tau \in \mathbb{R}$, and $\langle\,,\rangle = z_1\overline{\zeta_1} + \cdots + z_n\overline{\zeta_n}$.

After having defined the left translation by p as $L_p(q) = p\cdot q$, it is introduced the following basis of left invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

for $j = 1,\dots,n$ and where $(x_1,\dots,x_n,y_1,\dots,y_n,t) = (z,t)$. Then, the following notion of H-perimeter for a Lebesgue-measurable set $E \subset \mathbb{H}^n$ in an open set A is considered:

$$P(E,A) = \sup\left\{\int_E \operatorname{div}_H \phi \, d\mathcal{L}^{2n+1} : \phi \in C^1_c(A;\mathbb{R}^{2n}), \|\phi\|_\infty \le 1\right\},$$

where $\operatorname{div}_H \phi = \sum_{j=1}^n (X_j\phi_j + Y_j\phi_{n+j})$.

A major problem in this area is finding, studying and discussing the regularity of H-perimeter minimizing sets. This challenging research program takes place in the context of Calculus of Variations, Geometric Measure Theory and PDEs.

The most natural tools one can develop to understand the properties of minimal surfaces are area formulas and variation formulas: in particular minimizing sets will have vanishing first variation of its area functional and nonnegative second variation.

Various examples of variation formulas have been recently computed, for different kinds of sets. However, they all need some regularity to make sense: this is unsatisfactory if we are dealing, for example, with regularity theory.

R. Monti, along with D. Vittone, worked to obtain a first variation formula holding under the sharp, natural hypothesis of finiteness of H-perimeter: no other regularity assumptions are made. Actually, in such a family there are extremely non-regular sets; even sets with fractal boundaries. This first variation formula is obtained using a flow of *contact diffeomorphisms*, a special family of diffeomorphisms preserving the finiteness of H-perimeter. The vector fields generating these flows have the form

$$V_\psi = -4\psi T + \sum_{j=1}^{n}(Y_j\psi)X_j - (X_j\psi)Y_j, \quad \psi \in C^\infty,$$

for a C^∞ function ψ. The formula is deduced from the following:

Theorem 0.1. *Let $A \subset \mathbb{H}$. be an open set, and let $E \subset \mathbb{H}^n$ be a set with finite H-perimeter in A. Let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(\Psi, A)$, be the flow generated by $\psi \in C_c^\infty(A)$. Then*

$$\left|P(\Psi_s(E), \Psi_s(A)) - P(E, A) + s\int_A \Big(4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E)\Big)d\mu_E\right| \leq CP(E, A)s^2$$

In the inequality above, $\mathscr{Q}_\psi$ is a suitably defined quadratic form, ν_E is the *horizontal normal* to ∂E and μ_E is the *perimeter measure*. In fact, all these objects are proved to exist in our minimal hypothesis.

The goal of this work is to understand whether a generalization of Theorem 0.1 is possible, and, in this case, to provide it: this would in particular give a general second variation formula for the H-perimeter of sets with no regularity assumptions. Actually, in the thesis we show that such a result hold; it is the following

Theorem 0.2. *Let E be a set with finite H-perimeter in an open set $A \subset \mathbb{H}^n$, with horizontal normal ν_E. Let $\Psi : [-\delta, \delta] \times A \in \mathbb{H}^n$, $\delta = \delta(\psi, A)$, be the contact flow generated by $\psi \in C_c^\infty(A)$. Let $\mathscr{A}_\psi(\nu_E) = \mathscr{Q}_\psi(\nu_E) + 4(n+1)T\psi$. Then there exists $C = C(\psi, A)$ such that*

$$\Bigg|P(\Psi_s(E), \Psi_s(A)) - P(E, A) + s\int_A \mathscr{A}_\psi(\nu_E)d\mu_E - s^2\int_A \Big(\mathscr{S}_\psi(\nu_E) - \big(\mathscr{A}_\psi(\nu_E)\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big)d\mu_E\Bigg| \leq CP(E, A)s^3.$$

In the above formula, $\mathscr{S}_\psi$ is another quadratic form. Theorem 0.2 is the new and original contribution of my work.

The thesis is divided in two chapters. In Chapter 1, we first introduce Heisenberg groups and H-perimeter, and illustrate some of the most important related known facts. Then, we discuss area formulas for sets with some regularity, and we deduce standard first and second variation formulas for the H-perimeter of these sets. Examples of minimizers are provided too. A particular focus is on the issues of these formulas, concerning mainly their lack of sense in absence of regularity assumption. The main reference is [12].

Chapter 2 is devoted to contact diffeomorphisms and variation formulas by means of these deformations. In particular, Theorem 0.2 is proved in its full generality. This chapter is organized as follows. In the first part, we introduce contact diffeomorphisms, and prove the properties and characterizations that are needed to build variation formulas. The notion of contact maps is not new: it was introduced in [11] for different purposes. However, along this part of the thesis contact diffeomorphisms are described in a simple way adapted to best suit our setting. After that, we proceed to prove Theorem 0.2. The proof goes in the following way: we prove such a theorem first assuming C^∞ regularity $\partial E \cap A$, then, we drop this regularity assumption using the fact that the "relevant" part of the boundary of a set with locally finite H-perimeter is *H-rectifiable*: a fundamental property, introduced in [9], analogous, in Euclidean setting, to standard rectifiability. The proof proceeds by approximation, and is based on the techniques used by R. Monti and D. Vittone to prove Theorem 0.1.

Theorem 0.2 could have several applications, that can be object of future works. For example second variation formulas were the main tool used in [3] to give a positive answer to the Bernstein problem (the problem of understanding whether minimal global graphs must have affine parametrizations) assuming C^2 regularity of the parametrization of an important class of sets, the sets with intrinsic Lipshitz boundary, described also in this thesis, Chapter 1. However, those variation formulas cannot allow to extend deeply that result, because of the already remarked lack of meaning when regularity assumptions are dropped. Theorem 0.2 could then be used, for example, to the study of the Bernstein problem in the most general setting.

In general, many results about minimal sets in $\mathbb{H}^n$ are proved to hold only under regularity assumptions or other hypothesises made to ensure integrability of variation formulas, for example, about the structure of a particular subset of the boundary of a set called *characteristic locus*; the robust integrability of these new variation formulas do not need such requests, and could then be useful to generalize results of this type.

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Chapter 1

Heisenberg groups and H-perimeter

1.1 Heisenberg groups

We define the *Heisenberg group* of dimension $2n+1$ as the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ endowed with the group product

$$\begin{gathered} \cdot : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n \\ (z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\Im\langle z, \overline{z_2}\rangle), \end{gathered}$$

where $z_1, z_2 \in \mathbb{C}^n$, $t_1, t_2 \in \mathbb{R}$ and $\langle z_1, \overline{z_2}\rangle$ is the standard scalar product in $\mathbb{R}^{2n}$; by $\Im$ we mean the imaginary part. It is straightforward to see that $(\mathbb{H}^n, \cdot)$ is a non-commutative Lie group, where the identity element is $(0,0)$ and the inverse element of (z,t) is $(-z,t)$. We will always identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and then $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. The group product is defined accordingly.

Let $p = (z,t) \in \mathbb{H}^n$. We define the left translation by p as the mapping

$$\begin{gathered} L_p \colon \mathbb{H}^n \to \mathbb{H}^n \\ L_p(q) = p \cdot q. \end{gathered}$$

The differential map of L_p, that we denote by JL_p, is given by an upper triangular matrix with 1 along the principal diagonal. In particular, its determinant is 1.

We show that the Lebesgue measure is the Haar measure of the Heisenberg group. We denote by $|E|$ the Lebesgue measure of $E \subset \mathbb{H}^n$.

Proposition 1.1. *Let $E \subset \mathbb{H}^n$, $p \in \mathbb{H}^n$. Then*

$$|E| = |L_pE|,$$

that is, Lebesgue measure is the Haar measure of $\mathbb{H}^n$.

Proof. Let, for any $q \in E$, $q^p = L_p q$. By change of variable formula,

$$|L_p E| = \int_{L_p E} dq^p = \int_E \det JL_p dq = \int_E dq = |E|.$$

□

Define now, for $\lambda > 0$ the mapping

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$$
$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t).$$

We have that δ_λ is an automorphism of $\mathbb{H}^n$ with inverse $\delta_{\lambda^{-1}}$, that $\delta_1 = \mathrm{id}$, and, for $\lambda_1, \lambda_2 > 0$,

$$\delta_{\lambda_1}(\delta_{\lambda_2}(z,t)) = \delta_{\lambda_1 + \lambda_2}(z,t).$$

It means that $\{\delta_\lambda\}_{\lambda > 0}$ form a 1-parameter family of automorphisms of $\mathbb{H}^n$. Such automorphisms are called *dilations*, and Lie groups endowed with a family of automorphisms of this type are called *homogeneous Lie groups*. It is evident that $\det J\delta_\lambda = \lambda^Q$, with

$$Q = 2n + 2.$$

Such important number is called *homogeneous dimension* of $\mathbb{H}^n$, and we get, mimicking the proof of Proposition 1.1, that

$$|\delta_\lambda E| = \lambda^Q |E|.$$

1.1.1 Lie algebra on $\mathbb{H}^n$

We describe the natural Lie algebra of left invariant vector fields on $\mathbb{H}^n$. We always identify a smooth vector field V on an open set $A \subset \mathbb{H}^n$ both with a vector valued function in $C^1(A, \mathbb{H}^n)$ and a linear differential operator.

Definition 1.1. *A vector field V on $\mathbb{H}^n$ is* left invariant *if*

$$V(f \circ L_p) = (Vf) \circ L_p$$

for any $p \in \mathbb{H}^n$ and $f \in C^\infty(\mathbb{H}^n)$.

It is clear that left invariant vector fields form a Lie algebra on $\mathbb{H}^n$: the *Heisenberg Lie algebra* $\mathfrak{h}_n$. In order to find a basis for $\mathfrak{h}_n$, we prove the following characterization:

Proposition 1.2 (Characterization of left invariant vector fields)**.** *Let V be a vector field on $\mathbb{H}^n$. The following are equivalent:*

(i) V is left invariant.

(ii) $V(p \cdot q) = JL_p(q)V(q)$ *for any* $p, q \in \mathbb{H}^n$.

(iii) $V(p) = JL_p(0)V(0)$ *for any* $p \in \mathbb{H}^n$.

Proof. We show that (i) is equivalent to (ii). Let $f \in C^\infty(\mathbb{H}^n)$, $p, q \in \mathbb{H}^n$. We have that

$$(V(f \circ L_p))(q) = \langle \nabla(f \circ L_p)(q), V(q) \rangle = \langle \nabla f(L_p(q))JL_p(q), V(q) \rangle,$$

where by $\nabla f(L_p(q))JL_p(q)$ we mean the standard product between a row vector and a matrix. On the other hand,

$$Vf \circ L_p(q) = \langle \nabla f(p \cdot q), V(p \cdot q) \rangle.$$

By the above equalities, we deduce that (i) holds if and only if

$$\langle \nabla f(L_p(q))JL_p(q), V(q) \rangle = \langle \nabla f(p \cdot q), V(p \cdot q) \rangle. \tag{1.1}$$

Clearly condition (ii) implies (1.1), and thus (i). Conversely, assume (1.1) holds. Since it holds for any $f \in C^\infty(\mathbb{H}^n)$, choose $f = \sum_{j=1}^{2n+1} h_j q_j$, for h_j real constants and where by q_j we mean the j-th component of q. Thus, letting $h = (h_1, \dots, h_{2n+1})$, we get

$$\langle h, JL_p(q), V(q) \rangle = \langle h, V(L_p(q)) \rangle$$

for any $h \in \mathbb{R}^{2n+1}$, which implies (ii).

Condition (iii) follows by condition (ii) simply choosing $q = 0$.

Assume now (iii). Since it holds for any $p \in \mathbb{H}^n$, it holds also for $p \cdot q$, that is:

$$V(p \cdot q) = JL_{p \cdot q}V(0).$$

By associativity of the group product we have $L_{p \cdot q} = L_p \circ L_q$, and thus, differentiating both sides of this equality, we obtain on the other hand

$$JL_{p \cdot q}(0) = JL_p(q)JL_q(0).$$

We obtain then

$$V(p \cdot q) = JL_p(q)(JL_q(0)V(0)).$$

By condition (iii), the above equality is precisely condition (ii), and the proof is complete. $\square$

We build a system of generators for the Lie algebra $\mathfrak{h}_n$ on $\mathbb{H}^n$. Let $(z, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \in \mathbb{H}^n$. Then, we define, for $j = 1, \dots, n$

$$X_j(p) := JL_p(0)\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}$$

$$Y_j(p) := JL_p(0)\frac{\partial}{\partial y_j} = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$$

$$T(p) := JL_p(0)\frac{\partial}{\partial t} = \frac{\partial}{\partial t}$$

These vector fields are left invariant by condition (iii) of Proposition (1.1), and they are linearly independent for any $p \in \mathbb{H}^n$; thus, they span $\mathfrak{h}_n$.

Horizontal distribution. We define the *horizontal distribution* at a point $p \in \mathbb{H}^n$ as

$$H_p = \text{span}\{X_j(p), Y_j(p), \ j = 1, \dots, n\}.$$

A vector in H_p for some $p \in \mathbb{H}^n$ are called *horizontal vector*. A vector field V such that $V(p) \in H_p$ for any p is called *horizontal vector field*. The horizontal distribution, in fact, suffices to generate the Lie algebra $\mathfrak{h}_n$: indeed, denoting with $[\cdot, \cdot]$ the Lie brackets, or commutator, we have

$$[X_j, Y_j] = -4T \tag{1.2}$$

on $\mathbb{H}^n$, for $j = 1, \dots, n$. All other commutators among $\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$ vanish identically. Since it suffices just one commutator among the vector fields of the horizontal distribution to generate the whole of $\mathfrak{h}_n$, the horizontal distribution is said to be bracket generating of step 2. We write

$$Lie\{X_1, \dots, X_n, Y_1, \dots, Y_n\} = \mathfrak{h}_n.$$

Lie groups enojoying such a property are called *Carnot groups*. In particular, $\mathbb{H}^n$ is a Carnot group with $2n$ generators.

1.1.2 Metrics on the Heisenberg group

We want to construct a metric on $\mathbb{H}^n$ by means, in a sense, only of the horizontal distribution. We start with the following definition:

Definition 1.2 (Horizontal curve). *A Lipschitz curve $\gamma : \mathbb{R} \supset [a, b] \to \mathbb{H}^n$ is a* horizontal curve *if*

$$\dot\gamma(t) = \sum_{j=1}^{n} h_j(t)X_j + h_{n+j}(t)Y_j$$

a.e. in $[a, b]$ for suitable functions $h_j \in L^\infty[0, 1]$, $j = 1, \dots, 2n$.

We introduce the following notation for flows: let V be a vector field on $\mathbb{H}^n$, we let

$$\mathbb{R} \times \mathbb{H}^n \ni (s, p) \to \exp(sV)(p) \in \mathbb{H}^n$$

be the flow of the vector field V at time s starting from $p \in \mathbb{H}^n$. One can readily check that, for $j = 1, \dots, n$,

$$\exp(sX_j)(p) = (x_1, \dots, x_j + s, \dots, x_n, y_1, \dots, y_n, t + 2y_j s)$$
$$\exp(sY_j)(p) = (x_1, \dots, x_n, y_1, \dots, y_j + s, \dots, y_n, t - 2x_j s).$$

Clearly, the line flows starting from a point $p \in \mathbb{H}^n$ of horizontal vector fields are horizontal curves. The key step to provide our metric is the fact that we can join any couple of points with a horizontal curve, namely

Proposition 1.3. *For any $p, q \in \mathbb{H}^n$ there exists a horizontal curve $\gamma : [a, b] \to \mathbb{H}^n$ such that $\gamma(a) = p$ and $\gamma(b) = q$.*

Proof. We find such a γ by composition of line flows of horizontal vector fields: notice that, for $j = 1, \ldots, n$ and $s \in \mathbb{R}$

$$\begin{aligned}\exp(-sY_j)\exp(-sX_j)\exp(sY_j)\exp(sX_j)(x_1, \ldots, x_n, y_1, \ldots, y_n, t)\\ =(x_1, \ldots, x_n, y_1, \ldots, y_n, t - 4t^2)\end{aligned}$$

and

$$\begin{aligned}\exp(sY_j)\exp(sX_j)\exp(-sY_j)\exp(-sX_j)(x_1, \ldots, x_n, y_1, \ldots, y_n, t)\\ =(x_1, \ldots, x_n, y_1, \ldots, y_n, t + 4t^2);\end{aligned}$$

thus, we are able to find a horizontal curve joining any couple of point with the same first $2n$ coordinates.

Since we can also join any couple of point p, q with just the j-th component differing, simply by $\exp(sX_j)(p)$ if $j \in \{1, \ldots, n\}$, or $\exp(sY_{2n-j})(p)$, if $j \in \{n + 1, \ldots, 2n\}$, we are done. □

We can clearly deal, without loss of generality, only with horizontal curves defined on $[0, 1]$, up to a new parametrization. The following definition of *Carnot-Carathéodory distance* is thus well-posed:

Definition 1.3 (Carnot-Carathéodory distance)**.** *For any couple of points $p, q \in \mathbb{H}^n$, we define their* Carnot-Carathéodory distance $d(p, q)$ *as*

$$d(p, q) = \inf_{\gamma} \left\{ L(\gamma) := \int_0^1 |\dot{\gamma}| \right\},$$

where the infimum is taken over any horizontal curve $\gamma : [0, 1] \to \mathbb{H}^n$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

We omit the proof that the above defined function d is actually a distance. The Carnot-Carathéodory distance satisfies the following property: for any $p, q, w \in \mathbb{H}^n$, $\lambda > 0$, there hold

(*i*) $d(w \cdot p, w \cdot q) = d(p, q)$,

(*ii*) $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$.

The above equalities are respectively consequences of the left invariance of the horizontal distribution and of the elementary identity

$$X_j(f \circ \delta_\lambda) = \lambda(X_j f) \circ \delta_\lambda, \quad Y_j(f \circ \delta_\lambda) = \lambda(Y_j f) \circ \delta_\lambda \tag{1.3}$$

holding for any $f \in C^\infty(\mathbb{H}^n), \lambda > 0, j = 1, \ldots, n$. In light of (i) and (ii), the distance d is said to be *left-invariant* and 1-*homogeneous*.

We introduce now the following homogeneous "box norm". Let $p = (z, t) \in \mathbb{H}^n$, we define

$$\|p\|_\infty^b := \max\{|z|, |t|^{1/2}\}. \tag{1.4}$$

Such function satisfies

(i) $\|\delta_\lambda(p)\|_\infty^b = \lambda\|p\|_\infty$

(ii) $\|p \cdot q\|_\infty^b \leq \|p\|_\infty^b + \|q\|_\infty^b$

and it is actually a norm on $\mathbb{H}^n$. We define consequently the function

$$\begin{aligned} &\rho : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty) \\ &\rho(p, q) = \|p^{-1} \cdot q\|_\infty^b, \end{aligned} \tag{1.5}$$

that is a distance on $\mathbb{H}^n$. It is clear that also such distance is left invariant and 1-homogeneous. We now show that the distances d and ρ are equivalent.

Proposition 1.4. *There exists an absolute constant C such that*

$$C^{-1}d(p, q) \leq \rho(p, q) \leq Cd(p, q) \tag{1.6}$$

for any $p, q \in \mathbb{H}^n$.

