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Minimizing Clusters: existence and planar examples

Candidato:
Nicola Girotto
Matricola 1180030

Relatore:
Roberto Monti

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Introduction

The problem of enclosing a fixed area inside a figure in the plane with least perimeter was known since the times of ancient Greeks. They knew that the optimal solution was a circle, although they did not prove this fact precisely but just by approximation. Surprisingly, the first rigorous proof was found only in the 19^{th} century. First Steiner showed that, if a solution exists, then it is necessarily a ball and, some years later, Carathéodory completed the proof showing the existence of the minimizers. We could generalize this problem, for example, trying to find two sets of fixed areas which minimize the perimeter of their boundary. In general, this problem could be set with N subsets of $\mathbb{R}^n$. This is called *partitioning problem.*

In this thesis we are going to study exactly this problem. Namely, we define an N-cluster in $\mathbb{R}^n$ as a collection of N sets of finite perimeter and with finite and non null Lebesgue measure. Moreover, these sets have to intersect pairwise in null measure sets. Given a cluster $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$, we define its perimeter as the $\mathcal{H}^{n-1}$-measure of

$$\bigcup_{i=1}^{N} \partial^* \mathcal{E}_i$$

and we denote it by $P(\mathcal{E})$. Given $\mathbf{m} \in \mathbb{R}^N_+$, we want to find, among all the N-clusters of $\mathbb{R}^n$ with $|\mathcal{E}_i| = \mathbf{m}_i$, the one which minimizes the perimeter. In other words, we have to determine

$$\inf \big\{ P(\mathcal{E}) : \mathcal{E}\ N\text{-cluster},\ \mathbf{m}(\mathcal{E}) = \mathbf{m} \big\}$$

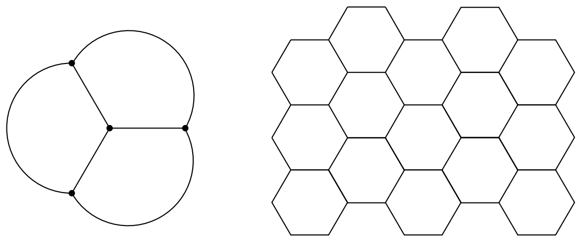
As we have already told, it is well known that the case $N = 1$ admits as unique minimizer the n-dimensional ball. Moreover, notice that it has very good geometric regularity properties. For the case $N \geq 2$, the first question to face is whether this problem admits a solution, that is if there exists (at least) a N-cluster of $\mathbb{R}^n$ which realizes the infimum. A cluster of this kind is called *minimizer*. The other aspect to study is if this minimizer has some relevant regularity properties.

We will proceed in this way. The first chapter will be entirely dedicated to the proof of the existence of the minimizers. We followed and detailed the version of [6]. The basic idea of the proof is to consider a minimizing sequence, i.e. which perimeter converges to the infimum value, and show that, up to extracting a subsequence, it converges to a certain admissible cluster. We'll start presenting in detail the problem, defining precisely a cluster, its perimeter and the partitioning problem. Immediately after we will deduce the basic properties of these quantities. The first important step for the proof is the compactness criterion stated in Section 1.2. In fact, under some suitable assumptions on the the minimizing sequence, this criterion will allow us to extract a converging subsequence. The two statements in Section 1.3 represent the second important step. Indeed, we will see that we can locally modify our cluster through a diffeomorphism around some interface points with a precise estimation on the volumes of the transformed chambers and on the perimeter of a generic $\mathcal{H}^{n-1}$-rectifiable set. In particular, if we modify our cluster changing also the volumes of the chambers, thanks to that theorem we can restore the original measures.

In the second chapter, instead, we are going to study the regularity of minimizers, starting from the general case and then analysing some particular ones. First of all, we will prove an important theorem for N-clusters in $\mathbb{R}^n$. We will discover that minimizers have constant mean curvature hypersurfaces as boundaries. Then we are going to focus just on planar examples. For these kind of clusters we will see that they are characterized by having a finite number of arcs or line segments in their boundaries and they satisfy the 120° rule. This says that the boundary arcs meet in threes at a finite number of points forming 120° angles. These facts were first proved by [7] in 1994. In this thesis, we detail that proof. Exploiting these informations for the case of $N = 2$, we can entirely characterize the 2-minimizer clusters as standard double bubbles, that is clusters formed by two connected chambers and three arcs meeting at two vertices. In fact, in this case it is possible to show that the chambers and the exterior have to be connected. This is a key point since it is not obvious for other kind of problems (see [2]). Lastly, we will consider the case of $N = 4$, recently developed in [9], [10]. We will prove that the optimal cluster with chambers of the same area admits a very curious configuration. It is formed by two quadrangular regions and two triangular ones. The firsts have a line segment in common and are adjacent to both the triangular regions. Moreover the entire cluster is symmetric w.r.t. both the line segment above and its axis.