Proof. By left invariance and homogeneity of the distance functions involved, we claim that there exists C such that (1.6) holds for $p = 0$ and any $q \in B_{CC}(0, 1) := \{q : d(0, q) = 1\}$. Indeed, once proved such a claim, if $d(0, q) = \lambda > 0$, letting $\tilde{q} \in B_{CC}(0, 1)$ such that $\delta_\lambda(\tilde{q}) = q$, by 1-homogeneity we have

$$C^1\lambda d(0, \delta_\lambda(\tilde{q})) \leq \lambda\rho(0, \delta_\lambda(\tilde{q})) \leq C\lambda d(0, \delta_\lambda(\tilde{q})),$$

and then, if p, q are arbitrary in $\mathbb{H}^n$, by applying the chain of inequalities to $(p \cdot 0, p \cdot (p^{-1} \cdot q))$, and by left invariance of the metrics, we are done.

Let

$$M := \sup\{\rho(0, q) : q \in B_{CC}(0, 1)\}, \quad m := \inf\{\rho(0, q) : q \in B_{CC}(0, 1)\}.$$

We have that $0 < m, M < +\infty$, by compactness of $B_{CC}(0, 1)$ and the fact that $d(0, \cdot)$ and $\rho(0, \cdot)$ are continuous and strictly positive functions in $\mathbb{H}^n \setminus \{0\}$.

Let finally $C = \max\{M, 1/m\}$, and notice that our claim is true for such a choice of C. □

For a more comprehensive introduction to Heisenberg groups in the more general setting of *stratified Lie groups*, see [5].

1.2 H-perimeter

We introduce a notion of perimeter in Heisenberg groups analogous to the standard Euclidean perimeter (see [8]) but taking into account the particular structure of $\mathbb{H}^n$ and its Lie algebra of left invariant vector fields.

Let V be a smooth horizontal vector field on $A \subset \mathbb{H}^n$, A open. We express it using the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$:

$$V = \sum_{j=1}^{n} \Big(\phi_j X_j + \phi_{n+j} Y_j \Big),$$

for suitable functions $\phi_i \in C^1(A)$, $i = 1, \ldots, 2n$. We define its *horizontal divergence* as follows:

$$\operatorname{div}_H V := \sum_{j=1}^{n} \Big(X_j \phi_j + Y_j \phi_{n+j} \Big).$$

We are identifying a horizontal vector field V with a vector valued functions $\phi \in C^1(A, \mathbb{R}^{2n})$. Notice that, for $p \in \mathbb{H}^n$, letting $\|\cdot\|$ the norm on H_p that makes $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ orthonormal, we have

$$\|V(p)\| = |\phi(p)|,$$

where we denote by $|\cdot|$ the standard norm on $\mathbb{R}^{2n}$.

The following definition is the starting point of the whole theory of minimal surfaces in $\mathbb{H}^n$, and has been introduced in [9].

Definition 1.4 (H-perimeter)**.** *Let $A \subset \mathbb{H}^n$ be open, and let $E \subset \mathbb{H}^n$ be Lebesgue measurable. Then the H-perimter of E in A is*

$$P(E, A) = \sup \left\{ \int_E \operatorname{div}_H \phi \, dp : \phi \in C_c^1(A, \mathbb{R}^{2n}), \|\phi\|_\infty \leq 1 \right\},$$

where by $\|\phi\|_\infty$ we mean the standard sup-norm of ϕ:

$$\|\phi\|_\infty = \sup_{p \in A} |\phi(p)|.$$

We say that E has finite *H-perimeter in A if $P(E, A) < \infty$. We say that E has* locally finite *H-perimeter in A if $P(E, A') < \infty$ for any open set $A' \Subset A$.*

By left invariance of the horizontal distribution, and by equalities (1.3), one proves the following elementary properties of the H-perimeter, with E and A as in the above definition, $p \in \mathbb{H}^n$, $\lambda > 0$:

(*i*) $P(L_p E, L_p A) = P(E, A)$

(*ii*) $P(\delta_\lambda E, \delta_\lambda A) = \lambda^{Q-1} P(E, A)$.

It is fundamental the following result:

Proposition 1.5 (Gauss-Green formula)**.** *Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in $A \subset \mathbb{H}^n$ open. Then there exists a positive Radon measure μ_E on A and a μ_E measurable function $\nu_E \colon A \to \mathbb{R}^{2n}$ such that*

$$\int_E \operatorname{div}_H \phi \, dp = -\int_A \langle \phi, \nu_E \rangle d\mu_E \tag{1.7}$$

for all $\phi \in C_c^1(A, \mathbb{R}^{2n})$.

Proof. Consider the following linear functional $T : C_c^1(A, \mathbb{R}^{2n}) \to \mathbb{R}$:

$$T(\phi) = \int_E \operatorname{div}_H \phi \, dp.$$

Since E has locally finite perimeter in A, for any open set $A' \Subset A$ we have

$$T(\phi) \leq \|\phi\|_\infty P(E, A') \tag{1.8}$$

for any $\phi \in C_c^1(A', \mathbb{R}^{2n})$. By density, T can be extended to a bounded linear operator on $C_c(A', \mathbb{R}^{2n})$ satisfying again (1.8). We finally deduce our result by Riesz' representation theorem (see e.g. [8]). □

With reference to the above result, we give the following

Definition 1.5. *The measure μ_E is called H-*perimeter measure*, or perimeter measure, and the function ν_E is called* measure theoretic inner horizontal normal *of E, or horizontal normal.*

We finally show that the perimeter measure $\mu_E(A)$ coincides with the H-perimeter of E in A:

Proposition 1.6. *Let E and A as before. Then for any open set $A' \Subset A$ we have $\mu_E(A') = P(E, A')$.*

Proof. Let $A' \Subset A$ be open. By definition of H-perimeter, the inequality $P(E, A') \leq \mu_E(A')$ follows. We prove the reverse inequality by a standard approximation argument. By Lusin's theorem, we find a compact set $K \subset A'$ such that $\mu_E(A' \setminus K) < \epsilon$ and ν_E is continuous on K. By Urysohn's lemma, we find a function $\psi \in C_c(A'\mathbb{R}^{2n})$ such that $\psi = \nu_E$ on K and $\|\psi\|_\infty \leq 1$. By mollification, there exists $\phi \in C_c^\infty(A', \mathbb{R}^{2n})$ such that $\|\phi - \psi\|_\infty < \epsilon$ and $\|\phi\|_\infty \leq 1$. Then we have, with this choice of ϕ and by (1.7)

$$P(E, A') \geq \int_E \operatorname{div}_H \phi \, dp = -\int_{A'} \langle \phi, \nu_E \rangle d\mu_E \geq (1 - \epsilon)\mu_E(A') - 2\epsilon,$$

and by arbitrariness of ϵ we conclude.

□

More explicit expressions for the horizontal normal and the perimeter measure will appear later.

1.3 Area Formulas

In this section we derive formulas for computing the H-perimeter of sets with some regularity. Our starting point is the area formula for sets with Lipschitz boundary.

1.3.1 Sets with Lipschitz Boundary

Let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary. By *Rademacher's Theorem*, the Euclidean outer normal N to ∂E is defined $\mathscr{H}^{2n}$ almost everywhere, where we denote by $\mathscr{H}^{2n}$ the $2n$-dimensional standard Hausdorff measure. But then also the vector field

$$N_H = \Big(\langle X_1, N \rangle, \dots, \langle X_n, N \rangle, \langle Y_1, N \rangle, \dots, \langle Y_n, N \rangle \Big).$$

is defined at $\mathscr{H}^{2n}$-a.e. point of ∂E.

We have the following

Proposition 1.7. *Let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary, and N be its standard Euclidean outer normal. Let $A \subset \mathbb{H}^n$ be open. Then*

$$P(E, A) = \int_{\partial E \cap A} |N_H| d\mathscr{H}^{2n}, \tag{1.9}$$

where N_H is defined above and $|N_H|$ is its standard Euclidean norm.

Proof. We recall that, given $\phi \in C_c^1(A, \mathbb{R}^{2n})$, then $\operatorname{div}_H \phi = \operatorname{div} V$, where $V = \sum_{j=1}^n \phi_j X_j + \phi_{n+j} Y_j$. By standard divergence theorem and Cauchy-Schwarz inequality

$$\begin{aligned} \int_E \operatorname{div}_H \phi dz dt = \int_E \operatorname{div} V dz dt &= \int_{\partial E} \langle V, N \rangle dz dt \\ &= \int_{\partial E} \sum_{j=1}^n \phi_j \langle X_j, N \rangle + \phi_{n+j} \langle Y_j, N \rangle d\mathscr{H}^{2n} \\ &\le \int_{\partial E} \sum_{j=1}^n |\phi| |N_H| d\mathscr{H}^{2n}; \end{aligned}$$

thus, taking the supremum on ϕ with $\|\phi\|_\infty \le 1$ we obtain that $P(E, A) \le \int_{\partial E \cap A} |N_H| d\mathscr{H}^{2n}$.

In order to obtain the reverse inequality, we approximate $N_H / |N_H|$ by a $C_c^1(A, \mathbb{R}^{2n})$ function with infinity norm less or equal than one. Let $\epsilon > 0$. By Lusin's theorem applied to the measure $|N_H| d\mathscr{H}^{2n}$, we find a compact set $K \subset \partial E \cap A$ such that

$$\int_{(\partial E \cap A) \setminus K} |N_H| d\mathscr{H}^{2n} < \epsilon,$$

and N_H is continuous and different from zero on K. By Urysohn's lemma, there exists $\psi \in C_c(A, \mathbb{R}^{2n})$ equal to $N_H / |N_H|$ on K such that $\|\psi\|_\infty \le 1$. Mollification then yields a function $\phi \in C^1_c(A, \mathbb{R}^{2n})$ with $\|\phi\|_\infty \leq 1$ and $\|\phi - \psi\|_\infty \leq \epsilon$. Then, for such a test function ϕ, again by divergence theorem we have

$$P(E, A) \geq \int_E \operatorname{div}_H \phi = \int_{\partial E} \sum_{k=1}^{n} \phi_j \langle X_j, N\rangle + \phi_{n+j}\langle Y_j, N\rangle d\mathscr{H}^{2n}$$
$$\geq (1-\epsilon)\int_{\partial E \cap A} |N_H| d\mathscr{H}^{2n} - 2\epsilon$$

and we conclude by arbitrariness of ϵ. □

Remark. The integral in formula 1.9 makes sense even if in place of $\partial E \cap A$ we have a "surface" S which is not the boundary of a set. In this case, we prefer to call such expression $\text{Area}_H(S)$.

Formula (1.9) allows us to write an explicit formula for the perimeter measure μ_E and for the horizontal inner normal ν_E for sets E with Lipshitz boundary. Indeed, let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary, and $\phi \in C^1_c(\mathbb{H}^n, \mathbb{R}^{2n})$; then, by standard divergence theorem and by Gauss-Green's formula (1.7), we have that

$$\int_{\partial E} \langle \phi, N_H\rangle d\mathscr{H}^{2n} = \int_E \operatorname{div}_H \phi dz dt = -\int_{\mathbb{H}^n} \langle \phi, \nu_E\rangle d\mu_E,$$

and so we get that

$$\mu_E = |N_H| \mathscr{H}^{2n} \llcorner \partial E \tag{1.10}$$

and

$$\nu_E = -\frac{N_H}{|N_H|}. \tag{1.11}$$

Our next task will be to exploit formula (1.9) to get area formulas for two important type of sets. The simplest sets we deal with are the so called t-epigraphs:

Definition 1.6. *Let $D \subset \mathbb{R}^{2n}$ be open and let $f\colon D \to \mathbb{R}$ be a function. Then we call the set*

$$E_f = \{(z,t) \in \mathbb{H}^{2n} : t > f(z), z \in D\} \tag{1.12}$$

the t-epigraph of f, *and its boundary*

$$gr(f) = \{(z,t) \in \mathbb{H}^{2n} : t = f(z), z \in D\}$$

the t-graph of f.

Then the area formula for sets with Lipschitz boundary we have just proved specializes as follows:

Proposition 1.8 (Area formula for t-graphs)**.** *Let D and f, E_f as in Definition 1.6. Let also f be Lipschitz. Then*

$$P(E_f, D \times \mathbb{R}) = \int_D |\nabla f(z) + 2z^{\perp}| dz, \tag{1.13}$$

where $z^{\perp} = (x, y)^{\perp} = (-y, x)$.

Proof. Since $\partial E_f \cap (D \times \mathbb{R}) = gr(f)$, we can write its Euclidean outer normal N as follows:

$$N = \frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}},$$

and thus, since

$$\langle N, X_j \rangle = \frac{\partial_{x_j} f - 2y_j}{\sqrt{1 + |\nabla f|^2}}, \quad \langle N, Y_j \rangle = \frac{\partial_{y_j} f + 2x_j}{\sqrt{1 + |\nabla f|^2}},$$

we get that

$$N_H = \frac{|\nabla f + 2z^{\perp}|}{\sqrt{1 + |\nabla f|^2}}.$$

Formula (1.9) and standard area formula for graphs finally yield the thesis:

$$P(E_f, D \times \mathbb{R}) = \int_{gr(f)} |N_H| d\mathscr{H}^{2n} = \int_D |\nabla f z + 2z^{\perp}| dz.$$

□

The following sets are more subtle: we introduce *intrinsic graphs*.

1.3.1.1 Intrinsic graphs and intrinsic Lipschitz functions

In order to introduce intrinsic graphs and intrinsic Lipschitz functions, we define C^1_H functions and C^1_H regular hypersurfaces. Such notions will be fundamental also in the last part of the thesis, when we will deal with H-rectifiability.

Definition 1.7 (C^1_H functions and horizontal gradient)**.** *Let $A \subset \mathbb{H}^n$ be an open set. A function $f \colon A \to \mathbb{R}$ is of class $C^1_H(A)$ if*

(i) $f \in C(A)$,

(ii) the derivatives $X_1 f, \dots, X_n f, Y_1 f, \dots Y_n f$ exist in distributional sense and are represented by continuous function defined on A.

We define the horizontal gradient *of a $C^1_H(A)$ function f as the vector valued function $\nabla_H f \in C(A, \mathbb{R}^{2n})$ such that*

$$\nabla_H f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f).$$

Let now, for $p \in \mathbb{R}$,

$$B_{CC}(p,r) = \{q \in \mathbb{H}^n : d(p,q) < r\},$$

where d is the Carnot-Carathéodory distance in $\mathbb{H}^n$- The following is the natural adaptation to our setting of the classical definition of smooth hypersurface:

Definition 1.8 (H-regular hypersurfaces)**.** *A set $S \subset \mathbb{H}^n$ is a H-regular hypersurface if for any $p \in S$ there exist $r > 0$ and a function $f \in C^1_H(B(p,r))$ such that*

(i) $S \cap B(p,r) = \{q \in B(p,r) : f(q) = 0\}$,

(ii) $|\nabla_H f| \neq 0$.

We can begin now to describe *intrinsic graphs* by a special but clearer case, that we later generalize. Consider $S \subset \mathbb{H}^n$ a C^1_H- hypersurface such that $S = \{f = 0\}$ with $f \in C^1_H$ and $|\nabla_H f| \neq 0$. Assume that $X_1 f > 0$ locally, this is always possible up to a rotation. Then S is locally a graph along X_1; this is the idea lying behind the definition of *intrinsic graphs*. Recall that the line flow of X_1 starting from a point $(z,t) \in \mathbb{H}^n$, that we denote

$$s \mapsto \exp(sX_1), s \in \mathbb{R},$$

is given by

$$\exp(sX_1)(z,t) = (z + se_1, t + 2y_1 s),$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^{2n}$, $z = (x,y) = (x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{R}^{2n}$. We choose as domain of initial points

$$W = \{(z,t) \in \mathbb{H}^n : x_1 = 0\};$$

we identify W with $\mathbb{R}^{2n}$ and we will then give a point $w \in W$ coordinates $w = (x_2,\dots,x_n,y_1,\dots,y_n,t)$. We can now give the following

Definition 1.9 (Intrinsic graphs along X_1)**.** *Let $D \subset W$, and let $\phi\colon D \to \mathbb{R}$ be a function. The set*

$$E_\phi = \{\exp(sX_1)(w) \in \mathbb{H}^n : s > \phi(w), w \in D\}$$

is called intrinsic epigraph *of ϕ along X_1. The set*

$$Gr(\phi) = \{\exp(\phi(w)X_1)(w) \in \mathbb{H}^n : w \in D\},$$

is the intrinsic graph *of ϕ along X_1.*

In order to provide an area formula for intrinsic graphs, we introduce the following nonlinear gradient:

Definition 1.10. *Let $\phi \in Lip_{loc}(D)$, where $D \subset W$ is open, then the* intrinsic gradient *of ϕ is the vector valued mapping*

$$\nabla^{\phi}\phi = \Big(X_2\phi, \dots, X_n\phi, \mathcal{B}\phi, Y_2\phi, \dots, Y_n\phi\Big),$$

where $\mathcal{B}\phi$ is the Burger's operator*, defined as follows:*

$$\mathcal{B}^{\phi}\phi = \frac{\partial \phi}{\partial y_1} - 4\phi\frac{\partial \phi}{\partial t}$$

By classic theorems on Lipschitz functions we have that $\nabla^{\phi}\phi \in L^{\infty}_{loc}(D, \mathbb{R}^{2n-1})$.

We can now state an area formula for intrinsic graphs. We define, for $D \subset W$,

$$\exp(\mathbb{R}X_1)(D) = \{\exp(sX_1) \in \mathbb{H}^n : w \in D, s \in \mathbb{R}\},$$

the cylinder over D along X_1. Since $w \mapsto \exp(sX_1)(w)$ is a diffeomorphism, with inverse $w \mapsto \exp(-sX_1)(w)$, it is open; thus if D is open in W then $\exp(\mathbb{R}X_1)(D)$ is open in $\mathbb{H}^n$, and then it makes sense to consider $P(E_\phi, \exp(\mathbb{R}X_1)(D))$. We have the following:

Proposition 1.9. *Let $\phi\colon D \to \mathbb{R}$ be Lipschitz, where $D \subset W$ is open. Then*

$$P(E_\phi, \exp(\mathbb{R}X_1)(D)) = \int_D \sqrt{1 + |\nabla^{\phi}\phi|}dw \tag{1.14}$$

Proof. We work out the proof only in the case $n = 1$. The boundary of E_ϕ, that is $Gr(\phi)$, is parametrized by the function $\Phi\colon D \to \mathbb{R}^3$

$$\Phi(y,t) = \exp(\phi(y,t)X)(0,y,t) = (\phi, y, t + 2y\phi),$$

and thus we can compute the outer Euclidean normal

$$N = -(\Phi_y \wedge \Phi_t)/|\Phi_y \wedge \Phi_t|.$$

A direct computation shows that

$$\Phi_y \wedge \Phi_t = (1 + 2y\phi_t)\frac{\partial}{\partial x} + (2\phi\phi_t - \phi_y)\frac{\partial}{\partial y} - \phi_t\frac{\partial}{\partial t},$$

and thus

$$\langle N, X\rangle = \frac{-1}{|\Phi_y \wedge \Phi_t|}, \quad \langle N, Y\rangle = \frac{\phi_y - 4\phi\phi_t}{|\Phi_y \wedge \Phi_t|} = \frac{\mathcal{B}\phi}{|\Phi_y \wedge \Phi_t|}.$$

Then, by formula (1.7) and standard area formula we get

$$\begin{aligned} P(E_\phi, \exp(\mathbb{R}X_1)(D)) &= \int_{\partial E_\phi \cap \exp(DX_1)(D)} |N_H| d\mathscr{H}^2 \\ &= \int_D \sqrt{\frac{1}{|\Phi_y \wedge \Phi_t|^2} + \frac{(\mathcal{B}\phi)^2}{|\Phi_y \wedge \Phi_t|^2}}|\Phi_y \wedge \Phi_t| dydt \\ &= \int_D \sqrt{1 + (\mathcal{B}^{\phi}\phi)^2} dydt. \end{aligned}$$

□

We now introduce an equivalent point of view for intrinsic graphs, that will allow us to generalize Definition (1.9) to intrinsic graphs along any direction; we will then define and briefly discuss intrinsic Lipschitz functions. We first give the following

Definition 1.11 (Vertical plane)**.** *For any $\nu \in \mathbb{R}^{2n}$, we call the set*

$$H_\nu = \{(z,t) \in \mathbb{H}^n : \langle \nu, z\rangle \geq 0, t \in \mathbb{R}\}$$

the vertical half-space *through $0 \in \mathbb{H}^n$ with inner normal ν. The boundary of H_ν, the set*

$$\partial H_\nu = \{(z,t) \in \mathbb{H}^n : \langle \nu, z\rangle = 0, t \in \mathbb{R}\},$$

is called vertical plane *orthogonal to ν passing through $0 \in \mathbb{H}^n$.*

Notice that $W = \partial H_{e_1}$ and for $w \in W$

$$\exp(\phi(w)X_1)(w) = w \cdot (\phi(w)e_1),$$

where by "$\cdot$" we mean the group product on $\mathbb{H}^n$. Thus, the intrinsic graph of ϕ along X_1 can be equivalently defined as follows:

$$Gr(\phi) = \{w \cdot (\phi(w)e_1) \in \mathbb{H}^n : w \in D\},$$

and it makes sense to write

$$D \cdot \mathbb{R} = \exp(\mathbb{R}X_1)(D).$$

These observations suggest the following

Definition 1.12 (Intrinsic graphs)**.** *Let $\nu \in \mathbb{R}^{2n}$ with $|\nu| = 1$. Let $D \subset H_\nu$, and $\phi \colon D \to \mathbb{R}$ be a function. Then the* intrinsic graph *of ϕ is*

$$Gr(\phi) = \{p \cdot \phi(p)(\nu, 0) \in \mathbb{H}^n : p \in D\}.$$

It is known by classic theory that standard graphs of Lipschitz functions are characterized by the so-called *cone condition* (see for example [2]). The notion of intrinsic Lipschitz function exploits this fact replacing standard graph with intrinsic graph and standard cones with a new notion of *intrinsic cones*, that we are now going to define.

Let $\nu \in \mathbb{R}^{2n}$ with $|\nu| = 1$, and identify it with $(\nu, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, let $\nu(p) = \langle p, \nu\rangle \nu \in \mathbb{H}^n$. We define $\nu^\perp(p)$ to be the unique point such that:

$$p = \nu^\perp(p) \cdot \nu(p).$$

Definition 1.13 (Intrinsic cones)**.** *The (intrinsic) cone with vertex $0 \in \mathbb{H}^n$, axis $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$, and aperture $\alpha \in (0, \infty)$ is the set*

$$C(0, \nu, \alpha) = \left\{p \in \mathbb{H}^n : \|\nu^\perp(p)\|^b_\infty < \alpha \|\nu(p)\|^b_\infty\right\},$$

where $\|\cdot\|^b_\infty$ is the box norm defined in (1.4).

The (intrinsic) cone with vertex $p \in \mathbb{H}^n$, axis ν and aperture α is the set

$$C(p, \nu, \alpha) = p \cdot C(0, \nu, \alpha).$$

We are ready to define *intrinsic Lipschitz functions*:

Definition 1.14 (Intrinsic Lipschitz functions)**.** *Let $D \subset \partial H_\nu$ with ν as above. Let $L > 0$. A function $\phi\colon D \to \mathbb{R}$ is called* intrinsic Lipschitz *of constant L if for any $p \in Gr(\phi)$ there holds*

$$Gr(\phi) \cap C(p, \nu, 1/L) = \emptyset,$$

where $Gr(\phi)$ and $C(p, \nu, 1/l)$ are defined respectively in Definition 1.12 and Definition 1.13.

To end this section, we state a theorem which extends the area formula (1.14) to intrinsic graphs of intrinsic Lipschitz functions, that we call *intrinsic Lipschitz graphs.* We need the following:

Definition 1.15. *Let $D \subset W = \mathbb{R}^{2n}$ be an open set and let $\phi \in C(D)$ be a continuous function.*

1. *We say that $\mathcal{B}^\phi\phi \in L^\infty_{loc}(D)$ in the sense of distributions if there exists a function $\psi \in L^\infty_{loc}$ such that for any test function $\theta \in C^1_c(D)$ there holds*

$$\int_D \theta\psi dw = -\int_D \phi\frac{\partial\theta}{\partial y_1} - 2\phi^2\frac{\partial\theta}{\partial t} dw.$$

2. *We say that $\nabla^\phi\phi \in L^\infty_{loc}(D, R^{2n-1})$ in the sense of distributions if $X_1\phi, \dots, X_n\phi$, $\mathcal{B}\phi, Y_2\phi, \dots, Y_n\phi$ belong to $L^\infty_{loc}(D)$ in the sense of distributions.*

We have the following fundamental theorem; it is the final result of many contributions, see [4] and [6]

Theorem 1.1. *Let $\nu = e_1$, $D \subset \partial H_\nu$ be an open set and $\phi\colon D \to \mathbb{R}$ be a continuous function. Then ϕ is intrinsic Lipschitz in any $D' \Subset D$ if and only if $\nabla^\phi\phi \in L^\infty_{loc}(D, \mathbb{R}^{2n-1})$ in the sense of distributions. Moreover, in this case, for any $D' \Subset D$, we have*

$$P(E_\phi, D' \cdot \mathbb{R}) = \int_{D'} \sqrt{1 + |\nabla^\phi\phi|^2} dw.$$

1.4 Minimality and Variation Formulas

We derive and discuss first and second variation formulas for the H-perimeter of t-graphs and intrinsic graphs. Examples of minimizers are provided too.

To avoid ambiguity, we clear up some terminology: let $A \subset \mathbb{H}^n$ be open, we say that E is a *H-perimeter minimizer* (or simply a *minimizer*) in A if, for any $F \subset \mathbb{H}^n$ such that $E\Delta F \Subset A$, we have $P(E, A) \leq P(F, A)$. We call instead *H-minimal* in A any set E such that the first variation of $P(E, A)$ is 0. The precise meaning of this property will thus depend on the functional and on the kind of variations we are using, and we will point this out case by case. Of course if E is a minimizer it is also H-minimal, while the converse is in general false, as we will see later in this section.

1.4.1 Variation Formulas for t-graphs

Let $D \subset \mathbb{R}^{2n}$, $f\colon D \to \mathbb{R}$ be a Lipschitz function. Suppose that its t-epigraph, defined as in (1.12) is a H-perimeter minimizer in the cylinder $A = D \times \mathbb{R}$. We define moreover

$$\Sigma(f) = \{z \in D : \nabla f(z) + 2z^{\perp} = 0\}.$$

The set $\Sigma(f)$ is called the *characteristic set* of f, and, geometrically, it is the set of z such that the point $p = (z, f(z)) \in \partial E$ belongs to $\Sigma(\partial E)$, where

$$\Sigma(\partial E) = \{p \in \partial E : T_p\partial E = H_p\}.$$

First Variation. Let f and A be as above, that is, $E = E_f$, the t-epigraph of f, is a minimizer in A. By (1.13), we have that

$$P(E, A) = \int_D |\nabla f(z) + 2z^{\perp}|dz = \int_{D\setminus\Sigma(f)} |\nabla f + 2z^{\perp}|dz.$$

By minimality of E, if $\phi \in C_c^1(D)$, then

$$\begin{aligned}\int_{D\setminus\Sigma(f)} |\nabla f + 2z^{\perp}| &\leq \int_D |\nabla f + \epsilon\nabla\phi + 2z^{\perp}|dz\\ &= \int_{D\setminus\Sigma(f)} |\nabla f + \epsilon\nabla\phi + 2z^{\perp}|dz\\ &+ |\epsilon| \int_{\Sigma(f)} |\nabla\phi|dz,\end{aligned}$$

for all $\epsilon \in \mathbb{R}$, and we call the above last quantity

$$\mathscr{A}_f(\epsilon) = \int_{D\setminus\Sigma(f)} |\nabla f + \epsilon\nabla\phi + 2z^{\perp}|dz + |\epsilon| \int_{\Sigma(f)} |\nabla\phi|dz. \tag{1.15}$$

Looking at the second summand of $\mathscr{A}_f(\epsilon)$, we notice that in order to differentiate $\mathscr{A}_f(\epsilon)$ in $\epsilon = 0$, for all $\phi \in C_c^1(D)$, we need the measure of $\Sigma(f)$ to be 0. A condition ensuring this is the C^2 regularity of f:

Proposition 1.10. *In the same setting as above, let $f \in C^2(D)$. Then its 2-dimensional Lebesgue measure $\mathcal{L}^2(\Sigma(f)) = 0$.*

Proof. A point $z = (x, y)$ belongs to $\Sigma(f)$ if and only the system of equations

$$\Phi(z) = \nabla f(z) + 2z^{\perp} = 0$$

is satisfied.

We claim that, for each $z_0 \in \Sigma(f)$, there exists a neighbourhood V of z_0 in $\Sigma(f)$ contained in the graph of a C^1 function, and thus, $\Sigma(f)$ has measure 0.

By implicit function theorem, it suffices to show that there exists a non-vanishing directional derivative of some component Φ_j of Φ in z_0.

We have that

$$|\partial_{y_1}\Phi_1(z_0)| + |\partial_{x_1}\Phi_{n+1}(z_0)| = |f_{x_1 y_1}(z_0) - 2| + |f_{x_1 y_1}(z_0) + 2| \neq 0.$$

Hence, either $\partial_{y_1}\Phi_1(z_0)$ or $\partial_{x_1}\Phi_{n+1}(z_0)$ is different from 0.

The C^2 regularity of f is used to ensure, by Schwarz, that $f_{x_1 y_1} = f_{y_1 x_1}$.

□

We assume hereafter $f \in C^2$. Then, differentiating $\mathscr{A}_f(\epsilon)$, we get

$$\mathscr{A}_f'(\epsilon) = \int_{D\setminus\Sigma(f)} \frac{\left\langle \nabla f + \epsilon\nabla\phi + 2z^\perp, \nabla\phi \right\rangle}{|\nabla f + \epsilon\nabla\phi + 2z^\perp|} dz\,; \tag{1.16}$$

that, computed in $\epsilon = 0$, yields

$$\mathscr{A}_f'(0) = \int_{D\setminus\Sigma(f)} \frac{\langle \nabla f + 2z^\perp, \nabla\phi \rangle}{|\nabla f + 2z^\perp|} dz, \quad \phi \in C_c^1(D). \tag{1.17}$$

This is a *first variation formula* for $P(E;A)$; by minimality we have $\mathscr{A}_f'(0) = 0$ for all $\phi \in C_c^1(D)$. We stress the important fact that the formula actually makes sense for all $\phi \in C_c^1(D)$, even with support intersecting $\Sigma(f)$. However, if we let $\phi \in C_c^1(D \setminus \Sigma(f))$ integration by parts yields

$$\int_{D\setminus\Sigma(f)} \operatorname{div}\Big(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big)\phi dz = 0, \quad \phi \in C_c^1(D \setminus \Sigma(f)),$$

that, by the Fundamental Lemma of Calculus of Variations, is equivalent to the *H-minimal surface* equation for f, that is

$$\operatorname{div}\Big(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big) = 0, \quad z \in D \setminus \Sigma(f). \tag{1.18}$$

Then a t-epigraph E_f is *H-minimal* if f solves (1.18). Notice that this PDE makes (classic) sense only if f belongs at least to C^2.

After the derivation of a second variation formula for the perimeter of t-epigraphs, we will clear up the relation between the H- minimal surface equation and the property of being a minimizer.

Second Variation. In this paragraph we work out the easiest example of second variation formula for the perimeter functional. We will get it deriving twice expression (1.15) for $\mathscr{A}_\phi(\epsilon)$. Recall that $f \in C^2$. Diffentiating (1.16) gives

$$\mathscr{A}_f''(\epsilon) = \int_{D\setminus\Sigma(f)} -\frac{\left\langle \nabla f + \epsilon\nabla\phi + 2z^\perp, \nabla\phi \right\rangle^2}{|\nabla f + \epsilon\phi + 2z^\perp|^3} + \frac{|\nabla\phi|^2}{|\nabla f + \epsilon\phi + 2z^\perp|} dz\,,$$

that, computed in $\epsilon = 0$, becomes

$$\mathscr{A}_f''(0) = \int_{D\setminus\Sigma(f)} -\frac{\left\langle \nabla f + 2z^\perp, \nabla\phi \right\rangle^2}{|\nabla f + 2^\perp|^3} + \frac{|\nabla\phi|^2}{|\nabla f + 2z^\perp|}\, dz\,; \quad \phi \in C_c^1(D\setminus\Sigma(f)). \tag{1.19}$$

The above expression is the *second variation formula* for t-epigraphs. Notice that, unlike first variation formula (1.17), (1.19) is not defined if $\mathrm{supp}(\phi) \cap \Sigma(f) \neq \emptyset$, since the integral may diverge: this second variation formula will give us information only outside the characteristic locus. We will see that this problem will be by-passed introducing variations by flows of contact diffeomorphisms, studied in the next chapter.

Anyway, minimality of E implies

$$\mathscr{A}_f''(0) \geq 0, \quad \phi \in C_c^1(D\setminus\Sigma(f)).$$

It becomes thus interesting studying the sign of the second variation formula; for example if f is such that (1.18) holds, but with $\mathscr{A}_f''(0) < 0$ for some $\phi \in C_c^1(D\setminus\Sigma(f))$, then E_f would not be a minimizer, although being a H-minimal set.

In general, H-minimal sets for which the second variation formula is negative for some test function are called *unstable*. Minimal sets for which this phenomenon does not occur are said to be *stable*.

In the case of t-graph, we have the following

Proposition 1.11. *Let $f \in C^2(D)$. Then the second variation formula* (1.19) *is strictly positive for all $\phi \in C_c^1(D)$. In particular, H-minimal t-graphs are* stable.

Proof. Cauchy-Schwarz inequality gives

$$\frac{\left\langle \nabla f + 2z^\perp, \nabla\phi \right\rangle}{|\nabla f + 2z^\perp|} \leq |\nabla\phi|, \tag{1.20}$$

and this is an equality if and only if $\nabla f + 2z^\perp$ and $\nabla\phi$ are proportional, that is, if and only if there exists $\lambda \neq 0$ such that

$$\nabla f + 2z^\perp = \lambda\nabla\phi,$$

or

$$\nabla(f - \lambda\phi) = -2z^\perp = (2y,\, -2x).$$

with $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. But this is would imply that the 1-differential form $2ydx - 2xdy$ is closed, but it is not. Then (1.20) is a strict inequality, and thus the argument of (1.19) is strictly positive. $\square$

The good behaviour of the area functional for t-graphs is in fact an instance of the convexity of the integrand

$$L_z : \mathbb{R}^{2n} \ni v \mapsto |v + 2z^{\perp}| \in \mathbb{R};$$

this allows to prove that indeed a H-minimal epigraph is actually a minimizer: let E_{f^*} be a H-minimal epigraph, that is, let f^* be a solution of (1.18). Let E_f be another t-epigraph. Both f and f^* are assumed to belong to $C^2(D)$, with $D \Subset \mathbb{R}^{2n}$. Let $A = D \times \mathbb{R}$. Assume that $f = f^*$ on ∂D. Then

$$\begin{aligned} P(E_f, A) - P(E_{f^*}, A) &= \int_{D \setminus \Sigma(f)} L_z(\nabla f) - L_z(\nabla f^\star) dz \\ &\geq \int_{D \setminus \Sigma(f)} \Big\langle \nabla L_z(\nabla f^*), (\nabla f - \nabla f^*) \Big\rangle dz \\ &= - \int_{D \setminus \Sigma(f)} \operatorname{div}\left(\frac{\nabla f^*(z) + 2z^{\perp}}{|\nabla f^*(z) + 2z^{\perp}|} \right)(f - f^*) dz \\ &= 0, \end{aligned}$$

where the inequality is due to convexity.

1.4.2 Variation formulas for intrinsic graphs

Recalling definions of Subsection 1.3.1.1, let $W = \{(z,t) : x_1 = 0\}$, $D \subset W$ be open and $\phi \colon D \to \mathbb{R}$ be an intrinsic Lipschitz function such that E_ϕ is a *minimizing* intrinsic epigraph along X_1. We have by formula (1.14) that

$$\mathscr{A}(\phi) := P(E_\phi, D \cdot \mathbb{R}) = \int_D \sqrt{1 + |\nabla^\phi \phi|^2} dw.$$

Hereafter in this paragraph, we denote by f_x the partial derivative of a function f in the variable x.

Notice the following important phenomenon: let ϕ be as above and ψ smooth; then formally we have

$$\mathcal{B}^{\phi+\psi}(\phi + \psi) = \mathcal{B}^\phi(\phi) + \mathcal{B}^\psi(\psi) - 4(\phi\psi_t + \psi\phi_t), \tag{1.21}$$

and the strict intrinsic Lipschitz regularity of ϕ doesn't ensure that ϕ_t is (in distributional sense represented by) a L^∞_{loc} function. This would cause the divergence to ∞ of $\mathscr{A}(\phi + \psi)$ even for a small smooth ψ. This is one of the motivations for building a more precise kind of variations.

However, this phenomenon is prevented assuming the (standard) Lipschitz regularity for ϕ.

First Variation. Let thus $\phi \in \mathrm{Lip}(D)$: by minimality of E_ϕ we have

$$\frac{d}{d\epsilon}\mathscr{A}(\phi+\epsilon\psi)\Big|_{\epsilon=0} = 0, \quad \psi \in C_c^1(D) \tag{1.22}$$

We compute the left hand side of (1.22).

By (1.21), we have

$$\begin{aligned}\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi) = \Big(&X_2(\phi+\epsilon\psi),\dots,X_n(\phi+\epsilon\psi),\\ &\mathcal{B}^\phi(\phi)+\mathcal{B}^{\epsilon\psi}(\epsilon\psi)-4\epsilon(\phi\psi)_t,\\ &Y_2(\phi+\epsilon\psi),\dots,Y_n(\phi+\epsilon\psi)\Big),\end{aligned} \tag{1.23}$$

where $\mathcal{B}^\phi(\epsilon\psi) = \epsilon\psi_y - 4\epsilon^2\psi\psi_t$. The derivative with respect to ϵ of (1.23) is

$$\frac{d}{d\epsilon}\Big(\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi)\Big) = (X_2\psi,\dots,X_n\psi,\psi_y-8\epsilon\psi\psi_t-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi),$$

and so

$$\begin{aligned}&\frac{d}{d\epsilon}\mathscr{A}(\phi+\epsilon\psi) =\\ &\int_D \frac{\Big\langle(\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi),(X_2\psi,\dots,X_n\psi,\psi_y-8\epsilon\psi\psi_t-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big\rangle}{\sqrt{1+|\nabla^\phi\phi+\nabla^{\epsilon\psi}\epsilon\psi|^2}}dw,\end{aligned} \tag{1.24}$$

that, computed in $\epsilon = 0$ gives the following first variation formula for (Lipschitz) intrinsic graphs, that we impose to be equal to 0 by the minimality of E_ϕ:

$$\int_D \frac{\Big\langle\nabla^\phi\phi,(X_2\psi,\dots,X_n\psi,\psi_y-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big\rangle}{\sqrt{1+|\nabla^\phi\phi|^2}}dw = 0, \quad \psi\in C_c^1(D).$$

If we assume also the C^2 regularity of ϕ, we can integrate by parts with no boundary contribution the above expression and obtain the following *minimal surface equation* for intrinsic graphs:

$$\left(\frac{\partial}{\partial y}-4\phi\frac{\partial}{\partial t}\right)\frac{\mathcal{B}^\phi\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}+\sum_{j=2}^n X_j\left(\frac{X_j\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\right)+Y_j\left(\frac{Y_j\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\right)=0. \tag{1.25}$$

In $\mathbb{H}^1$, formula (1.25) specializes as follows:

$$\mathcal{B}^\phi\left(\frac{\mathcal{B}^\phi\phi}{\sqrt{1+|\mathcal{B}^\phi\phi|^2}}\right)=0,$$

that is

$$\frac{\mathcal{B}^\phi(\mathcal{B}^\phi\phi)\sqrt{1+|\mathcal{B}^\phi\phi|^2}-\mathcal{B}^\phi\phi\frac{\mathcal{B}^\phi\phi\mathcal{B}^\phi(\mathcal{B}^\phi\phi)}{\sqrt{1+|\mathcal{B}^\phi\phi|^2}}}{1+|\mathcal{B}^\phi\phi|^2}=\frac{\mathcal{B}^\phi(\mathcal{B}^\phi\phi)}{(1+|\mathcal{B}^\phi\phi|^2)^{3/2}}=0,$$

solved if and only if

$$\mathcal{B}^\phi(\mathcal{B}^\phi\phi)=0. \tag{1.26}$$

Notice that affine functions of the form $\phi(y,t)=yk+c$ for constants $k,c\in\mathbb{R}$ trivially solve (1.26), that is, they are H-minimal intrinsic graphs in $\mathbb{H}^1$.