Although we are going to deal only with the these matters, other very important results about the characterization of some minimizing clusters have been proved in the last twenty years. For example, in 2002 Wichiramala proved that the standard triple bubble is the unique 3-minimizing cluster in the plane (see [12]). This cluster is formed by three connected regions and its boundary is composed by six circular arcs, joining in four points with the 120° rule (see Figure 1a).

A different way of looking at the problem is given by Wichiramala in [13] with the so called *weak approach*. As we have already told, one of the



(a) (b)

Figure 1: The standard triple bubble with three equal areas in figure **(a)**. The *honeycomb* formed by the hexagon tile in figure **(b)**.

biggest obstacles is proving that every chamber is connected. With the weak approach, we consider also clusters with chambers of areas greater than the correct ones. In this way, for example, we can easily reduce to the case of an exterior connected. In fact we can incorporate the bounded connected components of the exterior inside other chambers. Then, the area of some bubble increases, the cluster remains admissible thanks to the weak assumption, but its perimeter decreases.

Some results were found also in the three-dimensional space. One of the most important is the proof of the double bubble conjecture. It states that the standard double bubble in $\mathbb{R}^3$, formed by three spherical surfaces meeting at angles of 120° along a common circle, is the optimal 2-cluster. A proof of this conjecture was given in [5] in 2002.

Finally also the case $N = \infty$ has been studied. This is the so called *Honeycomb conjecture* [3] and it affirms that, in a certain sense, the honeycomb represent the way of enclosing infinitely many (equal) areas with the least perimeter (see Figure 1b). Precisely, let $\mathcal{T}$ be a network in $\mathbb{R}^2$ such that $T = \mathbb{R}^2 \setminus \mathcal{T}$ has infinitely many connected components with the same area 1. Then

$$\limsup_{r\to 0^+} \frac{P(T \cap B_r)}{\text{area}(T \cap B_r)} \geq \sqrt[4]{12}$$

The equality is attained exactly for the regular hexagonal tile.

Notation

With the following list we want to fix some notation that we are going to use in this thesis.

$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$.
$\|E\|$	$\mathcal{L}^n$-measure of the set $E \subseteq \mathbb{R}^n$.
ω_n	$\mathcal{L}^n$-measure of a ball with unitary radius.
$\mathcal{H}^{n-1}$	$(n-1)$-Hausdorff measure on $\mathbb{R}^n$.
$\#I, \mathcal{H}^0(I)$	cardinality of the set I.
$\approx$	equivalence of two $(n-1)$-dimensional sets. We say that E, F are $\mathcal{H}^{n-1}$-equivalent whenever $\mathcal{H}^{n-1}(E \Delta F) = 0$, that is if they differ on a set of null $\mathcal{H}^{n-1}$-measure.
$\theta_s(E)(x)$	s-dimensional density of the set $E \subseteq \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$, i.e. $\theta_s(E)(x) = \lim_{r \to 0^+} \dfrac{\mathcal{H}^s(E \cap B(x,r))}{\omega_s r^s}$.
$E^{(t)}$	set of points of $\mathbb{R}^n$ with density w.r.t. E equal to t.
$\partial^* E$	reduced boundary of the set of finite perimeter E.
$\nu_E(x)$	outer unit normal vector to E at x.
$P(E;F)$	perimeter of the set E of finite perimeter inside F.
$\subset\subset$	compactly contained.
C^k_c	set of the C^k function with compact support.
B_r	n-dimensional ball of radius r centered in the origin.

CHAPTER 1

Existence of the minimum

The aim of this chapter is to start explaining in detail the theory of the minimizing cluster. First we'll define an N-cluster, its perimeter and what we mean by a partitioning problem and a minimizing cluster. In particular, we will devote most of the chapter to the long proof of the existence of the minimum in a partitioning problem.