Second Variation. We start computing a second variation formula deriving expression (1.24). We obtain, for $\psi\in C^1_c(D)$

$$\begin{aligned}
&\frac{d^2}{d\epsilon^2}\mathscr{A}(\phi+\epsilon\psi)=\\
&\int_D\Bigg(-\frac{\Big\langle\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi),(X_2\psi,\dots,X_n\psi,\psi_y-8\epsilon\psi\psi_t-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big\rangle^2}{\Big(1+|\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi)|^2\Big)^{3/2}}\\
&+\frac{\Big|(X_2\psi,\dots,X_n\psi,\psi_y-8\epsilon\psi\psi_t-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big|^2}{\sqrt{1+|\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi)|^2}}\\
&+\frac{\Big\langle\nabla^\phi\phi+\nabla\epsilon\psi(\epsilon\psi),(0,\dots,0,-8\psi\psi_t,0,\dots,0)\Big\rangle}{\sqrt{1+|\nabla^{\phi+\epsilon\psi}(\phi+\epsilon\psi)|^2}}\Bigg)dw
\end{aligned}$$

that, computed in $\epsilon=0$, gives the *second variation formula* for intrinsic graphs:

$$\begin{aligned}
&\frac{d}{d\epsilon}\mathscr{A}(\phi+\epsilon\psi)\Big|_{\epsilon=0}=\\
&\qquad\int_D\Bigg(-\frac{\Big\langle(\nabla^\phi\phi),(X_2\psi,\dots,X_n\psi,\psi_y-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big\rangle^2}{\Big(1+|\nabla^\phi\phi|^2\Big)^{3/2}}\\
&\qquad+\frac{\Big(1+\nabla^\phi\phi\Big)^2\Big|(X_2\psi,\dots,X_n\psi,\psi_y-4(\phi\psi)_t,Y_2\psi,\dots,Y_n\psi)\Big|^2}{\Big(1+|\nabla^\phi\phi|^2\Big)^{3/2}}\\
&\qquad+\frac{\mathcal{B}^\phi\phi(-8\psi\psi_t)}{\Big(1+|\nabla^\phi\phi|^2\Big)^{3/2}}\Bigg)dw.
\end{aligned}$$

In $\mathbb{H}^1$, the above formula assumes the form

$$
\begin{aligned}
&\frac{d}{d\epsilon}\mathscr{A}(\phi+\epsilon\psi)\Big|_{|\epsilon=0} = \\
&= \int_D \frac{-(\mathcal{B}^\phi\phi)^2(\psi_y-4(\phi\psi)_t)^2+(1+(\mathcal{B}^\phi\phi)^2)((\psi_y-4(\phi\psi)_t)^2+\mathcal{B}^\phi\phi(-8\psi\psi_t))}{(1+(\mathcal{B}\phi)^2)^{3/2}} \\
&= \int_D \frac{(\psi_y-4(\phi\psi)_t)^2+(1+(\mathcal{B}^\phi\phi)^2)(\mathcal{B}^\phi\phi)(-4(\psi^2)_t)}{(1+(\mathcal{B}^\phi\phi)^2)^{3/2}}dw.
\end{aligned}
$$

Exploiting again the C^2 regularity of ϕ, we can integrate by parts with no boundary contribution and obtain, by minimality of E_ϕ, the following condition:

$$
\int_D \frac{(\psi_y-4(\phi\psi)_t)^2}{(1+(\mathcal{B}^\phi\phi)^2)^{3/2}}+4\psi^2\frac{\partial}{\partial t}\left(\frac{\mathcal{B}^\phi\phi}{(1+(\mathcal{B}^\phi\phi)^2)^{1/2}}\right)dw \geq 0 \quad \psi\in C^1_c(D). \tag{1.27}
$$

We immediately see that solutions to the minimal surface equation (1.26) of the affine form $\phi(y,t)=yk+c$ satisfy the above condition (1.27); in fact, they are H-perimeter minimizing. To prove this claim it is enough to proceed by a standard calibration argument after noticing that the horizontal inner unit normal to $\partial E_\phi = Gr(\phi)$

$$
\nu = \left(\frac{1}{\sqrt{1+(\mathcal{B}^\phi\phi)^2}}, -\frac{\mathcal{B}^\phi\phi}{\sqrt{1+(\mathcal{B}^\phi\phi)^2}}\right) = \left(\frac{1}{\sqrt{1+k^2}}, -\frac{k}{\sqrt{1+k^2}}\right)
$$

is constant and in particular divergence free.

We can then ask whether, as in the case of t-graphs, every solution to (1.26) actually parametrizes a minimizer, and, in particular, whether H-minimal intrinsic graphs (1.14) are always *stable* or not. We show that the answer is negative.

Consider indeed the intrinsic epigraph E_ϕ whose boundary is parametrized by the function

$$
\phi\colon W\equiv\mathbb{R}^2\ni(y,t)\mapsto -\frac{yt}{1+2y^2}\in\mathbb{R}.
$$

One can check that $\mathcal{B}^\phi(\mathcal{B}^\phi\phi)=0$, i.e., E_ϕ is a H-minimal set. However, it is proved in [3] that affine functions are the only C^2 entire functions (that is, defined on the whole $\mathbb{R}^2$) parametrizing intrinsic H-minimizing epigraph in $\mathbb{H}^1$. In particular, E_ϕ is *unstable*. The problem of understanding whether entire parametrizations of perimeter minimizing set must have an affine form is known in literature as the *Bernstein problem*.

Chapter 2

Variations by contact diffeomorphisms

Along the discussion of variation formulas for intrinsic graphs (see Section 1.4.2), we got the first clue that standard variations (of "$+\epsilon\psi$ type") are not suitable variations to use in $\mathbb{H}^n$ to obtain general results: regularity assumptions were necessary to control convergence of area functionals. The reason is the following: in general, finiteness of H-perimeter is not preserved under diffeomorphisms.

We build here an explicit example of such a phenomen.

Example. Let p_i denote the i-th coordinate of a point p in $\mathbb{H}^n$. Let, for $j \in \mathbb{N}$

$$S_j = \{q \in \mathbb{H}^1 : |(q_1, q_2)| < r_j, q_3 = 1/j\},$$

where

$$r_j = \frac{1}{j^\alpha}, \quad \alpha > 0.$$

Consider thus the following set S in $\mathbb{H}^1$; it is a pile of circles with centres on the t-axis:

$$S = \bigcup_{j=1}^{\infty} S_j.$$

We compute the H-area of S in $\mathbb{H}^1$ making use of (1.9); let N^j be the Euclidean normal to S_j; we have that $N^j = (0, 0, 1)$, so that $|N_H^j| = 2\sqrt{x^2 + y^2}$. Thus, by area formula

$$\begin{aligned}\text{Area}_H(S) = \sum_{j=1}^{\infty} \text{Area}_H(S_j) = \sum_{j=1}^{\infty} \int_{S_j} |N_H^j| = \sum_{j=1}^{\infty} \int_0^{r_j} \Big(\int_{|(x,y)|=r} 2r d\mathscr{H}^1 \Big) dr \\ = \frac{4}{3}\pi \sum_{j=1}^{\infty} r_j^3.\end{aligned}$$

Let now $\Psi \colon \mathbb{H}^n \mapsto \mathbb{H}^n$ be the rotation that maps the t-axis on the x-axis; it is of course a diffeomorphism, of the simplest kind.

Letting N^{Ψ_j} be the Euclidean outer normal to $\Psi(S_j)$, we have $N^{\Psi_j} = (1,0,0)$ and then $|N_H^{\Psi_j}| = \sqrt{2}$. Thus

$$\mathrm{Area}_H\Big(\Psi(S)\Big) = \sum_{j=1}^{\infty} \mathrm{Area}_H\Big(\Psi(S_j)\Big) = \sqrt{2}\int_0^{r_j} 2\pi r dr = \sqrt{2}\pi \sum_j r_j^2.$$

It suffices then to choose $1/3 < \alpha < 1/2$ in the definition of r_j to have $\mathrm{Area}_H(S) < \infty$ and $\mathrm{Area}_H\Big(\Psi(S)\Big) = +\infty$.

2.1 Contact diffeomorphisms

The suitable class of diffemorphisms preserving the finiteness of H-perimeter are *contact diffeomorphisms*. They have been introduced in [11] for different purposes.

Definition 2.1 (Contact diffemorphisms)**.** *Let $A \subset \mathbb{H}^n$ be an open set, let*

$$\Psi \colon A \to \Psi(A) \subset \mathbb{H}^n$$

be a C^∞ diffeomorphism. Then Ψ is a contact diffeomorphism *if, for any $p \in A$, the differential*

$$J\Psi : T_pA \to T_{\Psi(p)}\Psi(A)$$

is such that

$$J\Psi(H_p) = H_{\Psi(p)}. \tag{2.1}$$

Notice that, actually, the linear diffeomorphism used in the example above is not a contact one.

We show now that contact diffemorphisms fulfil our requirement: they deform sets with finite H-perimeter into sets still with finite H-perimeter:

Proposition 2.1. *Let $E \subset \mathbb{H}^n$ be a Lebesgue measurable set, $A \subset \mathbb{H}^n$ be open, such that*

$$P(E,A) < \infty,$$

Let $\Psi \colon A \to \Psi(A)$ be a contact diffeomorphism *with compact support in A. Then*

$$P(\Psi(E), \Psi(A)) < \infty.$$

Proof. Let $\phi \in C_c^1(\Psi(A), \mathbb{R}^{2n})$, with $\|\phi\|_\infty \le 1$. Then, by the change of variables $p = \Psi(q)$,

$$\int_{\Psi(E)} \mathrm{div}_H\, \phi(p) dp = \int_E (\mathrm{div}_H(\phi))(\Psi(q))|\mathrm{det} J\Psi(q)| dq.$$

We can suppose that A is bounded. Write $\phi = (\phi \circ \Psi) \circ \Psi^{-1}$, and let $\rho = \phi \circ \Psi$. Thus, there holds

$$(\mathrm{div}_H\, \phi) \circ \Psi = \Big(\sum_{j=1}^{n} X_j(\rho_j \circ \Psi^{-1}) + Y_j(\rho_{n+j} \circ \Psi^{-1}) \Big) \circ \Psi = \sum_{j=1}^{n} \langle \nabla \rho, J\Psi^{-1} X_j \rangle + \langle \nabla \rho, J\Psi^{-1} Y_j \rangle,$$

and thus, since Ψ^{-1} is a contact diffemorphism, there exist smooth functions f_j, $j \in \{1, \ldots, 2n\}$, such that

$$(\mathrm{div}_H\, \phi) \circ \Psi |\mathrm{det} J\Psi| = \sum_{j=1}^{n} f_j X_j \rho_j + f_{n+j} Y_j \rho_j.$$

Write, for arbitrary $K_j \in \mathbb{R}$,

$$\begin{aligned} f_j X_j \rho_j &= K_j X_j \Big(\frac{f_j \rho_j}{K_j} \Big) - (X_j f_j) \rho_j, \\ f_{n+j} j Y_j \rho_j &= K_{n+j} Y_j \Big(\frac{f_{n+j} \rho_{n+j}}{K_{n+j}} \Big) - (Y_j f_{n+j}) \rho_{n+j}, \quad j = 1, \ldots, n. \end{aligned}$$

Now, for K_j large enough

$$\sum_{j=1}^{2n} \Big(\frac{f_j \rho_j}{K_j} \Big)^2 \leq 1,$$

and notice that K_j can be chosen not depending on ρ since $\|\rho\|_\infty \leq 1$. Thus, we can find $K = K(\Psi, A)$ such that the function $\psi \in C^1_c(A, \mathbb{R}^{2n})$ defined by the components $\psi_j = f_j \rho_j / K$ satisisfies $\|\rho\|_\infty \leq 1$. Finally, we have

$$\begin{aligned} \Big| \int_E (\mathrm{div}_H(\phi)) \circ \Psi |\mathrm{det} J\Psi| dq \Big| &\leq K \Big| \int_E \mathrm{div}_H\, \psi dq \Big| + \Big| \int_E \sum_{j=1}^{n} \Big((X_j f_j) \rho_j + (Y_j f_{n+j}) \rho_{n+j} \Big) dq \Big| \\ &\leq K \Big| \int_E \mathrm{div}_H\, \psi dq \Big| + C, \end{aligned}$$

where $C = C(\Psi, A)$ and $K = K(\Psi, A)$. We have used again the fact that $\|\rho\|_\infty \leq 1$. By taking the supremum on ϕ and recalling that $P(E, A) < +\infty$, we conclude. □

In order to get general variation formulas, we want to deform our sets under a *flow* $\{\Psi_s\}_{s \in \mathbb{R}}$ of contact diffeomorphisms. Then, we need to know the structure of vector fields generating such *contact flows*.

Before to proceed with a structure result for these contact vector fields, we define the following differential 1-form:

$$\theta_c = dt + 2 \sum_{j=1}^{n} x_j dy_j - y_j dx_j.$$

Such form will be useful because of the following characterization: it is the unique differential 1-form in $\mathbb{H}^n$ such that

$$\begin{aligned} &\theta_c(X_j) = \theta_c(Y_j) = 0 \quad j = 1, \dots, n, \\ &\theta_c(T) = 1. \end{aligned} \tag{2.2}$$

We are ready to characterize contact vector fields:

Theorem 2.1 (Characterization of contact vector fields)**.** *Let V be a vector field defined on $A \subset \mathbb{H}^n$. The following are equivalent:*

(i) V is a contact vector field, that is, it generates a flow of contact diffeomorphisms.

(ii) The following holds:

$$[V, X_j](p) \in H_p \quad \text{and} \quad [V, Y_j](p) \in H_p \quad p \in A,\ j = 1, \dots n$$

(iii) There exists $\psi \in C^\infty(A)$ such that

$$V = V_\psi := -4\psi T + \sum_{j=1}^{n} (Y_j \psi) X_j - (X_j \psi) Y_j \tag{2.3}$$

Proof. We prove first that $(i) \implies (ii)$.

Let $\{\Psi_s\}_{s \in \mathbb{R}}$ be the flow generated by V. Then, by definition of contact diffeomorphism, $J(\Psi)_s X_j(p) \in H_{\Psi_s(p)}$, with $p \in A$. Hereafter in the proof we will omit dependency on p. But then, by the property (2.2) of θ_c, we have that

$$\Psi_s^* \theta_c(X_j) = \theta_c(J\Psi_s X_j) = 0, \quad s \in \mathbb{R}, \tag{2.4}$$

where, by $\Psi_s^* \theta_c$ we mean the pull-back of θ_c by Ψ_s. Hence, differentiating (2.4) with respect to s, we obtain

$$0 = \frac{\partial}{\partial s} \Psi_s^* \theta_c\big(X_j\big) = \theta_c\Big(\frac{\partial}{\partial s}(J\Psi_s)(X_j)\Big), \quad s \in \mathbb{R}. \tag{2.5}$$

Now, recalling the definition of the Lie derivative $\mathcal{L}_W U$ of a vector field U along a vector field W,

$$\frac{\partial}{\partial s} J\Psi_s X_j \Big|_{s=0} = \mathcal{L}_V X_j = [V, X_j],$$

and so, computing (2.5) in $s = 0$,

$$\theta_c([V, X_j]) = 0.$$

But this means, again by (2.2), that $[V, X_j]$ is horizontal. A completely analogous argument for Y_j completes the proof of (ii).

Assume now (ii). We claim that (i) holds. We have at $s = 0$

$$\frac{\partial}{\partial s}\Psi_s^*\theta_c(X_j)\Big|_{s=0} = \theta_c\Big([V, X_j]\Big) = 0.$$

We also have for any $s \in \mathbb{R}$

$$\frac{\partial}{\partial s}\Psi_s^*\theta_c(X_j) = 0.$$

Indeed, by the property of flows,

$$\begin{aligned}\Psi_{s+\delta}^*\theta_c(X_j) &= \theta_c(J\Psi_{s+\delta}(X_j)) \\ &= \theta_c(J(\Psi_\delta \circ \Psi_s)(X_j)) \\ &= \theta_c(J\Psi_\delta J\Psi_s X_j) \\ &= \Psi_\delta^*(J\psi_s X_j) \\ &= \Psi_s^*\Psi_\delta^*(X_j),\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial}{\partial s}\Psi_s^*\theta_c(X_j) = \lim_{\delta\to 0}\frac{\Psi_s^*\Psi_\delta^*\theta_c(X_j) - \Psi_s^*\theta_c(X_j)}{\delta} &= \Psi_s^*\Big(\lim_{\delta\to 0}\frac{\Psi_\delta^*\theta_c(X_j) - \theta_c(X_j)}{\delta}\Big) \\ &= \Psi_\delta^*\Big(\frac{\partial}{\partial s}\Psi_s^*\theta_c(X_j)\Big|_{s=0}\Big) = 0.\end{aligned}$$

This means that $\theta_c(J\Psi_s X_j)$ is constant in s, but since $\Psi_0 = \mathrm{id}$ and $\theta_c(X_j) = 0$, we conclude that

$$\theta_c(J\Psi_s X_j) = 0 \quad s \in \mathbb{R},$$

that is, $J\Psi_s X_j$ is horizontal. Repeating these computations with Y_j in place of X_j proves (i).

We show now that (ii) is equivalent to (iii), completing the proof. Recall that, with f scalar valued function, U, W vector fields,

$$[fU, W] = f[U, W] - (Wf)U. \tag{2.6}$$

Write V of our statement as follows

$$V = \sum_{j=1}^{n} u_j X_j + v_j Y_j + zT,$$

for suitable smooth functions u_j, v_j, T, $j = 1, \ldots, n$. Then, recalling that the only non-vanishing commutators among X_i, Y_j, T are $[X_j, Y_j] = -4T$, and making use of (2.6), we get that the T-component of $[V, X_j]$ is

$$[V, X_j]_T = 4v_j - X_j z \tag{2.7}$$

and the T-component of $[V, Y_j]$ is

$$[V, Y_j]_T = 4u_j + Y_j z. \tag{2.8}$$

Now, (ii) holds if and only if (2.7) and (2.8) vanish. Thus, defining $\psi := z/4$, we are done.

□

Finally, we prove the following important property of contact diffeomorphisms: they are (locally) Lipschitz in the Carnot-Carathéodory metric of Heisenberg groups. We call such functions *H-Lipschitz*. Namely

Proposition 2.2. *Let d be the Carnot-Carathéodory metric on $\mathbb{H}^n$ of Definition 1.3. Let $A \subset \mathbb{H}^n$ be open, and let $\Psi\colon A \to \Psi(A)$ be a* contact diffeomorphism. *Then, for each subset $K \Subset A$ there exists $L_K > 0$ such that*

$$d(\Psi(p), \Psi(q)) \leq L_K\, d(p, q)$$

for every $p, q \in K$.

Proof. Fix $K \Subset A$, and let $p, q \in K$. Let $\gamma \in C^1([0,1], K)$ with $\gamma(0) = p$ and $\gamma(1) = q$ be a horizontal curve, that is, there exist functions h_j, $j = 1, \dots, 2n$, such that

$$\dot\gamma = \sum_{j=1}^{n} h_j X_j + h_{n+j} Y_j.$$

Let $\epsilon > 0$. We can assume, by definition of d, that

$$L(\gamma) = \int_0^1 |h| \leq d(p, q) + \epsilon, \tag{2.9}$$

where h is the $\mathbb{R}^{2n}$-valued function with components h_j.

Consider now $\tilde\gamma := \Psi \circ \gamma$. We have $\tilde\gamma(0) = \Psi(p)$ and $\tilde\gamma(1) = \Psi(q)$. We claim that $\tilde\gamma$ is a horizontal curve. Indeed,

$$\begin{aligned}
\dot{\tilde\gamma} = J\Psi\dot\gamma &= J\Psi \sum_{j=1}^{n} h_j X_j + h_{n+j} Y_j \\
&= \sum_{j=1}^{n} h_j J\Psi X_j + h_{n+j} J\Psi Y_j \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \Big(h_j f_{jk} + h_{n+j} f_{(n+j)k}\Big) X_k + \Big(h_j f_{j(n+k)} + h_{n+j} f_{(n+j)(n+k)}\Big) Y_k
\end{aligned}$$

for suitable regular functions f_{ij} with $i, j = 1, \dots, 2n$; we have used the definition of contact diffeomorphisms (see (2.1)) in the final equality.

Let now $\tilde{h}$ be the $\mathbb{R}^{2n}$-valued function whose k-th component is

$$\tilde{h}_k = \sum_{j=1}^{n} \Big(h_j f_{jk} + h_{n+j} f_{(j+n)k} \Big);$$

in this way $\dot{\tilde{\gamma}} = \sum_{k=1}^{n} \tilde{h}_k X_k + \tilde{h}_{n+k} Y_k$.

Since there exists $C_K > 0$ such that

$$\sup_{x \in K, i,j \in \{1,\dots,2n\}} |f_{ij}(x)| \le C_K < \infty,$$

we deduce that there exists $L = L_K$ such that

$$|\tilde{h}| \le L_K |h|.$$

This implies that

$$d(\Psi(p), \Psi(q)) \le L(\tilde{\gamma}) = \int_0^1 |\tilde{h}| \le L_K \int_0^1 |h| = L_K L(\gamma) \le L_K d(p,q) + L_K \epsilon,$$

where we used (2.9) in the final inequality. We conclude by arbitrariness of ϵ. □

2.2 Variation formulas: the smooth case

We obtain variation formulas fo1r the H-perimeter in an open set A of a set $E \subset \mathbb{H}^n$ by a Taylor formula for $P(\Psi_s(E), \Psi_s(A))$, where $\{\Psi_s\}_{s \in \mathbb{R}}$ is a *contact flow*. We compute its first and second term in order to get first and second variation term, respectively.

We need a Taylor expansion, up to the second order, of the Jacobian determinant

$$\begin{aligned} \mathscr{J}\Psi \colon \mathbb{R} &\to \mathbb{R} \\ s &\mapsto \mathscr{J}\Psi_s(p) = \sqrt{\det (J\Psi_s)|^*_{\partial E} \circ J\Psi_s|_{\partial E}}(p), \end{aligned}$$

where $\{\Psi\}_{s \in \mathbb{R}}$ is *any* flow of diffeomorphisms in $\mathbb{H}^n$ and $p \in \partial E \cap A$.

We prove here such a Taylor formula up to the first order; the second order will follow easily from this proof and it is postponed to Subsection 2.2.2, where it will be needed.

Lemma 2.1. *Let $E \subset \mathbb{H}^n$ and $A \subset \mathbb{H}^n$ be an open set such that $\partial E \cap A$ is a smooth hypersurface. Let $p \in \partial E \cap A$. Let $\{\Psi\}_{s \in \mathbb{R}}$ be a flow of diffemorphisms in A , generated by the vector field V. Then*

$$\mathscr{J}\Psi_s(p) = \mathscr{J}\Psi_s(p) \Big(\operatorname{div} V(\Psi_s(p)) - \langle JVN_s, N_s \rangle (\Psi_s)(p) \Big), \tag{2.10}$$

where $N_s(\Psi_s(p))$ is the standard Euclidean normal to $\Psi_s(\partial E \cap A)$ at $\Psi_s(p)$ and JV is the Jacobian matrix of V.
In particular,

$$\mathscr{J}\Psi_s(p) = 1 + s(\operatorname{div} V - \langle JVN, N \rangle)(p) + O(s^2) \tag{2.11}$$

Proof. Let

$$F\colon \mathbb{R}^{2n} \supset D \to \mathbb{R}^{2n+1}$$

be a C^1 function such that $\partial E \cap A = \{F(x) \in \mathbb{R}^{2n+1} : x \in D\}$. Consequently, $\Psi_s(\partial E \cap A) = \{\Psi_s(F(x)), x \in D\}$. Fix on $\partial E \cap A$ a frame of tangent fields $U_1, \dots, U_{2n}$. For $s \in \mathbb{R}$, $J\Psi_s U_1, \dots, J\Psi_s U_{2n}$ is a tangent frame to $\Psi_s(\partial E \cap A)$. For $s \in \mathbb{R}$, $q = \Psi_s(p) \in \Psi_s(\partial E \cap A)$ the symmetric square matrix $g^s(q)$ be defined by

$$g^s_{ij}(q) = \langle J\Psi_s U_i(p), J\Psi_s U_j(p)\rangle,$$

where $i, j \in \{1, \dots, 2n\}$, and where by $\langle \cdot, \cdot \rangle$ we mean the standard scalar product in $\mathbb{R}^{2n+1}$. We also let $g = g^0$.

We claim that:

$$\mathscr{J}\Psi_s(F(x)) = \frac{\sqrt{\det g^s(\Psi_s(F(x)))}}{\sqrt{\det g(F(x))}}, \quad x \in D, s \in \mathbb{R}. \tag{2.12}$$

By the standard area formula, we have

$$\mathscr{H}^{2n}(\Psi_s(\partial E \cap A)) = \int_D \sqrt{\det g^s(F(x))}dx;$$

on the other hand, by the change of variable formula, and again the area formula

$$\mathscr{H}^{2n}(\Psi_s(\partial E \cap A)) = \int_{\partial E \cap A} \mathscr{J}\Psi_s(p) d\mathscr{H}^{2n} = \int_D \mathscr{J}\Psi_s(F(x))\sqrt{\det g_{ij}(F(x))}dx,$$

and thus

$$\int_D \sqrt{\det g^s(\Psi_s(F(x)))}dx = \int_D \mathscr{J}\Psi_s(F(x))\sqrt{\det g(F(x))}dx. \tag{2.13}$$

Repeating the above arguments for arbitrary subsets of $\Psi_s(\partial E \cap A)$, we get that in fact (2.13) holds for arbitrary subsets of D, and this implies the validity of (2.12) pointwise in D, for any $s \in \mathbb{R}$, proving our claim.

Next, we compute the derivative of $s \mapsto \sqrt{\det g^s_{ij}(\Psi_s(F(x)))}$, for fixed $x \in D$. We can assume, for $p = F(x) \in \partial E \cap A$ fixed, that $\{J\Psi_s U_1(q), \dots, J\Psi_s U_{2n}(p)\}$ is an orthogonal family. This implies that $g^s_{kk}(p) = \langle J\Psi_s U_k(p), J\Psi_s U_k(p)\rangle$ for $k = 1, \dots, 2n$, while all other entries vanish. At the point p, we have

$$\frac{\partial}{\partial s}\sqrt{\det g^s} = \frac{\partial}{\partial s}\sqrt{\prod_{k=1}^{2n} g^s_{kk}} = \frac{1}{2\sqrt{\det g^s}}\sum_{k=1}^{2n}\left(\prod_{h=1}^{2n} g^s_{hh}\right)\frac{\frac{\partial}{\partial s}g^s_{kk}}{g^s_{kk}} = \frac{\sqrt{\det g^s}}{2}\sum_{k=1}^{2n}\frac{\frac{\partial}{\partial s}g^s_{kk}}{g^s_{kk}}, \tag{2.14}$$

where,

$$\begin{aligned}\frac{\partial}{\partial s}g^s_{kk}(p) = \frac{\partial}{\partial s}\langle J\Psi_s U_k(p), J\Psi_s U_k(p)\rangle &= 2\Big\langle J\Big(\frac{\partial}{\partial s}\Psi_s\Big)U_k(p), J\Psi_s U_k(p)\Big\rangle \\ &= 2\langle J\Big(V \circ \Psi_s)U_k(p), J\Psi_s U_k(p)\rangle \\ &= 2\langle JVJ\Psi_s U_k(p), J\Psi_s U_k(p)\rangle.\end{aligned}$$

Thus, since $\{J\Psi_s U_1(p), \dots, J\Psi_s U_{2n}(p), N_s(\Psi_s(p))\}$ is an orthogonal basis of $\mathbb{R}^{2n+1}$, we get

$$\begin{aligned}\sum_{k=1}^{2n} \frac{\frac{\partial}{\partial s} g^s_{kk}}{g^s_{kk}} &= 2\Big(\sum_{k=1}^{2n} \frac{\langle JVJ\Psi_s U_k(p), J\Psi_s U_k(p)\rangle}{|J\Psi_s U_k(p)|^2} + \langle JVN_s, N_s\rangle - \langle JVN_s, N_s\rangle \Big) \\ &= 2\Big(\operatorname{div} V - \langle JVN_s, N_s\rangle\Big)(\Psi_s(p)).\end{aligned} \tag{2.15}$$

Formula (2.10) follows from (2.12), (2.14) and (2.15). Taylor formula (2.11) is straightforward. □

2.2.1 First variation formula

Before to proceed with the derivation of first variation formula, we define the following real quadratic form. Let $\psi \in C^2(\mathbb{H}^n)$, $p \in \mathbb{H}^n$.

$$\begin{aligned} \mathscr{Q}_\psi \colon H_p &\to \mathbb{R} \\ v \ \mapsto \mathscr{Q}_\psi(v) &= \sum_{i,j} u_i u_j X_j Y_i \psi + u_j w_i (Y_i Y_j \psi - X_j X_i \psi) - w_i w_j Y_j X_i \psi, \end{aligned}$$

where ψ and its derivatives are evaluated at p. We identify a horizontal vector field $v = \sum_{j=1}^n v_j X_j + v_{n+j} Y_j$ with the vector $(v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}) \in \mathbb{R}^{2n}$.

We are ready to state and prove the following important theorem; we postpone its a discussion at the end of the proof:

Theorem 2.2. *Let $A \subset \mathbb{H}^n$ be an open set, and let $E \subset \mathbb{H}^n$ be a smooth hypersurface with finite H-perimeter in A, and let ν_E be horizontal normal. Let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(\Psi, A)$, be the contact flow generated by $\psi \in C^\infty(A)$. Then there exists a constant $C = C(\psi, A)$ such that*

$$\begin{aligned}\Big| P(\Psi_s(E), \Psi_s(A)) - P(E, A) + s \int_A \Big(4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E)\Big) d\mu_E \Big| \\ \le CP(E, A)s^2\end{aligned} \tag{2.16}$$

for any $s \in [-\delta, \delta]$.

Proof. We introduce first some notation. Let $E_s = \Psi_s(E)$, and $A_s = \Psi_s(A)$. Let also N be the Euclidean unit normal to $\partial E \cap A$, N_s be the Euclidean unit normal to $\partial E_s \cap A_s$. Define then

$$\begin{aligned} K &= \Big(\sum_{j=1}^n \langle X_j, N\rangle^2 + \langle Y_j, N\rangle^2 \Big)^{1/2}, \\ K_s &= \Big(\sum_{j=1}^n \langle X_j, N_s\rangle^2 + \langle Y_j, N_s\rangle^2 \Big)^{1/2}. \end{aligned}$$

Thus, by formula (1.9) (notice that $E_s \cap A_s$ is a smooth hypersurface), we have

$$P(E_s, A_s) = \int_{\partial E_s \cap A_s} K_s d\mathscr{H}^{2n}.$$

By the change of variables formula,

$$\int_{\partial E_s \cap A_s} K_s d\mathscr{H}^{2n} = \int_{\partial E \cap A} K_s \circ \Psi_s \mathscr{J} \Psi_s d\mathscr{H}^{2n}.$$

We will get (2.16) from the Taylor expansion (in s) of $K_s \circ \Psi_s$ and $\mathscr{J}\Psi_s$.

In the sequel we will often write N_s in place of $N_s(\Psi_s)$ and, consequently, K_s in place of $K_s(\Psi_s)$. We will omit dependency on $p \in \partial E \cap A$.

We compute the derivative of K_s with respect to s, and we start by computing the derivative of N_s with respect to s, that we indicate with N_s'. Fix a frame of vector fields $V_1, \ldots, V_{2n}$ tangent to $\partial E \cap A$. Thus, $\left(J\Psi_s V_1, \ldots, J\Psi_s V_{2n}\right)$ are tangent to $\partial E_s \cap A_s$, and so

$$\langle J\Psi_s V_i, N_s\rangle = 0 \quad i = 1, \ldots, n.$$

Differentiating the above identity yields with respect to s yields

$$\langle JV_\psi J\Psi_s V_i, N_s\rangle + \langle J\Psi_s V_i, N_s'\rangle = 0, \quad i = 1 \ldots, 2n, \tag{2.17}$$

where V_ψ is the contact field generating the flow Ψ, taking thus the form (2.3). On the other hand, differentiating with respect s the identity $|N_s|^2 = 1$, we get

$$\langle N_s', N_s\rangle = 0,$$

that is, N_s' is tangent to $\partial E_s \cap A_s$.

We deduce that, letting N_0' be the derivative of N_s' at $s = 0$,

$$\begin{aligned} N_0' = \sum_{i=1}^{2n} \langle V_i, N_0'\rangle &= -\sum_{i=1}^{2n} \langle JV_\psi V_i, N\rangle V_i \\ &= -\sum_{i=1}^{2n} \langle V_i, (JV_\psi)^* N\rangle V_i \\ &= \langle (JV_\psi)^* N, N\rangle N - (JV_\psi)^* N, \end{aligned} \tag{2.18}$$

where the second equality is due to (2.17) computed in $s = 0$. By the property of flows, one can prove that (2.18) holds for all $s \in \mathbb{R}$, that is

$$N_s' = \langle JV_\psi N_s, N_s\rangle N_s - JV_\psi^* N_s. \tag{2.19}$$

Let W be a smooth vector field in $\mathbb{H}^n$, and define

$$F_W(s) = \langle W, N_s\rangle(\Psi_s).$$

Then

$$F'_W(s) = \langle JWV_\psi, N_s\rangle + \langle W, N'_s\rangle,$$

that, by (2.19) and the definition of the adjoint map, becomes

$$\begin{aligned} F'_W(s) &= \langle JWV_\psi - JV_\psi W, N_s\rangle + \langle JV_\psi N_s, N_s\rangle\langle W, N_s\rangle \\ &= \langle [V_\psi, W], N_s\rangle + \langle JV_\psi N_s, N_s\rangle\langle W, N_s\rangle. \end{aligned} \tag{2.20}$$

We are ready to compute the derivative of $K_s \circ \Psi_s$. By definition of contact diffeomorphism, $K(p) \neq 0$ if and only if $K_s(\Psi_s(p)) \neq 0$. Thus, assuming $K(p) \neq 0$, we get

$$\frac{dK_s \circ \Psi_s}{ds} = \frac{1}{K_s}\sum_{j=1}^{n}\langle X_j, N_s\rangle F'_{X_j} + \langle Y_j, N_s\rangle F'_{Y_j}(s), \tag{2.21}$$

that is, by (2.20) with $W = X_j$ and $W = Y_j$,

$$\frac{dK_s \circ \Psi_s}{ds} = K_s\langle JV_\psi N_s, N_s\rangle + \frac{1}{K_s}\sum_{j=1}^{n}\Big\langle \langle X_j, N_s\rangle[V_\psi, X_j] + \langle Y_j, N_s\rangle[V_\psi, Y_j], N_s\Big\rangle, \tag{2.22}$$

evaluated at Ψ_s.

Notice that there exists $C_1 = C_1(\psi, A)$ such that

$$\left|\frac{dK_s \circ \Psi_s}{ds}\right| \leq C_1 K_s; \tag{2.23}$$

this is due to the characterization (ii) in Theorem 2.1 of contact vector fields: since $\langle X_j, N_s\rangle$ and $\langle Y_j, N_s\rangle$ are horizontal, the second summand of the right hand side of (2.22) is homogeneous of degree 1 in $(X_1, \ldots X_n, Y_1, \ldots, Y_n)$.

Thus we can interchange integral and derivative in s:

$$\frac{d}{ds}\int_{\partial S\cap A} K_s \circ \Psi_s \mathscr{J}\Psi_s d\mathscr{H}^{2n} = \int_{\partial E\cap A}\frac{d}{ds}\Big(K_s \circ \Psi_s \mathscr{J}\Psi_s\Big)d\mathscr{H}^{2n}.$$

Computing (2.22) in $s = 0$, and by (2.11), we get the following Taylor expansion for $K_s \circ \Psi_s$:

$$K_s \circ \Psi_s \mathscr{J}\Psi_s = K + s\Big(K \operatorname{div} V_\psi + \frac{1}{K}\sum_{j+1}^{n}\langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle\Big) + \Theta(s), \tag{2.24}$$

where we set $N_{X_j} = \langle X_j, N\rangle$, and $N_{Y_j} = \langle Y_j, N\rangle$ and where the function Θ is $O(s^2)$, for s tending to 0.

Now, by definition, there exists $C_2 = C_2(\psi, A)$ such that $\Theta(s^2) \leq C_2 s^2$ for $s \in [-\delta, \delta]$. However, we claim that we can find a constant C such that,

$$\Theta(s^2) \leq CKs^2 + O(s^3)$$

In order to show this, we use the following fact. There exists a constant $C_3 = C_3(\psi, A)$ such that

$$\left|\frac{d^2 K_s \circ \Psi_s}{ds^2}\right| \leq C_3 K_s. \tag{2.25}$$

Inequality (2.25) can be checked directly by expression (2.43) below, or can be deduced by a homogeneity argument. Thus, by Taylor formula with Lagrange remainder, (2.25) and (2.23), we have, evaluating K_s at Ψ_s,

$$\Theta(s^2) = \frac{1}{2}\frac{d^2}{ds^2}K_s\Big|_{s=\overline{s}} s^2 \leq CK_{\overline{s}}s^2 = C\Big((Ks^2 + \frac{d}{ds}K_s\Big|_{s=\overline{\overline{s}}}\overline{s}s^2\Big) \leq CKs^2 + O(s^3),$$

where $0 \leq \overline{s} \leq s$ and $\overline{s} \leq \overline{\overline{s}} \leq s$.