1.1 Partitioning problem and basic properties

An N-*cluster* $\mathcal{E}$ in $\mathbb{R}^n$ is a collection $\{\mathcal{E}(h)\}_{h=1}^N$ of sets in $\mathbb{R}^n$ of finite perimeter with $N \in \mathbb{N}, N \geq 1$ and

$$\begin{aligned} 0 < \left|\mathcal{E}(h)\right| < +\infty, \qquad & h = 1, \ldots, N \\ \left|\mathcal{E}(h) \cap \mathcal{E}(k)\right| = 0, \qquad & h, k = 1, \ldots, N,\ h < k \end{aligned}$$

Thus if $h \neq k$, $\mathcal{E}(h), \mathcal{E}(k)$ may intersect in a non-empty set but its $\mathcal{L}^n$-measure is null.

We call the sets $\mathcal{E}(1), \ldots, \mathcal{E}(N)$ chambers of the cluster $\mathcal{E}$. We define also the exterior chamber $\mathcal{E}(0)$ as

$$\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^{N} \mathcal{E}(h)$$

In this way, $\{\mathcal{E}(h)\}_{h=0}^N$ is a partition of $\mathbb{R}^n$ up to a set of Lebesgue measure null; we notice that $\left|\mathcal{E}(0)\right| = \infty$. By convenience, we set

$$\mathbf{m}(\mathcal{E}) = (\left|\mathcal{E}(h)\right|)_{h=1}^N \in \mathbb{R}_+^N$$

and we call it measure vector, or volume vector. Its entries are exactly the volumes of the chambers of $\mathcal{E}$. Clearly it belongs to $\mathbb{R}_+^N$ because, by definition of N-cluster, $\left|\mathcal{E}(h)\right| > 0$ for every $h = 1, \ldots, N$.

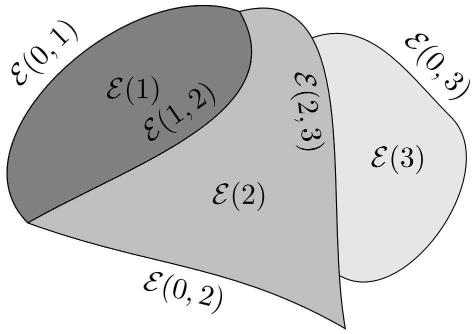


Figure 1.1: In figure it is represented an example of a 3-cluster in the plane.

Now we define the interfaces as the sets given by the intersection of the reduced boundaries of two different chambers, namely

$$\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k), \qquad h,k = 0,\dots,N,\ h \neq k$$

We notice that the interfaces are $\mathcal{H}^{n-1}$-rectifiable sets because, by De Giorgi's structure theorem, we know that the reduced boundary of any set of finite perimeter is of that kind.

Now we are ready to define the perimeter of a cluster. The perimeter of $\mathcal{E}$ in $F \subseteq \mathbb{R}^n$ is

$$P(\mathcal{E};F) = \sum_{0\le h<k\le N} \mathcal{H}^{n-1}\left(\mathcal{E}(h,k)\cap F\right)$$

and its perimeter is

$$P(\mathcal{E}) = P(\mathcal{E};\mathbb{R}^n) = \sum_{0\le h<k\le N} \mathcal{H}^{n-1}\left(\mathcal{E}(h,k)\right)$$

Now let's explain what a partitioning problem and a minimizing cluster are. Given $\mathbf{m} \in \mathbb{R}^N_+$, the *partitioning problem* in $\mathbb{R}^n$ associated to $\mathbf{m}$ is finding a cluster with prescribed chambers volumes and which minimizes the perimeter. Namely, we want to determine

$$\inf\{P(\mathcal{E}) : \mathbf{m}(\mathcal{E}) = \mathbf{m}\} \tag{1.1}$$

where $\mathcal{E}$ is a N-cluster in $\mathbb{R}^n$ with $\mathrm{spt}\mu_{\mathcal{E}(h)} = \partial\mathcal{E}(h)$ for every $h = 1,\dots,N$. If $\mathcal{E}$ is a cluster which perimeter realizes the infimum, we say that $\mathcal{E}$ is a minimizing, or minimal, cluster for the problem defined by $\mathbf{m}$.

This first chapter will be entirely devoted to the proof of the following theorem, which states the existence of the minimizers.

THEOREM 1.1. *Given non-negative integers $n, N \ge 2$ and $\mathbf{m} \in \mathbb{R}^N_+$, there exist minimizing N-clusters in $\mathbb{R}^n$ for the partitioning problem associated* *to* $\mathbf{m}$*, that is the problem* (1.1) *admits minimum. Moreover, if* $\mathcal{E}$ *is such a minimizer,* $\mathcal{E}$ *is bounded, i.e. there exists* $R > 0$ *such that*

$$\mathcal{E}(h) \subseteq B_R, \qquad h = 1, \dots, N$$

As usual, we would like to define a convergence of clusters. In order to do this, we define the "distance" of two clusters. Given N-clusters $\mathcal{E}, \mathcal{E}'$, their distance in $F \subseteq \mathbb{R}^n$ is

$$d_F(\mathcal{E}, \mathcal{E}') = \sum_{h=1}^{N} \left| F \cap \big(\mathcal{E}(h) \Delta \mathcal{E}'(h)\big) \right|$$

and their (simple) distance is

$$d(\mathcal{E}, \mathcal{E}') = d_{\mathbb{R}^n}(\mathcal{E}, \mathcal{E}') = \sum_{h=1}^{N} \left| \mathcal{E}(h) \Delta \mathcal{E}'(h) \right|$$

It is easily seen that d is not a distance in the usual sense. Indeed, if $\mathcal{E}(h)$ and $\mathcal{E}'(h)$ differ by a null $\mathcal{L}^n$ measure set for some $h = 1, \dots, N$, their distance is zero although they are not equal. Using these definitions of distance, we say that a sequence of N-clusters $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}^n$ *locally converges* to $\mathcal{E}$, and we write $\mathcal{E}_k \xrightarrow{\text{loc}} \mathcal{E}$, if for every compact set $K \subseteq \mathbb{R}^n$, $d_K(\mathcal{E}_k, \mathcal{E}) \to 0$ as $k \to \infty$. We simply say that $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ *converges* to $\mathcal{E}$, and we write $\mathcal{E}_k \to \mathcal{E}$, if $d(\mathcal{E}_k, \mathcal{E}) \to 0$ as $k \to \infty$. This means that there is convergence whenever the measure of the sets difference goes to zero. Thus we can say that this is a convergence in a measure sense.

The following proposition is interesting for two reasons. The first is that it tells us how we can express the perimeter of a cluster through the perimeters of its chambers. In this way, we can use all the known properties of the perimeter of the sets of finite perimeter. The second is that it provides the lower semicontinuity of the relative perimeter of a cluster into an open set. This is crucial for the existence of the minimizer.

PROPOSITION 1.2. *Given an* N*-cluster* $\mathcal{E}$ *in* $\mathbb{R}^n$ *and a subset* $F \subseteq \mathbb{R}^n$ *it holds*

$$P(\mathcal{E}; F) = \frac{1}{2} \sum_{h=0}^{N} P(\mathcal{E}(h); F) \tag{1.2}$$

Moreover if A *is an open set of* $\mathbb{R}^n$ *and* $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ *is a sequence of* N*-clusters such that* $\mathcal{E}_k \xrightarrow{loc} \mathcal{E}$*, then*

$$P(\mathcal{E}; A) \leq \liminf_{k \to \infty} P(\mathcal{E}_k; A) \tag{1.3}$$

Proof. Let's prove that $P(\mathcal{E};F) = \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}(h);F)$. We claim that the interfaces $\{\mathcal{E}(h,k)\}_{0\leq h\neq k\leq N}$ are disjoint and that $\partial^*\mathcal{E}(h) \approx \cup_{k=0,k\neq h}^{N}\mathcal{E}(h,k)$. We are going to prove them later. If these statements are true then we have

$$\begin{aligned}\sum_{0\leq h<k\leq N} \mathcal{H}^{n-1}\big(\mathcal{E}(h,k)\cap F\big) &= \frac{1}{2}\sum_{0\leq h\neq k\leq N} \mathcal{H}^{n-1}\big(\mathcal{E}(h,k)\cap F\big)\\ &= \frac{1}{2}\sum_{h=0}^{N}\mathcal{H}^{n-1}\left(\bigcup_{k=0,k\neq h}^{N}\mathcal{E}(h,k)\cap F\right)\\ &= \frac{1}{2}\sum_{h=0}^{N}\mathcal{H}^{n-1}\big(\partial^*\mathcal{E}(h)\cap F\big)\\ &= \frac{1}{2}\sum_{h=0}^{N}P(\mathcal{E}(h);F)\end{aligned}$$

Let's prove the two previous statements.