Hence, formula (2.24) reads

$$K_s \circ \Psi_s \mathscr{J}\Psi_s = K\Big(1 + s\Big(\operatorname{div} V_\psi + \frac{1}{K^2}\sum_{j=1}^{n}\langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle\Big) + O(s^2)\Big). \tag{2.26}$$

Routine computations give

$$\sum_{j=1}^{n}\langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle = -\mathscr{Q}_\psi\Big(N_{X_1}, \dots, N_{Y_n}\Big), \tag{2.27}$$

where we used the form (2.3) of V_ψ. Hence, by formula (1.11) for the horizontal normal ν_E, and since $\mathscr{Q}_\psi$ is a quadratic form, we obtain

$$\frac{1}{K^2}\sum_{j=1}^{n}\langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle = -\mathscr{Q}_\psi(\nu_E). \tag{2.28}$$

Now, using (1.2) $\operatorname{div} V_\psi$ is computed as follows:

$$\operatorname{div} V_\psi = -4T\psi + \sum_{j=1}^{n} X_j Y_j \psi - Y_j X_j \psi = -4(n+1)T\psi. \tag{2.29}$$

Hence, integrating (2.26), by (2.28), (2.29), and recalling (see formula (1.10)) that

$$\mu_E = K\mathscr{H}^{2n} \llcorner \partial E,$$

we obtain formula (2.16).

□

Remarks:

(i) Formula (2.16) immediately yields the first variation of the area functional:

$$\frac{d}{ds}P(\Psi_s(E), \Psi_s(A))\Big|_{s=0} = -\int_A \Big(4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E)\Big)d\mu_E,$$

and that, in particular, a (smooth, by now) H-perimeter minimizing set E in A satisfies the necessary condition

$$\int_A \Big(4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E)\Big) d\mu_E = 0 \quad \text{for all}\, \psi \in C^\infty(A).$$

However, formula (2.16) is something more. It gives the exact estimate of how the first variation approximates the difference $|P(\Psi_s(E), \Psi_s(A)) - P(E, A)|$, that is $O(s^2)$, and it shows that it is controlled also by $P(E, A)$.

(iii) Notice that the objects appearing in formula 2.16 make sense also in the minimal hypothesis of just finiteness of the H-perimeter of E in A. Indeed, recall from Proposition (1.5) that the horizontal normal ν_E and the perimeter measure μ_E do exist assuming only finiteness of H-perimeter.

(iii) The integral in (2.16) (locally) converges for any ψ. Formula (2.28) in the proof says that

$$\mathscr{Q}_\psi(\nu_E) = -\frac{1}{\Big(\sum_{j=1}^n \langle X_j, N\rangle^2 + \langle Y_j, N\rangle^2\Big)} \sum_{j+1}^n \langle \langle X_j, N\rangle [V_\psi, X_j] + \langle Y_j, N\rangle [V_\psi, Y_j], N\rangle,$$

hence, by the characterization (ii) in Theorem 2.1, the numerator and the denominator in the above right-hand side quantity have the same order, and integrability of $\mathscr{Q}_\psi(\nu_E)$ follows.

2.2.2 Second variation formula

We introduce some definitions and notation, in order to deal more easily with second derivatives of contact fields. .

Let $\mathcal{M}\Big(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ be the space of functions from $\mathbb{R}^m$ to $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, the space of linear mappings from $\mathbb{R}^m \to \mathbb{R}^n$. Let also $(z_1, \ldots, z_m)$ be the usual coordinates on $\mathbb{R}^m$. We define the following operator, that generalizes the role of the Hessian matrix to any dimension $n \geq 1$:

Definition 2.2 (Hessian operator)**.** *We call (generalized)* Hessian operator

$$\begin{aligned} \mathscr{H} \colon C^2(\mathbb{R}^m, \mathbb{R}^n) \times C(\mathbb{R}^m, \mathbb{R}^n) &\to \mathcal{M}\Big(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)\Big) \\ (V, W) &\mapsto (\mathscr{H} V)W, \end{aligned}$$

where $(\mathscr{H}V)W$, computed at the point z in $\mathbb{R}^m$, is the $n \times m$ matrix with components

$$\Big(\Big(\Big(\mathscr{H} V\Big) W\Big)(z)\Big)_{i,j} = \sum_{k=1}^m \frac{\partial^2 V_i}{\partial z_j \partial z_k} W_k(z).$$

When $n = 1$ Definition 2.2 gives the standard Hessian matrix. It is clear by Schwarz's theorem on second derivatives that $\mathscr{H}$ enjoys the following property:

$$((\mathscr{H}V)W)U = ((\mathscr{H}V)U)W.$$

Notation. In the sequel we will omit parentheses: $\mathscr{H}VWU$ represents the vector valued function obtained letting the matrix $(\mathscr{H}V)W$ act on the vector U. Moreover, we wll always have $n = m$.
The Hessian $\mathscr{H}$ satisfies the following index-free Leibniz formula. Let $V, W \in C^2(\mathbb{R}^n, \mathbb{R}^n)$; elementary computations show that

$$J\Big((JV)W\Big) = \mathscr{H}VW + JVJW, \tag{2.30}$$

where by $JVJW$ we mean the standard product of matrices.

As a first application of $\mathscr{H}$, we complete Lemma 2.11 with the second order term in the Taylor expansion of the Jacobian, involving indeed $\mathscr{H}$:

Lemma 2.2. *Let $E \subset \mathbb{H}^n$, $A \subset \mathbb{H}^n$ an open set such that $\partial E \cap A$ is a smooth hypersurface, and let N be its standard Euclidean normal. Let $\{\Psi\}_{s\in\mathbb{R}}$ be a flow of diffemorphisms in A, generated by the vector field V. Then*

$$\begin{aligned} \mathscr{J}\Psi_s = 1 &+ s(\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle) \\ &+ \frac{s^2}{2}\Big(\Big(\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle\Big)^2 + \operatorname{div}(JV_\psi V_\psi) \\ &\qquad - \langle \mathscr{H}V_\psi V_\psi N, N\rangle - 2\langle JV_\psi N, N\rangle^2 + |JV_\psi N|^2\Big) \\ &+ O(s^3), \end{aligned} \tag{2.31}$$

where the functions are evaluated at a point $p \in \partial E \cap A$.

Proof. We derive (2.10) with respect to s. We have, omitting dependency on p, that

$$\frac{\partial}{\partial s}(\operatorname{div} V)(\Psi_s) = \operatorname{div}\Big(\frac{\partial}{\partial s}(V \circ \Psi_s)\Big) = \operatorname{div}(JVV)(\Psi_s). \tag{2.32}$$

Moreover, by standard Leibniz formula for scalar products and by (2.30), and by formula (2.18) for the derivative of the normal, we get

$$\begin{aligned} \frac{d}{ds}\langle JVN_s, N_s\rangle &= \langle J(JV_\psi V_\psi)N_s, N_s\rangle + \langle (JV_\psi + JV_\psi^*)N_s', N_s\rangle \\ &= \langle \mathscr{H}V_\psi V_\psi N_s, N_s\rangle + 2\langle JV_\psi N_s, N_s\rangle^2 - |JV_\psi^* N_s|^2. \end{aligned} \tag{2.33}$$

Composition with Ψ_s was implicit in the above computation. By (2.32) and (2.33), and by formula (2.10) for the first derivative of the Jacobian, we obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial s^2}\Big(\mathscr{J}\Psi_s\Big) &= \frac{\partial}{\partial s}\Big(\mathscr{J}\Psi_s\Big)\Big((\operatorname{div}V(\Psi_s) - \langle JVN_s, N_s\rangle(\Psi_s)\Big) \\
&\quad + \mathscr{J}\Psi_s \frac{\partial}{\partial s}\Big((\operatorname{div}V)(\Psi_s) - \langle JVN_s, N_s\rangle(\Psi_s)\Big) \\
&= \Big((\operatorname{div}V)(\Psi_s) - \langle JVN_s, N_s\rangle(\Psi_s)\Big)^2 \\
&\quad + \mathscr{J}\Psi_s\Big(\operatorname{div}(JVV)(\Psi_s) - \langle \mathscr{H}VVN_s, N_s\rangle \\
&\qquad\qquad - 2\langle JVN_s, N_s\rangle^2 + |JV^*N_s|^2\Big)(\Psi_s).
\end{aligned} \tag{2.34}
$$

The second order term in (2.31) follows from (2.34).

□

Second variation formula will display the following ψ-related quadratic forms. For $p \in \mathbb{H}^n$, we define

$$
\begin{aligned}
\mathscr{R}_\psi \colon H_p &\to \mathbb{R} \\
v = \sum_{j=1}^n u_j X_j + w_j Y_j &\mapsto \mathscr{R}_\psi(v),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{R}_\psi(v) = \sum_{i,j=1}^n u_i w_j &\Bigg(\Big(\sum_{k=1}^n X_k Y_i\psi X_j Y_k\psi - Y_k Y_i\psi X_j X_k\psi - X_k X_j Y_i\psi Y_k\psi + Y_k X_j Y_i\psi X_k\psi\Big) \\
&\quad + 4\psi X_j Y_i T\psi\Bigg) \\
+ u_j w_i &\Bigg(\Big(\sum_k X_k(X_j X_i\psi - Y_i Y_j\psi)Y_k\psi - Y_k(X_j X_i\psi - Y_i Y_j\psi)X_k\psi \\
&\quad - X_k(X_i\psi - Y_j\psi)(X_j - Y_i)Y_k\psi + Y_k(X_i\psi - Y_j\psi)(X_j - Y_i)X_k\psi\Big) \\
&\quad - 4\psi(X_j X_i - Y_j Y_i)T\psi\Bigg) \\
- w_i w_j &\Bigg(\Big(\sum_k Y_k X_i\psi Y_j X_k\psi - X_k X_i\psi Y_k Y_k\psi - Y_k Y_j X_i\psi X_k\psi + X_k Y_j X_i\psi Y_k\psi\Big) \\
&\quad - 4\psi Y_j X_i T\psi\Bigg),
\end{aligned}
$$

and

$$\mathscr{P}_\psi \colon H_p \to \mathbb{R}$$
$$v = \sum_{j=1}^n u_j X_j + w_j Y_j \mapsto \mathscr{P}_\psi(v) = \sum_{j=1}^n \Big(\Big(\sum_{i=1}^n -u_i Y_j Y_i \psi + w_i Y_j X_i \psi \Big)^2 + \Big(\sum_{i=1}^n -u_i X_j Y_i \psi + w_i X_j X_i \psi \Big)^2 \Big).$$

Finally, we define $\mathscr{S}_\psi$ as follows:

$$\mathscr{S}_\psi \colon H_p \to \mathbb{R} \qquad v \to \mathscr{R}_\psi(v) + \mathscr{P}_\psi(v). \tag{2.35}$$

In the above expressions, ψ and its derivatives are computed at p. As usual, we identify a vector $(v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}) \in \mathbb{R}^{2n}$ with the horizontal vector $\sum_j^n v_j X_j + v_{n+j} Y_j$.

We can state and prove the following second variation formula for the H-perimeter; we postpone again comments and remarks at the end of the proof.

For a smooth hypersurface ∂E, we let

$$\mathscr{A}_\psi(\nu_E) = \mathscr{Q}_\psi(\nu_E) + 4(n+1)T\psi. \tag{2.36}$$

This is the integrand function of the first variation formula (2.16),

Theorem 2.3. *Let $A \subset \mathbb{H}^n$ be an open set, and let $E \subset \mathbb{H}^1$ be a smooth hypersurface with finite H-perimeter in A, and let ν_E be its horizontal normal. Let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(\Psi, A)$, be the contact flow generated by $\psi \in C^\infty(\mathbb{H}^n)$. Then there exists a constant $C = C(\psi, A)$ such that, for $s \in [-\delta, \delta]$, we have*

$$\Big| P(\Psi_s(E), \Psi_s(A)) - P(E, A) + s \int_A \mathscr{A}_\psi(\nu_E) d\mu_E - s^2 \int_A \Big(\mathscr{S}_\psi(\nu_E) - \Big(\mathscr{A}_\psi(\nu_E)\Big)^2 + 32\Big((n+1)T\psi\Big)^2 + \operatorname{div}(JV_\psi V_\psi) \Big) d\mu_E \Big| \le C P(E, A) s^3, \tag{2.37}$$

The quantity $\operatorname{div}(JV_\psi V_\psi)$ can be made explicit as follows:

$$\operatorname{div}(JV_\psi V_\psi) = \sum_i \Big(\Big(\sum_j 2X_j Y_i \psi X_i Y_j \psi - 2X_i X_j \psi Y_i Y_j \psi \Big) - 4(n+2) X_i \psi Y_i T\psi + 4(n+2) Y_i \psi X_i T\psi \Big) + 16(n+1)\Big((T\psi)^2 + \psi T^2 \psi \Big). \tag{2.38}$$

Proof. Recall from the proof of Theorem 2.2 the notation $E_s = \Psi_s(E)$, $A_s = \Psi_s(A)$, the definition of K_s , K and N_s, and the area formula

$$P(E_s, A_s) = \int_{\partial E\cap A} K_s \circ \Psi_s \mathscr{J}\Psi_s d\mathscr{H}^{2n}.$$

We compute the Taylor series up to the second order of $K_s \circ \Psi_s \mathscr{J}\Psi_s$.

We start by formula (2.21), that holds assuming $K(p) \neq 0, p \in \partial E \cap A$:

$$\frac{dK_s \circ \Psi_s}{ds} = \frac{1}{K_s}\sum_{j=1}^{n}\langle X_j, N_s\rangle F'_{X_j}(s) + \langle Y_j, N_s\rangle F'_{Y_j}(s), \tag{2.39}$$

where, for a vector field W, $F_W(s) = \langle W, N_s\rangle(\Psi_s)$. Dependency on Ψ_s will be often omitted.

Differentiating the right-hand side of (2.39), we get

$$\begin{aligned}\frac{d^2K_s \circ \Psi_s}{ds^2} &= -\frac{1}{K_s^3}\Big(\langle X, N_s\rangle F'_X(s) + \langle Y, N_s\rangle F'_Y(s)\Big)^2 \\ &+ \frac{1}{K_s}(F'_X(s)^2 + F'_Y(s)^2 + \langle X, N_s\rangle F''_X(s) + \langle Y, N_s\rangle F''_Y(s)).\end{aligned} \tag{2.40}$$

We compute $F''_W(s)$, for any vector field W in $\mathbb{H}^n$. We found out in (2.20) that

$$F'_W(s) = \langle [V_\psi, W], N_s\rangle + \langle JV_\psi N_s, N_s\rangle\langle W, N_s\rangle. \tag{2.41}$$

Thus, we first compute the derivative of $\langle [V_\psi, W], N_s\rangle$, evaluated at Ψ_s. We have

$$\frac{d}{ds}[V_\psi, W] = \Big(J[V_\psi, W]\Big)V_\psi,$$

evaluated ad $\Psi_s(p)$.

The derivative of $N_s(\Psi_s)$ was already found in (2.19):

$$N'_s = \langle JV_\psi N_s, N_s\rangle N_s - JV_\psi^* N_s.$$

Hence, we get by Leibniz rule and the above formulas

$$\begin{aligned}\frac{d}{ds}\langle [V_\psi, W], N_s\rangle &= \langle J[V_\psi, W]V_\psi, N_s\rangle + \Big\langle [V_\psi, W], \langle JV_\psi N_s, N_s\rangle N_s - JV_\psi^* N_s\Big\rangle \\ &= \langle J[V_\psi, W]V_\psi - JV_\psi[V_\psi, W], N_s\rangle + \langle JV_\psi N_s, N_s\rangle\langle [V_\psi, W], N_s\rangle.\end{aligned}$$

Notice now that

$$J[V_\psi, W]V_\psi - JV_\psi[V_\psi, W] = \Big[V_\psi, [V_\psi, W]\Big],$$

and thus

$$\frac{d}{ds}\Big(\langle [V_\psi, W], N_s\rangle\Big) = \Big\langle \Big[V_\psi, [V_\psi, W]\Big], N_s \Big\rangle + \langle JV_\psi N_s, N_s\rangle\langle [V_\psi, W], N_s\rangle.$$

We already computed the derivative of $\langle JV_\psi N_s, N_s\rangle$ in (2.33):

$$\frac{d}{ds}\langle JV_\psi N_s, N_s\rangle = \langle \mathscr{H} V_\psi V_\psi N_s, N_s\rangle + 2\langle JV_\psi N_s, N_s\rangle^2 - |JV_\psi^* N_s|^2.$$

Hence, recalling (2.41), we obtain

$$\begin{aligned} F''_W(s) &= \frac{d}{ds}\langle [V_\psi, W], N_s\rangle + \frac{d}{ds}\langle JV_\psi N_s, N_s\rangle F_X(s) + \langle JV_\psi, N_s\rangle F'_X(s) \\ &= \Big\langle \Big[V_\psi, [V_\psi, W]\Big], N_s \Big\rangle + \langle JV_\psi N_s, N_s\rangle\langle [V_\psi, W], N_s\rangle \\ &\quad + (\langle \mathscr{H} V_\psi V_\psi N_s, N_s\rangle + 2\langle JV_\psi N_s, N_s\rangle^2 - |JV_\psi^* N_s|^2)\langle W, N_s\rangle \\ &\quad + \langle JV_\psi N_s, N_s\rangle\langle [V_\psi, W], N_s\rangle + \langle JV_\psi N_s, N_s\rangle^2\langle W, N_s\rangle \\ &= \Big\langle \Big[V_\psi, [V_\psi, W]\Big], N_s \Big\rangle + 2\langle JV_\psi N_s, N_s\rangle\langle [V_\psi, W], N_s\rangle \\ &\quad + (\langle \mathscr{H} V_\psi V_\psi N_s, N_s\rangle + 3\langle JV_\psi N_s, N_s\rangle^2 - |JV_\psi^* N_s|^2)\langle W, N_s\rangle. \end{aligned} \tag{2.42}$$

Write now K'_s and K''_s in place of the first and the second derivative of $K_s \circ \Psi_s$, and let K'_0 and K''_0 be such derivatives computed in $s = 0$. Since $K'_s = \Big(\langle X, N_s\rangle F'_X(s) + \langle Y, N_s\rangle F'_Y(s)\Big)/K_s$, we can write (2.40) in the following way:

$$K''_s = -\frac{K'^2_s}{K_s} + \frac{F(s)}{K_s}, \tag{2.43}$$

where we define $F(s)$ to be the numerator of the second summand of (2.40). By (2.41) and (2.42), for $W = X_j$ and $W = Y_j$, we get

$$\begin{aligned} F(s) = &\sum_{j=1}^{n} \Bigg(\bigg(\Big\langle \Big[V_\psi, [V_\psi, X_j]\Big], N_s \Big\rangle + 4\langle [V_\psi, X_j], N_s\rangle\langle JV_\psi N_s, N_s\rangle\bigg)\langle X_j, N_s\rangle \\ &+ \bigg(\Big\langle \Big[V_\psi, [V_\psi, Y_j]\Big], N_s \Big\rangle + 4\langle [V_\psi, Y_j], N_s\rangle\langle JV_\psi N_s, N_s\rangle\bigg)\langle Y_j, N_s\rangle \\ &+ \langle [V_\psi, X_j], N_s\rangle^2 + \langle [V_\psi, Y_j], N_s\rangle^2\Bigg) \\ &+ (\langle \mathscr{H} V_\psi V_\psi N_s, N_s\rangle - |JV_\psi^* N_s|^2 + 4\langle JV_\psi N_s, N_s\rangle^2)K_s^2. \end{aligned} \tag{2.44}$$

We have then computed all terms involved in the second derivative of $K_s \circ \Psi_s$.