1. First of all, we show that two distinct interfaces are disjoint. Let $x\in\mathcal{E}(h,k)\cap\mathcal{E}(h,j)$ for different indices h,k,j in $\{0,\dots,N\}$. Then, by Federer's theorem, $x\in\mathcal{E}(h)^{(1/2)}\cap\mathcal{E}(k)^{(1/2)}\cap\mathcal{E}(j)^{(1/2)}$ and so

$$\begin{aligned}1 &\geq \frac{\big|B(x,r)\cap(\mathcal{E}(h)\cup\mathcal{E}(k)\cup\mathcal{E}(j))\big|}{\omega_n r^n} =\\ &= \frac{\big|B(x,r)\cap\mathcal{E}(h)\big|}{\omega_n r^n} + \frac{\big|B(x,r)\cap\mathcal{E}(k)\big|}{\omega_n r^n} + \frac{\big|B(x,r)\cap\mathcal{E}(j)\big|}{\omega_n r^n}\\ &\to \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}\end{aligned}$$

which is clearly a contradiction. Then

$$\mathcal{E}(h,k)\cap\mathcal{E}(h,j) = \emptyset \tag{1.4}$$

2. Now we verify that

$$\partial^*\mathcal{E}(h) = M \cup \bigcup_{k=0,k\neq h}^{N}\mathcal{E}(h,k) \tag{1.5}$$

for a certain set M with $\mathcal{H}^{n-1}(M) = 0$. Clearly, all the interfaces $\{\mathcal{E}(h,k)\}_{k=0,\dots,N,k\neq h}$ are subsets of $\partial^*\mathcal{E}(h)$ and so

$$\bigcup_{k=0,k\neq h}^{N}\mathcal{E}(h,k)\subseteq\partial^*\mathcal{E}(h)$$

Viceversa, if $x \in \partial^*\mathcal{E}(h)$ then $x \in \mathcal{E}(h)^{(1/2)} = (\mathbb{R}^n \setminus \mathcal{E}(h))^{(1/2)} = \left(\bigcup_{k=0,k\neq h}^{N} \mathcal{E}(k)\right)^{(1/2)}$. According to the following lemma 1.3 and Federer's theorem, there exist sets $M', M'' \subseteq \mathbb{R}^n$ of null $\mathcal{H}^{n-1}$ measure such that

$$\begin{aligned}\left(\bigcup_{k=0,k\neq h}^{N} \mathcal{E}(k)\right)^{(1/2)} &\subseteq M' \cup \bigcup_{k=0,k\neq h}^{N} \mathcal{E}(k)^{(1/2)}\\ &\subseteq M'' \cup \bigcup_{k=0,k\neq h}^{N} \partial^*\mathcal{E}(k)\end{aligned}$$

Then we conclude that

$$\begin{aligned}\partial^*\mathcal{E}(h) &\subseteq \left(\bigcup_{k=0,k\neq h}^{N} \mathcal{E}(k)\right)^{(1/2)} \cap \partial^*\mathcal{E}(h)\\ &\subseteq M'' \cup \bigcup_{k=0,k\neq h}^{N} \big(\partial^*\mathcal{E}(k) \cap \partial^*\mathcal{E}(h)\big)\\ &= M'' \cup \bigcup_{k=0,k\neq h}^{N} \mathcal{E}(h,k)\end{aligned}$$

This proves (1.5).

Finally we have to demonstrate (1.3). This follows quite immediately by (1.2) and the lower semicontinuity of $P(\,\cdot\,;A)$ with respect to the convergence of finite perimeter sets "in measure". Indeed, since $\mathcal{E}_k \to \mathcal{E}$, we have $\mathcal{E}_k(h) \to \mathcal{E}(h)$ for every $h = 0, \dots, N$. Then, by the semicontinuity, we get $P(\mathcal{E}(h);A) \leq \liminf_{k\to\infty} P(\mathcal{E}_k(h);A)$ and so we can deduce that

$$\begin{aligned}P(\mathcal{E};A) &= \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}(h);A)\\ &\leq \frac{1}{2}\sum_{h=0}^{N} \liminf_{k\to\infty} P(\mathcal{E}_k(h);A)\\ &\leq \liminf_{k\to\infty}\left(\frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}_k(h);A)\right)\\ &= \liminf_{k\to\infty} P(\mathcal{E}_k;A)\end{aligned}$$

which proves (1.3). □

LEMMA 1.3. *Let $E, F \subseteq \mathbb{R}^n$ two sets of finite perimeter with $|E \cap F| = 0$. Then*

$$(E \cup F)^{1/2} \subseteq M \cup \left(\left(E^{(1/2)} \cap F^{(0)} \right) \cup \left(F^{(1/2)} \cap E^{(0)} \right) \right) \tag{1.6}$$

for some null $\mathcal{H}^{n-1}$-measure set $M \subseteq \mathbb{R}^n$.