Recalling now Lemma 2.2, and writing

$$K_s = K + sK'_0 + \frac{s^2}{2}K''_0 + O(s^3),$$

we get the following Taylor expansion

$$K_s \mathscr{J}\Psi_s = K + s(K(\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle) + K_0') + \frac{s^2}{2}\Big(K_0'' + K\Big(\operatorname{div}(JV_\psi V_\psi) - \langle \mathscr{H} V_\psi V_\psi N, N\rangle - 2\langle JV_\psi N, N\rangle^2 + |JV_\psi^* N|^2 + (\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle)^2\Big) + 2K_0'(\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle)\Big) + o(s^2);$$

the second derivative of $K_s \mathscr{J}\Psi_s$ will be thus the term multiplying $s^2/2$. Recall, by (2.22) and (2.27) in the proof of Theorem (2.2), that

$$\begin{aligned} \frac{dK_s \circ \Psi_s}{ds} &= K_s\langle JV_\psi N_s, N_s\rangle + \frac{1}{K_s}\sum_{j=1}^n \Big\langle \langle X_j, N_s\rangle [V_\psi, X_j] + \langle Y_j, N_s\rangle [V_\psi, Y_j], N_s\Big\rangle \\ &= K_s\langle JV_\psi N_s, N_s\rangle - \mathscr{Q}_\psi\Big(\langle X_1, N_s\rangle, \dots, \langle Y_n, N_s\rangle\Big)/K_s. \end{aligned} \tag{2.45}$$

Thus, by (2.45), (2.43) and (2.44), we get, omitting the argument of $\mathscr{Q}_\psi$,

$$\begin{aligned} \left(\frac{d^2 K_s \mathscr{J}\Psi_s}{ds^2}\right)\bigg|_{s=0} &= \Big(K_0'' + K(\operatorname{div}(JV_\psi V_\psi) - \langle \mathscr{H} V_\psi V_\psi N, N\rangle - 2\langle JV_\psi N, N\rangle^2 + |JV_\psi N|^2) \\ &\quad + 2K_0'\Big(\operatorname{div} V_\psi - \langle JV_\psi N, N\rangle\Big)\Big) \\ &= K\Bigg\{\sum_{j=1}^n \frac{1}{K^2}\Big(\Big\langle\Big[V_\psi, [V_\psi, X_j]\Big], N\Big\rangle\langle X_j, N\rangle + \Big\langle\Big[V_\psi, [V_\psi, Y_j]\Big], N\Big\rangle\langle Y_j, N\rangle \\ &\quad + \langle [V_\psi, X_j], N\rangle^2 + \langle [V_\psi, Y_j], N\rangle^2\Big) \\ &\quad - \frac{\mathscr{Q}_\psi^2}{K^4} - 2\frac{\mathscr{Q}_\psi}{K^2}\operatorname{div} V_\psi + (\operatorname{div} V_\psi)^2 + \operatorname{div} JV_\psi V_\psi\Bigg\}. \end{aligned}$$

Tedious computations show that

$$\sum_{j=1}^n \Big\langle\Big[V_\psi, [V_\psi, X_j]\Big], N\Big\rangle\langle X_j, N\rangle + \Big\langle\Big[V_\psi, [V_\psi, Y_j]\Big], N\Big\rangle\langle Y_j, N\rangle = \mathscr{R}_\psi(\langle X_1, N\rangle, \dots \langle Y_n, N\rangle),$$

and

$$\sum_{j=1}^n \langle [V_\psi, X_j], N\rangle^2 + \langle [V_\psi, Y_j], N\rangle^2 = \mathscr{P}_\psi(\langle X_1, N\rangle, \dots \langle Y_n, N\rangle).$$

Moreover, we write

$$-\frac{\mathscr{Q}_\psi^2}{K^4} - 2\frac{\mathscr{Q}_\psi}{K^2}\operatorname{div} V_\psi = -(\mathscr{A}_\psi(\nu_E))^2 + (\operatorname{div} V_\psi)^2$$

by completing the squares.

By (2.43) and (2.44) we deduce that there exists $C = C(\psi, A)$ such that

$$\left|\frac{d^2 K_s \circ \Psi_s}{ds^2}\right| \leq C K_s,$$

and then we can interchange integral and derivative. Moreover, there exists $C_1 = C_1(\psi, A)$ such that

$$\left|\frac{d^3 K_s \circ \Psi_s}{ds^3}\right| \leq C_1 K_s,$$

and we conclude the proof invoking Taylor formula with Lagrange remainder as in the proof of Theorem 2.2. This last inequality can be proved by direct computation or by a homogeneity argument. □

Remarks:

(i) By Theorem 2.3, we get that a smooth H-perimeter minimizing set E in A satisfies, for all $\psi \in C_c^\infty(A)$:

$$\int_A \Big(\mathscr{S}_\psi(\nu_E) - \big(\mathscr{A}_\psi(\nu_E)\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big) d\mu_E \geq 0,$$

along with vanishing first variation, whose formula was already proved in Theorem 2.2

$$\int_A \mathscr{A}_\psi(\nu_E) d\mu_E = 0 \quad \text{for all } \psi \in C_c^\infty(A).$$

Moreover, formula (2.37) improves the information about the difference $|P(\Psi_s(E), \Psi_s(A)) - P(E, A)|$ for small s discussed in the first remark at Theorem 2.2.

(ii) As for formula (2.16), formula (2.37) displays objects making sense also for non-smooth sets, actually, it makes sense for sets with just finite H-perimeter.

(iii) The integral corresponding to the second variation in formula (2.37) is well defined for all test functions ψ: this is again an instance of the characterization of contact vector fields, and can be seen analogously to the third remark at Theorem 2.2.

This is particularly interesting, since we saw in formula (1.19) that the characteristic set of ∂E could cause problems of integrability, and we were bound to restrict the set of admissible test functions. Contact diffeomorphisms prevent such an issue. In our setting, the characteristic set of ∂E is

$$\Sigma(\partial E) = \{p \in \partial E : \nu_E(p) = 0\}.$$

2.3 Variation formulas: the general case

We are going to prove a theorem that generalizes Theorem 2.3 to any set with finite H-perimeter, and that, in particular, gives first and second variation formulas for such sets.

Actually, we are proving this result for a slightly more general family of sets: *H-rectifiable sets*. Since such sets are usually not boundaries of other sets, it does not make sense talking of perimeter: we will thus deal with the *spherical Hausdorff measure*. Our result will follow by the important fact that the "relevant" part of the boundary of sets with finite H-perimeter is H-rectifiable and that, up to a constant, the perimeter measure coincide with the spherical Hausdorff measure. We now clear up all of these notions and preliminary results.

Reduced boundary and measure theoretic boundary. Recall the distance ρ defined in (1.5): $\rho(p,q) = \|p^{-1} \cdot q\|_{\infty}^{b}$, for $p, q \in \mathbb{H}^n$. We define the balls of center p and radius r with respect to ρ in the following way:

$$B(p,r) = \{q \in \mathbb{H}^n : \|p^{-1} \cdot q\|_{\infty} < r\}.$$

In analogy with the Euclidean setting, we define the *reduced boundary*:

Definition 2.3 (Reduced boundary)**.** *Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in an open set A. Let ν_E its horizontal normal. We call* reduced boundary *of E the set $\partial^* E$ of all points $p \in A$ such that the following hold:*

(i) $\mu_E(B(p,r)) > 0$ *for all* $r > 0$.

(ii) *There holds*

$$\lim_{r\to 0} \fint_{B(p,r)} \nu_E d\mu_E = \nu_E(p).$$

(iii) *There holds* $|\nu_E(p)| = 1$

The most important property of this set is the fact that the perimeter measure is *concentrated* on it: this is the content of the following

Theorem 2.4. *Let E, A as in the above definition. Then $\mu_E(A \setminus \partial^* E) = 0$.*

Definition 2.3 is introduced and studied in [9], where it is also proved Theorem 2.4. We now give the definition of *measure theoretic boundary*; its connection with the reduced boundary will be given in the next paragraph.

Definition 2.4. *Let $E \in \mathbb{H}^n$ be measurable. The* measure theoretic boundary *$\partial_* E$ of E is the set of $p \in \mathbb{H}^n$ such that*

$$|E \cap B(p,r)| > 0 \text{ and } |B(p,r) \setminus E| > 0 \text{ for all } r > 0.$$

Hausdorff measures in Heisenberg groups We consider again the distance ρ od Definition 1.5. We define the *diameter* of a set $U \subset \mathbb{H}^n$ as

$$\operatorname{diam} U = \sup_{p,q\in U} \rho(p,q)$$

In particular, for $p \in \mathbb{H}^n$ the diameter of $B(p,r) = 2r$. Let $E \subset \mathbb{H}^n$ be a set. We define, for $s \geq 0$ and $\delta > 0$ the following premeasures:

$$\mathcal{H}^s_\delta(E) = \inf \Big\{ \sum_{j\in\mathbb{N}} (\operatorname{diam} U_j)^s : E \subset \bigcup_{j\in\mathbb{N}} U_j, U_j \subset \mathbb{H}^n, \operatorname{diam} U_j < \delta\}$$
$$\mathcal{S}^s_\delta(E) = \inf \Big\{ \sum_{j\in\mathbb{N}} (\operatorname{diam} B_j)^s : E \subset \bigcup_{j\in\mathbb{N}} B_j, B_j \subset \mathbb{H}^n \rho - \text{balls}, \operatorname{diam} B_j < \delta\}.$$

Letting $\delta \to 0$ we define

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$$
$$\mathcal{S}^s(E) = \lim_{\delta\to 0} \mathcal{S}^s_\delta(E).$$

By Carathéodory's criterion one can prove that $\mathcal{H}^s$ and $\mathcal{S}^s$ are actually Borel measures; we call the first s-dimensional *Hausdorff measure* and the latter s-dimensional *spherical Hausdorff measures.* Such measures are equivalent, in the sense that for $E \subset \mathbb{H}^n$ there holds

$$\mathcal{H}^s(E) \leq \mathcal{S}^s(E) \leq 2^s \mathcal{H}^s(E).$$

One can prove that $\mathcal{H}^Q$ and $\mathcal{S}^Q$, where we recall that $Q = 2n + 2$, are Haar measures in $\mathbb{H}^n$, and then, they coincide with the Lebesgue measure $\mathcal{L}^{2n+1}$ up to a multiplicative constant factor. It follows that the natural dimension to measure hypersurfaces in $\mathbb{H}^n$ is $Q - 1$. In fact, we have the following fundamental theorem, proved in [9].

Theorem 2.5. *Let $E \subset \mathbb{H}^n$, be a set with locally finite H-perimeter in A, with A open in $\mathbb{H}^n$. Then we have*

$$\mu_E = c_n \mathcal{S}^{Q-1} \llcorner \partial^* E \cap A,$$

where $c_n > 0$ is an absolute constant.

Even though $\mathcal{H}^s$ and $\mathcal{S}^s$ are equivalent measures, it is still an open problem establishing the validity of the above theorem for $\mathcal{S}^s$.

We illustrate now some general facts we will use in the proof of our variation variation formula in the non-regular case. The following lemma can be proved by the same techniques used in [8] adapted to our non-Euclidean case.

Lemma 2.3. *Let E a set with finite H-perimeter in $\mathbb{H}^n$. Then there hold*

(i) $\partial^* E \subset \partial_* E$,

(ii) $\mathcal{S}^{Q-1}(\partial_* E \setminus \partial^* E) = 0$

We close this paragraph with the following easy result involving H-Lipschitz functions: in particular it applies to contact diffeomorphisms (recall Proposition 2.2).

Proposition 2.3. *Let A be an open and bounded set in $\mathbb{H}^n$, and let $F : A \to \mathbb{H}^n$ be a H-Lipschitz function, that is, there exists L such that*

$$d(F(p), F(q)) \leq L\, d(p, q)$$

for $p, q \in A$ and where d is the Carnot-Carathéodory distance in $\mathbb{H}^n$. Let $s > 0$. Then

$$\mathcal{S}^s(F(A)) \leq L^s \mathcal{S}^s(A).$$

Proof. Recall the equivalence between ρ and d established in Proposition 1.4 .Let $\delta > 0$. Let, for $j \in \mathbb{N}$, $B_j := B(0, r_j)$ such that $A \subset \cup_j B_j$ and $r_j \leq \delta$. Thus, $\operatorname{diam} F(B_j) \leq L \operatorname{diam} B_j \leq L\delta/2$, and $F(A) \subset \cup_j F(B_j)$. Thus

$$\mathcal{S}^s_{L\delta}(F(A)) \leq \sum_{j=1}^{\infty} (\operatorname{diam} F(B_j))^s \leq L^s \sum_{j=1}^{\infty} (\operatorname{diam} B_j)^s,$$

and thus, taking the infimum on $\{B_j\}_{j \in \mathbb{N}}$ covering A we get

$$\mathcal{S}^s_{L\delta} F(A) \leq L^s \mathcal{S}^s_\delta(A).$$

Taking the limit as $\delta \to 0$ in the above inequality ends the proof. □

For more details on Hausdorff measures, see e.g. [8], where such a theory is carried out in Euclidean setting.

***H*-rectifiability** It is proved in [9] that the reduced boundary of a set with finite H-perimeter is rectifiable in an intrinsic sense: a property called *H-rectifiability*. Roughly speaking, it can be covered, up to null sets, with C^1_H hypersurfaces, introduced in Definition 1.8. As previously remarked, the natural measure for hypersurfaces in $\mathbb{H}^n$ is the Hausdorff spherical measure $\mathcal{S}^{Q-1}$.

Definition 2.5. *A set $R \subset \mathbb{H}^n$ is H-rectifiable if there exists a sequence of H-regular hypersurfaces $\{S_j\}_{j \in \mathbb{N}}$ with $\mathcal{S}^{Q-1}(S_j) < \infty$ such that*

$$\mathcal{S}^{Q-1}\Big(R \setminus \bigcup_{j \in \mathbb{N}} S_j\Big) = 0.$$

In particular, it is clear that $R \subset \mathbb{H}^n$ is H-rectifiable if and only if there exists a $\mathcal{S}^{Q-1}$-negligible set N and a sequence of sets $S_j = \{p \in U_j, f_j(p) = 0\}$, $j \in \mathbb{N}$ with U open and bounded in $\mathbb{H}^n$ and $f_j \in C^1_H$ such that $\mathcal{S}^{Q-1}(S_j) < \infty$ and

$$R \subset N \cup \bigcup_{j \in \mathbb{N}} S_j.$$

This is actually the characterization we will use in the proof of our variation formulas by contact diffeomorphisms in the general case.

We can define $\mathcal{S}^{Q-1}$ a.e. a horizontal normal for a H-rectifiable set R in the following way: let $p \in R \cap \cup_j S_j$, then we define the horizontal normal to R at p as

$$\nu_R(p) = \nu_{S_{\tilde{j}}}(p) \tag{2.46}$$

where $\tilde{j}$ is the unique integer such that $p \in S_{\tilde{j}} \setminus \cup_{j<\tilde{j}} S_j$.

Such a notion is well-defined up to a sign: namely if $\{S^1_j\}_{j\in\mathbb{N}}$ and $\{S^2_j\}_{j\in\mathbb{N}}$ are two sequences of H-regular hypersurfaces such that

$$\mathcal{S}^{Q-1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S^1_j\Big) = 0, \quad \mathcal{S}^{Q-1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S^2_j\Big) = 0,$$

then,

$$\nu^1_R(p) = \pm \nu^2_R(p)$$

for $\mathcal{S}^{Q-1}$ almost every point of R, where ν^1_R and ν^2_R are defined as in (2.46) respectively by means of $\{S^1_j\}_{j\in\mathbb{N}}$ and $\{S^2_j\}_{j\in\mathbb{N}}$. The proof of this fact can be found in [13].

We now state the theorem making H-rectifiability so important. The proof of the following is in [9].

Theorem 2.6. *Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter. Then its reduced boundary $\partial^* E$ is H-rectifiable.*

We finally state and prove the main theorem of the thesis. The proof of such a theorem, up to the first variation, was carried out and never published by R. Monti and D. Vittone. The following is an original result due to the author, along with the preparatory Theorem 2.3.

Theorem 2.7. *Let $R \subset \mathbb{H}^n$ be H-rectifiable and bounded, and let A be a bounded open set containing R. Let ν_R be the horizontal normal to ν_R. Let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(\Psi, A)$, be the contact flow generated by $\psi \in C^\infty_c(\mathbb{H}^n)$. Then there exists a constant $C = C(\psi, A)$ such that*

$$\begin{aligned} &\Big|\mathcal{S}^{Q-1}(\Psi_s(R)) - \mathcal{S}^{Q-1}(R) + s\int_R \mathscr{A}_\psi(\nu_R)\, d\mathcal{S}^{Q-1} \\ &- s^2 \int_R \Big(\mathscr{S}_\psi(\nu_R) - \big(\mathscr{A}_\psi(\nu_R)\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big) d\mathcal{S}^{Q-1}\Big| \le C\mathcal{S}^{Q-1}(R)s^3, \end{aligned} \tag{2.47}$$

where $\mathrm{div}(JV_\psi V_\psi)$ *can be computed as in* (2.38) *and where* $\mathscr{S}_\psi$ *and* $\mathscr{A}_\psi$ *are defined respectively in* (2.35) *and* (2.36).

In particular if E *is a set with finite* H*-perimeter in* A, *with horizontal normal* ν_E, *there exists* $C = C(\psi, A)$ *such that*

$$\Big| P(\Psi_s(E), \Psi_s(A)) - P(E, A) + s \int_A \mathscr{A}_\psi(\nu_E) d\mu_E \\ - s^2 \int_A \Big(\mathscr{S}_\psi(\nu_E) - \Big(\mathscr{A}_\psi(\nu_E)\Big)^2 + 32\Big((n+1)T\psi\Big)^2 + \mathrm{div}(JV_\psi V_\psi)\Big) d\mu_E \Big| \le C P(E, A) s^3. \tag{2.48}$$

Before proceeding with the proof, we list some results needed in the proof.

The following are two technical lemmas, trivial adaptations of results and proofs contained in [14].

Lemma 2.4. *Let* S *be a* H*-regular hypersurface such that* $S = \{p \in U : f(p) = 0\}$ *with* $U \subset \mathbb{H}^n$ *open and bounded and* $f \in C^1_H(U)$ *such that* $\nabla_H f(p) \neq 0$ *for any* p *in* S.

Then there exist $\tilde{U} \subset \mathbb{H}^n$ *open and bounded,* $\tilde{f} \in C^1_H(\tilde{U})$ *such that* $S = \{q \in \tilde{U} : \tilde{f}(q) = 0\}$, $\nabla_H \tilde{f}(q) \neq 0$ *for any* q *in* S *and*

$$\tilde{f} \in C^\infty(\tilde{U} \setminus S).$$

Proof. It is an adaptation of Lemma 4.4 in [14]. □

Lemma 2.5. *Let* $A \in \mathbb{H}^n$ *be an open set, and let* $C_b(A)$ *the space of continuous and bounded functions on* A. *Let* $\{E_j\}_{j\in\mathbb{N}}$ *be a sequence of sets with finite* H*-perimeter in* A *such that* $\chi_{E_j} \to \chi_E$ *in* $L^1(A)$ *and* $\mu_{E_j}(A) \to \mu_E(A)$. *Let* $F : \mathbb{R}^{2n} \to \mathbb{R}$ *be a continuous* 1*-homogeneous function. Then* $F(\nu_{E_j}) d\mu_{E_j} \rightharpoonup F(\nu_E) d\mu_E$ *in the weak* convergence of measures, that is*

$$\lim_{j\to\infty} \int_A \rho F(\nu_{E_j}) d\mu_{E_j} = \int_A \rho F(\nu_E) d\mu_E \quad \text{for any } \rho \in C_b(A).$$

Proof. It is a special case of Lemma 2.5 in [14] and of its proof. □

The following is an useful continuity theorem of Reshetnyak, adapted to best suit our setting.