Proof. Let $x \in (E \cup F)^{(1/2)}$; then, as $r \to 0^+$, it holds that

$$\frac{1}{2} \leftarrow \frac{\left|B(x,r) \cap (E \cup F)\right|}{\omega_n r^n} = \frac{\left|B(x,r) \cap E\right|}{\omega_n r^n} + \frac{\left|B(x,r) \cap F\right|}{\omega_n r^n} \tag{1.7}$$

and so $x \in E^{(1/2)}$ if and only if $x \in F^{(0)}$. So let $x \notin E^{(1/2)} \cap F^{(1/2)}$, that is $x \notin F^{(0)} \cap E^{(0)}$. For sure $x \notin E^{(1)}$ and $x \notin F^{(1)}$ because it would contradict (1.7). Then $x \in \partial^e E \setminus E^{(1/2)}$ and $x \in \partial^e F \setminus F^{(1/2)}$. By Federer's theorem, we know that

$$\mathcal{H}^{n-1}\left(\partial^e E \setminus \partial^* E\right) = 0, \qquad \mathcal{H}^{n-1}\left(\partial^e F \setminus \partial^* F\right) = 0$$

and so, necessarily, $x \in M$ for some $M \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(M) = 0$. $\square$

By the previous proposition, some interesting consequences follow.

REMARK 1.4. The equality (1.2) is useful because we can deduce some easy and interesting facts. Indeed the perimeter of a cluster $\mathcal{E}$ in $\mathbb{R}^n$ is invariant with respect to rigid motions and, choosen $\lambda > 0$, it holds true that $P(\lambda\mathcal{E}) = \lambda^{n-1} P(\mathcal{E})$. These are clear consequences of (1.2) and the analogous formulas valid for any sets of finite perimeter.

REMARK 1.5. It is easy to see that $P(\mathcal{E}) = \mathcal{H}^{n-1}\left(\bigcup_{h=1}^N \partial^* \mathcal{E}(h)\right)$. In fact, since we have shown that $\partial^*\mathcal{E}(h) \approx \cup_{k \neq h} \mathcal{E}(h,k)$, then

$$\bigcup_{h=1}^{N} \partial^* \mathcal{E}(h) \approx \bigcup_{h=1}^{N} \bigcup_{k \neq h} \mathcal{E}(h,k) \approx \bigcup_{0 \leq h < k \leq N} \mathcal{E}(h,k)$$

This proves the equivalence of perimeter initially stated.

We end this section with some important remarks.

REMARK 1.6. If $x \in \mathcal{E}(h,k)$, $0 \leq h < k \leq N$ and $j \neq h, k$ then

$$\nu_{\mathcal{E}(h)}(x) = -\nu_{\mathcal{E}(k)}(x) \tag{1.8}$$

$$\theta_n(\mathcal{E}(j))(x) = 0 \tag{1.9}$$

Moreover, there exists a set $M \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(M) = 0$ such that for every $x \in \mathcal{E}(h,k) \setminus M$

$$\theta_{n-1}(\partial^* \mathcal{E}(j))(x) = 0 \tag{1.10}$$

Indeed, it is known that if E, F are sets of locally finite perimeter, $E \subseteq F$ and $x \in \partial^* E \cap \partial^* F$ then $\nu_E(x) = \nu_F(x)$. Then, considering $E = \mathcal{E}(h), F = \mathbb{R}^n \setminus \mathcal{E}(k)$, up to a set of null $\mathcal{L}^n$-measure, $E \subseteq F$ and so

$$\nu_{\mathcal{E}(h)}(x) = \nu_{\mathbb{R}^n \setminus \mathcal{E}(k)}(x) = -\nu_{\mathcal{E}(k)}(x)$$

which demonstrates (1.8). Now let's prove (1.9). As $x \in \mathcal{E}(h,k) = \partial^* \mathcal{E}(h) \cap \partial^* \mathcal{E}(k)$, then $x \in \mathcal{E}(h)^{(1/2)} \cap \mathcal{E}(k)^{(1/2)}$ and so

$$\frac{\left|B(x,r) \cap \mathcal{E}(h)^c\right|}{\omega_n r^n} = \frac{\left|B(x,r) \cap \mathcal{E}(k)\right|}{\omega_n r^n} + \frac{\left|B(x,r) \cap \left(\cup_{i \neq h,k} \mathcal{E}(i)\right)\right|}{\omega_n r^n}$$

provides, as $r \to 0^+$,

$$\frac{\left|B(x,r) \cap \left(\cup_{i \neq h,k} \mathcal{E}(i)\right)\right|}{\omega_n r^n} \to 0$$

In particular

$$\frac{\left|B(x,r) \cap \mathcal{E}(j)\right|}{\omega_n r^n} \to 0, \quad r \to 0^+$$

that is $\theta_n(\mathcal{E}(j))(x) = 0$. We finally prove (1.10). We recall corollary 6.5 of [6]: if E is a Borel set, $s \in (0,n)$, and $\mathcal{H}^s(E \cap K) < \infty$ for all the compact sets K in $\mathbb{R}^n$, then for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n \setminus E$,

$$\theta_s(E)(x) = \lim_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\omega_s r^s} = 0$$

Since $\partial^* \mathcal{E}(j)$ is a Borel set and $\mathcal{H}^{n-1}\left(\partial^* \mathcal{E}(j) \cap K\right) = P(\mathcal{E}(j); K) < \infty$ for each compact set $K \subseteq \mathbb{R}^n$, then for $\mathcal{H}^{n-1}$-almost every $x \in \mathbb{R}^n \setminus \partial^* \mathcal{E}(j)$, $\theta_{n-1}(\partial^* \mathcal{E}(j))(x) = 0$. Reminding that $\mathcal{E}(h,k) \subseteq \mathbb{R}^n \setminus \partial^* \mathcal{E}(j)$, we conclude that, for some $M \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(M) = 0$ and for every $x \in \mathcal{E}(h,k) \setminus M$, $\theta_{n-1}(\partial^* \mathcal{E}(j))(x) = 0$.

The next remark gives us a slightly generalization of what we have already proved in (1.5).

REMARK 1.7. Consider $\Lambda \subseteq \{0, \ldots, N\}$. Then

$$\mathcal{H}^{n-1}\left(\partial^*\left(\bigcup_{h \in \Lambda} \mathcal{E}(h)\right) \setminus \bigcup_{h \in \Lambda, k \notin \Lambda} \mathcal{E}(h,k)\right) = 0$$

Just by convenience, we will prove the remark with $\Lambda = \{1,2\}$; the general case has the same basic idea. We already know that

$$\partial^*\left(\mathcal{E}(1) \cup \mathcal{E}(2)\right) \approx (\mathcal{E}(1) \cup \mathcal{E}(2))^{(1/2)} \approx (\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)}) \cup (\mathcal{E}(1)^{(0)} \cap \mathcal{E}(2)^{(1/2)})$$

Let's consider just the term $\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)}$. We prove that

$$\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)} \approx \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(j)^{(1/2)} \approx \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \partial^*\mathcal{E}(1) \cap \partial^*\mathcal{E}(j)$$

Let $x \in \mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(j)^{(1/2)}$ for some $j = 0, \ldots, N$, $j \neq 1, 2$. Clearly $x \in \mathcal{E}(1)^{(1/2)}$. Moreover $x \in \mathcal{E}(2)^{(0)}$ because by

$$1 = \frac{\left|B(x,r) \cap \mathcal{E}(1)\right|}{\omega_n r^n} + \frac{\left|B(x,r) \cap \mathcal{E}(j)\right|}{\omega_n r^n} + \sum_{\substack{i=0 \\ i\neq 1,j}}^{N} \frac{\left|\mathcal{E}(i) \cap B(x,r)\right|}{\omega_n r^n}$$

we find that

$$\sum_{\substack{i=0 \\ i\neq 1,j}}^{N} \frac{\left|\mathcal{E}(i) \cap B(x,r)\right|}{\omega_n r^n} \to 0, \quad r \to 0^+$$

and so $x \in \mathcal{E}(2)^{(0)}$. Hence we have

$$\bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(j)^{(1/2)} \subseteq \mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)}$$

Viceversa, from (1.6) and the above argument, for some null $\mathcal{H}^{n-1}$-measure sets $M_3, M_4, \ldots, M$, we get

$$\begin{aligned}
\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)} \subseteq \left(\bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \mathcal{E}(j) \right)^{(1/2)} &= \left(\mathcal{E}(3) \cup \bigcup_{\substack{j=0 \\ j\neq 1,2,3}}^{N} \mathcal{E}(j) \right)^{(1/2)} \\
&\subseteq M_3 \cup \mathcal{E}(3)^{(1/2)} \cup \left(\bigcup_{\substack{j=0 \\ j\neq 1,2,3}}^{N} \mathcal{E}(j) \right)^{(1/2)} \\
&\subseteq M_4 \cup \mathcal{E}(3)^{(1/2)} \cup \mathcal{E}(4)^{(1/2)} \cup \left(\bigcup_{\substack{j=0 \\ j\neq 1,2,3,4}}^{N} \mathcal{E}(j) \right)^{(1/2)} \\
&\subseteq \cdots \subseteq M \cup \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \mathcal{E}(j)^{(1/2)}
\end{aligned}$$

and so

$$\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)} \subseteq M \cup \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \left(\mathcal{E}(j)^{(1/2)} \cap \mathcal{E}(1)^{(1/2)} \right)$$