Theorem 2.8 (Reshetnyak continuity theorem)**.** *Let* $A \in \mathbb{H}^n$ *be an open set. Let* $\{E_j\}_{j\in\mathbb{N}}$ *be a sequence of sets with finite* H*-perimeter in* A *such that* $\mu_{E_j}(A) \to \mu_E(A)$. *Suppose moreover that* $\nu_{E_j}\mu_{E_j} \rightharpoonup \nu_E \mu_E$ *weakly*. Then*

$$\lim_{j\to\infty} \int_A f(\nu_{E_j}) d\mu_{E_j} = \int_A f(\nu_E) d\mu_E$$

for any $f \in C^0(\mathbb{S}^{2n-1})$.

Proof. Theorem 2.8 follows from the general version of Reshetnyak continuity theorem (see e.g. [2], Theorem 2.39). □

We are ready to prove Theorem 2.7.

Proof of Theorem 2.7. We start by two preliminary remarks:

(*i*) Recall that ν_R, the horizontal normal to R, exists $\mathcal{S}^{Q-1}$-almost everywhere up to a sign. However, both $\mathscr{Q}_\psi$ and $\mathscr{S}_\psi$, appearing in formula (2.47) are quadratic forms: it implies that the integrals in such formula are well defined.

(*ii*) We assume with no loss of generality that all the open sets U, U_s, C_j, U_j, V_k appearing later in the proof are all contained in A: in this way we can let $\delta = \delta(\psi, A)$ and $C = C(\psi, A)$ uniform along the proof.

The proof is organized as follows: In *Step 1* we prove formula (2.47) for a H-regular surface S as the one in Lemma 2.4; in *Step 2* we generalize such formula for arbitrary subsets S, and finally in *Step 3* to a H-rectifiable set R. At the end, we will deduce formula (2.48).

Step 1. Let S be a H-regular surface with the following properties: $\mathcal{S}^{Q-1}(S) < \infty$, $S = \{p \in U : f(p) = 0\}$ for a suitable open and bounded set $U \in \mathbb{H}^n$ and

$$f \in C^1_H(U), \quad \nabla_H f \neq 0 \quad \text{in } U. \tag{2.49}$$

By Lemma 2.4, we can assume $f \in C^\infty(U \setminus S)$.

Let now, for $s \in [-\delta, \delta]$,

$$U_s := \Psi_s(U), \qquad f_s = f \circ \Psi_s, \qquad S_s = \{q \in U_s : f_s(q) = 0\}$$

We claim that f_s enjoys the same properties of f with U_s in place of U. Clearly, U_s is an open and bounded subset of $\mathbb{H}^n$ and $f_s \in C^\infty(U_s \setminus S_s)$. Moreover, by the contact structure of $(\Psi_s)^{-1}$, we have, for $j = 1, \dots 2n$,

$$X_j(f_s) = \langle \nabla f, J(\Psi_s)^{-1} X_j \rangle = \sum_{i=1}^n g_{ij} X_i(f) + g_{(n+i)j} Y_i(f)$$
$$Y_j(f_s) = \langle \nabla f, J(\Psi_s)^{-1} X_j \rangle = \sum_{i=1}^n g_{(n+i)j} X_i(f) + g_{(n+i)(n+j)} Y_i(f)$$

for suitable smooth functions g_{ij}, $i, j = 1, \dots, 2n$. Thus, since (2.49) holds, we deduce our claim, and, in particular, S_s is a H-regular surface.

Define now, for $r \in \mathbb{R}$

$$E^r := \{p \in U : f(p) < r\}, \quad E := E^0$$
$$E^r_s := \{q \in U_s : f_s(q) < r\}, \quad E_s = E^0_s.$$

Notice that $\partial E = S$, and $\partial E_s = S_s$. By Sard's theorem, ∂E^r and ∂E^r_s are smooth hypersurfaces for almost every $r \in \mathbb{R}$; let then, for $s \in [-\delta, \delta]$ fixed, $\{r_j\}_{j\in\mathbb{N}}$ be an infinitesimal real sequence such that E^{r_j} and $E^{r_j}_s$ are smooth hypesurfaces. Since we are going to deal with local convergences, let now $K \subset U$, and $K_s := \Psi_s(K)$. Thus, by Theorem (2.37), we have

$$\begin{aligned}
&\Big| P(E^{r_j}_s, K_s) - P(E^{r_j}, K) + s\int_K \mathscr{A}_\psi(\nu_{E_{r_j}})d\mu_{E_{r_j}} \\
&- s^2\int_K \Big(\mathscr{S}_\psi(\nu_{E_{r_j}}) - \big(\mathscr{A}_\psi(\nu_{E_{r_j}})\big)^2 + 32\big((n+1)T\psi\big)^2 + \mathrm{div}(JV_\psi V_\psi)\Big)d\mu_{E_{r_j}}\Big| \\
&\le CP(E^{r_j}, K)s^3.
\end{aligned} \tag{2.50}$$

It is easy to see that

$$E^r \to E \quad \text{in } L^1_{\mathrm{loc}}(U). \tag{2.51}$$

We claim that the following convergences hold too:

$$P(E^r, K) \to P(E, K), \qquad P(E^r_s, K_s) \to P(E_s, K_s). \tag{2.52}$$

We sketch a proof of the first of the (2.52): the second is obtained in a completely analogous way. Since $|\nabla_H f|$ is non vanishing in K, we can assume, up to a rotation, that $X_1(f) \ge \theta > 0$. By an implicit function theorem in Heisenberg groups, that can be found in [9], there exists a compact set $I \Subset \mathbb{R}^{2n}$ such that, for r_j in a neighbourhood of 0, there exists a continuous function $\Phi^{r_j} : I \to \mathbb{H}^n$ such that

$$P(E^{r_j}, K) = \int_I \frac{|\nabla_H f|}{X_1 f}(\Phi^{r_j}(a))d\mathcal{L}^{2n}(a).$$

By continuity of Φ^{r_j}, $\nabla_H f$ and $X_1 f$, one can prove that

$$\lim_{j\to\infty} \frac{|\nabla_H f|}{X_1 f}(\Phi^{r_j}(a)) = \frac{|\nabla_H f|}{X_1 f}(\Phi^0(a))$$

and this, by exchanging integral and limit, proves our claim.

In particular, Lemma 2.5 applies and we get the weak* convergences

$$\nu_{E^{r_j}}\mu_{E_{r_j}} \rightharpoonup \nu_{E^r}\mu_{E_r}.$$

By Reshetnyak continuity theorem 2.8, and Theorem 2.5, we get

$$\lim_{j\to\infty}\int_K \mathscr{A}_\psi(\nu_{E_{r_j}})d\mu_{E_{r_j}} = \int_K \mathscr{A}_\psi(\nu_E)d\mu_E = c_n\int_{S\cap K} \mathscr{A}_\psi(\nu_S)d\mathcal{S}^{Q-1} \tag{2.53}$$

and

$$\begin{aligned}\lim_{j\to\infty}\int_K &\Big(\mathscr{S}_\psi(\nu_{E_{r_j}})-\Big(\mathscr{A}_\psi(\nu_{E_{r_j}})\Big)^2+32\Big((n+1)T\psi\Big)^2+\operatorname{div}(JV_\psi V_\psi)\Big)d\mu_{E_{r_j}}\\ =&\int_K \Big(\mathscr{S}_\psi(\nu_E)-\Big(\mathscr{A}_\psi(\nu_E)\Big)^2+32\Big((n+1)T\psi\Big)^2+\operatorname{div}(JV_\psi V_\psi)\Big)d\mu_E\\ =&c_n\int_{S\cap K}\Big(\mathscr{S}_\psi(\nu_S)-\Big(\mathscr{A}_\psi(\nu_S)\Big)^2+32\Big((n+1)T\psi\Big)^2+\operatorname{div}(JV_\psi V_\psi)\Big)d\mathcal{S}^{Q-1}\end{aligned}\tag{2.54}$$

Now, we let $r_j\to 0$ in (2.50): by (2.51), (2.52), (2.53) and (2.54) we obtain that there exists a constant $C=C(\psi,A)$ such that

$$\begin{aligned}&\Big|\mathcal{S}^{Q-1}(\Psi_s(S\cap K))-\mathcal{S}^{Q-1}(S\cap K)+s\int_{S\cap K}\mathscr{A}_\psi(\nu_S)d\mathcal{S}^{Q-1}\\ &-s^2\int_{S\cap K}\Big(\mathscr{S}_\psi(\nu_S)-\Big(\mathscr{A}_\psi(\nu_S)\Big)^2+32\Big((n+1)T\psi\Big)^2+\operatorname{div}(JV_\psi V_\psi)\Big)d\mathcal{S}^{Q-1}\Big|\\ &\le C\mathcal{S}^{Q-1}(S\cap K)s^3,\end{aligned}\tag{2.55}$$

Finally, let $\{K_j\}_{j\in\mathbb{N}}\subset U$ be a sequence of compact sets invading U. Applying formula (2.55) to $S\cap K_j$ and passing to the limit as $j\to\infty$ by monotone convergence theorem completes *Step 1.*

Step 2. Let now $\Sigma\subset S$, with S a H-regular surface such that there exists $U\in A$ such that $\mathcal{S}^{Q-1}(S_j)<\infty$, $S=\{p\in U: f(p)=0\}$ and f satisfies (2.49). Since $\mathcal{S}^{Q-1}$ is a Radon measure, we have

$$\mathcal{S}^{Q-1}(\Sigma)=\inf\{\mathcal{S}^{Q-1}(S\cap O, O\subset\mathbb{H}^n\text{open},\Sigma\subset O\},$$

and then, we fix a sequence of open sets O_j such that $\Sigma\subset O_j$ and

$$\lim_{j\to\infty}\mathcal{S}^{Q-1}(S\cap O_j)=\mathcal{S}^{Q-1}(\Sigma).$$

Analogously, for $s\in[-\delta,\delta]$ fixed, we let B_j such that $\Psi_s(\Sigma)\subset B_j$ and

$$\lim_{j\to\infty}\mathcal{S}^{Q-1}(\Psi_s(S)\cap B_j)=\mathcal{S}^{Q-1}(\Psi_s(\Sigma)).$$

Set then $C_j=O_j\cap\Psi_s^{-1}(B_j)$; we have

$$\Sigma\subset C_j\subset O_j,\quad \Psi_s(\Sigma)\subset\Psi_s(C_j)\subset B_j$$

and thus

$$\lim_{j\to\infty}\mathcal{S}^{Q-1}(S\cap C_j)=\mathcal{S}^{Q-1}(\Sigma),\qquad \lim_{j\to\infty}\mathcal{S}^{Q-1}(\Psi_s(S\cap C_j))=\mathcal{S}^{Q-1}(\Psi_s(\Sigma)).\tag{2.56}$$

Set $S_j = S \cap C_j$. We have that $S_j = \{p \in U \cap C_j : f(p) = 0\}$, and thus, by *Step 1*,

$$\begin{aligned}&\Big|\mathcal{S}^{Q-1}(\Psi_s(S_j)) - \mathcal{S}^{Q-1}(S_j) + s\int_{S_j} \mathscr{A}_\psi(\nu_{S_j})d\mathcal{S}^{Q-1}\\&- s^2\int_{S_j}\Big(\mathscr{S}_\psi(\nu_{S_j}) - \big(\mathscr{A}_\psi(\nu_{S_j})\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big)d\mathcal{S}^{Q-1}\Big|\\&\le C\mathcal{S}^{Q-1}(S_j)s^3,\end{aligned} \tag{2.57}$$

By dominated convergence theorem,

$$\begin{aligned}\lim_{j\to\infty}\int_{S_j}\mathscr{A}_\psi(\nu_{S_j})d\mathcal{S}^{Q-1} &= \lim_{j\to\infty}\int_S \chi_{S\cap C_j}\mathscr{A}_\psi(\nu_S)d\mathcal{S}^{Q-1}\\&= \int_S \chi_\Sigma \mathscr{A}_\psi(\nu_S)d\mathcal{S}^{Q-1}\\&= \int_\Sigma \mathscr{A}_\psi(\nu_\Sigma)\mathscr{A}_\psi(\nu_\Sigma)d\mathcal{S}^{Q-1},\end{aligned} \tag{2.58}$$

and, in a completely analogous way,

$$\begin{aligned}&\lim_{j\to\infty}\int_{S_j}\Big(\mathscr{S}_\psi(\nu_{S_j}) - \big(\mathscr{A}_\psi(\nu_{S_j})\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big)d\mathcal{S}^{Q-1}\\&= \int_\Sigma\Big(\mathscr{S}_\psi(\nu_\Sigma) - \big(\mathscr{A}_\psi(\nu_\Sigma)\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big)d\mathcal{S}^{Q-1}.\end{aligned} \tag{2.59}$$

Step 2 is achieved by taking into account (2.56), (2.58) and (2.59) in passing to the limit as $j \to \infty$ in (2.57).

Step 3. Finally, we prove formula (2.47) in its generality. Let then R be a H-rectifiable set, that is,

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j,$$

with $\mathcal{S}^{Q-1}(N) = 0$, $\mathcal{S}^{Q-1}(S_j) < \infty$ and $S_j = \{p \in U_j : f_j(p) = 0\}$ for suitable $U_j \in \mathbb{H}^n$ open and bounded and f_j satisfying (2.49). We can clearly assume

$$N \cup \bigcup_{j=1}^{\infty} S_j \subset A.$$

Up to replacing S_j with $S_j \setminus \bigcup_{i=1}^{j-1} S_i$, we can assume that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. However, there might exist sequences of points points $\{p_k\}_{k\in\mathbb{N}} \in S_i$ and $\{q_k\}_{k\in\mathbb{N}} \in S_j$ such that $d(p_k, q_k) \to 0$ as $k \to \infty$, where d is any metric: that, in general, prevents us to define $S_i \cup S_j$ as the level set of a function satisfying (2.49). We bypass such an issue in the following standard way. Let, for $k \geq 1$,

$$S_j^k := \left\{ p \in \Sigma_j : d(p, \cup_{i=1}^{j-1} S_i) > \frac{1}{k} \right\} \subset S_j,$$

and let $S^k = \cup_{j=1}^{\infty} S_j^k$: since the sets S_j^k, $j \in \mathbb{N}$ are at positive distance, it is clear that we can find for each k open and bounded sets V_k and functions g_k satisfying (2.49) such that $S^k = \{p \in V_k : g_k(p) = 0\}$. Since we are assuming that each S_j is contained in the bounded set A, and for each $i \neq j$ we have $d(S_i, S_j) \geq 1/k$, we deduce that $S_j^k \neq \emptyset$ for only finitely many indexes j: it implies that $\mathcal{S}^{Q-1}(S^k) < \infty$. Thus, we can apply *Step 2* to the set $R \cap S^k$ and obtain

$$\begin{aligned} &\Big| \mathcal{S}^{Q-1}(\Psi_s(R \cap S^k)) - \mathcal{S}^{Q-1}(R \cap S^k) + s \int_{R \cap S^k} \mathscr{A}_\psi(\nu_R) d\mathcal{S}^{Q-1} \\ &- s^2 \int_{R \cap S^k} \Big(\mathscr{S}_\psi(\nu_R) - \Big(\mathscr{A}_\psi(\nu_R)\Big)^2 + 32\Big((n+1)T\psi\Big)^2 + \operatorname{div}(JV_\psi V_\psi) \Big) d\mathcal{S}^{Q-1} \Big| \\ &\leq C \mathcal{S}^{Q-1}(R \cap S^k) s^3. \end{aligned} \tag{2.60}$$

Letting $S^\infty := \cup_{j=1}^{\infty} S_j$, we clearly have $S^k \nearrow S^\infty$ as $k \to \infty$. Thus, passing to the limit in (2.60), by monotone convergence theorem, we get

$$\begin{aligned} &\Big| \mathcal{S}^{Q-1}(\Psi_s(R \cap S^\infty)) - \mathcal{S}^{Q-1}(R \cap S^\infty) + s \int_{R \cap S^\infty} \mathscr{A}_\psi(\nu_R) d\mathcal{S}^{Q-1} \\ &- s^2 \int_{R \cap S^\infty} \Big(\mathscr{S}_\psi(\nu_R) - \Big(\mathscr{A}_\psi(\nu_R)\Big)^2 + 32\Big((n+1)T\psi\Big)^2 + \operatorname{div}(JV_\psi V_\psi) \Big) d\mathcal{S}^{Q-1} \Big| \\ &\leq C \mathcal{S}^{Q-1}(R \cap S^\infty) s^3. \end{aligned} \tag{2.61}$$

Finally, by H-rectifiability of R, we have $R \setminus S^\infty \subset N$ and thus

$$\mathcal{S}^{Q-1}(R \cap S^\infty) = \mathcal{S}^{Q-1}(R). \tag{2.62}$$

Moreover, by H-Lipschitz continuity of Ψ_s, and Proposition 2.3,

$$\mathcal{S}^{Q-1}(\Psi_s(R \setminus S^\infty)) \leq \mathcal{S}^{Q-1}(\Psi_s(N)) \leq L_s^{Q-1} \mathcal{S}^{Q-1}(N) = 0$$

where L_s is the Lipschitz constant of Ψ_s. It follows that

$$\mathcal{S}^{Q-1}(\Psi_s(R \cap S^\infty) = \mathcal{S}^{Q-1}(\Psi_s(R)). \tag{2.63}$$

Taking into account (2.62) and (2.63) in (2.61) completes the proof of (2.47).

We are left to prove formula (2.48). Let E be a set with finite H-perimeter in A, and consider its measure theoretic boundary defined in Definition 2.4:

$$\partial_* E = \{p \in \mathbb{H}^n : |E \cap B(p,r)| > 0 \text{ and } |E \setminus B(p,r) > 0|\}.$$

By Lemma 2.3, the reduced boundary of Definition 2.3 satisfies $\partial^* E \subset \partial_* E$ and $\mathcal{S}^{Q-1}(\partial_* E \setminus \partial^* E) = 0$. In particular, by Theorem 2.6 $\partial_* E$ is H-rectifiable and $\mu_E \llcorner A = c_n \mathcal{S}^{Q-1} \llcorner \partial_* E$, by Theorem 2.5. By H-Lipschitzianity of Ψ_s, it is clear that $\Psi_s(\partial_* E) = \partial_*(\Psi_s(E))$. Formula (2.48) finally follows applying formula (2.47) to $R = \partial_* E \cap A$. □

Theorem 2.7 yields variation formulas for any H-rectifiable set, involving in this way also highly non-regular sets, even with fractional Hausdorff dimension, see [10], Theorem 3.1. When a H-rectifiable set R is perimeter minimizing, or better $\mathcal{S}^{Q-1}$-minimizing, we have thus the following necessary conditions, holding for any test function $\psi \in C_c^\infty(R)$:

$$\int_R \mathscr{A}_\psi(\nu_R) d\mathcal{S}^{Q-1} = 0$$

$$\int_R \Big(\mathscr{S}_\psi(\nu_R) - \big(\mathscr{A}_\psi(\nu_R)\big)^2 + 32\big((n+1)T\psi\big)^2 + \operatorname{div}(JV_\psi V_\psi)\Big) d\mathcal{S}^{Q-1} \geq 0.$$

All the observations about the integrability of such expressions hold unchanged in the general case too.

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