Thus we conclude that

$$\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(2)^{(0)} \approx \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \left(\mathcal{E}(1)^{(1/2)} \cap \mathcal{E}(j)^{(1/2)}\right) \approx \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \left(\partial^*\mathcal{E}(1) \cap \partial^*\mathcal{E}(j)\right)$$

So finally we get

$$\begin{aligned} \partial^*(\mathcal{E}(1) \cup \mathcal{E}(2)) &\approx \big(\mathcal{E}(1) \cup \mathcal{E}(2)\big)^{(1/2)} \approx \\ &\approx \bigcup_{i=1,2} \bigcup_{\substack{j=0 \\ j\neq 1,2}}^{N} \left(\partial^*\mathcal{E}(i) \cap \partial^*\mathcal{E}(j)\right) = \bigcup_{i\in\Lambda, j\notin\Lambda} \mathcal{E}(i,j) \end{aligned}$$

1.2 Compactness criterion and some technical lemmas

In order to prove the existence of the minimum and reminding the Direct Method of the Calculus of Variation, it is very important to have a compactness criterion. The following proposition ensures us exactly this: given a sequence of N-clusters satisfying some quite restrictive hypotheses, we are able to extract a subsequence converging to another N-cluster $\mathcal{E}$.

Proposition 1.8. *Let $R > 0$ and $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ a collection of N-clusters in $\mathbb{R}^n$ such that*

$$\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < \infty \tag{1.11}$$

$$\mathcal{E}_k(h) \subseteq B_R, \qquad h = 1, \dots, N,\ k \in \mathbb{N} \tag{1.12}$$

$$\inf_{k\in\mathbb{N}} \min_{h=1,\dots,N} \big|\mathcal{E}_k(h)\big| > 0 \tag{1.13}$$

Then there exists a subsequence $\{\mathcal{E}_{k(l)}\}_{l\in\mathbb{N}}$ and an N-cluster $\mathcal{E}$ such that $\mathcal{E}_{k(l)} \to \mathcal{E}$ as $l \to \infty$.

Proof. We recall a similar proposition holding true for sequences of sets of finite perimeter. If $\{E_k\}_{k\in\mathbb{N}}$ is a collection of sets of finite perimeter such that for some $R > 0$

$$\begin{aligned} &\sup_{k\in\mathbb{N}} P(E_k) < \infty \\ &E_k \subseteq B_R, \quad \forall k \in \mathbb{N} \end{aligned} \tag{1.14}$$

then there exists a subsequence $\{E_{k(l)}\}_{l\in\mathbb{N}}$ and a set of finite perimeter E such that

$$E_{k(l)} \to E \subseteq B_R, \quad l \to \infty \tag{1.15}$$

We notice that each sequence $\{\mathcal{E}_k(h)\}_{k\in\mathbb{N}}$, $h = 1, \dots, N$ satisfies the hypotheses of the statement. Indeed, as

$$\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) = \sup_{k\in\mathbb{N}} \left(\frac{1}{2} \sum_{h=0}^{N} P(\mathcal{E}_k(h)) \right) < \infty$$

then, for all $h = 1, \dots, N$, $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k(h)) < \infty$. Moreover, by assumption (1.12) it holds true also $\mathcal{E}_k(h) \subseteq B_R$ and so we can conclude that, for $h = 1, \dots, N$, there exist subsequences converging. In order to find the subsequence $\{\mathcal{E}_{k(l)}\}_{l\in\mathbb{N}}$, we proceed in this way. Let $h = 1$. We extract a subsequence $\{\mathcal{E}_{k_1(l)}(1)\}_{l\in\mathbb{N}}$ from $\{\mathcal{E}_k(1)\}_{k\in\mathbb{N}}$ with

$$\mathcal{E}_{k_1(l)}(1) \to \mathcal{E}(1)$$

for some $\mathcal{E}(1) \subseteq \mathbb{R}^n$ of finite perimeter. Now let $h = 2$. As $\{\mathcal{E}_{k_1(l)}(2)\}_{l\in\mathbb{N}}$ satisfies hypothesis (1.14), we extract a subsequence $\{\mathcal{E}_{k_2(l)}(2)\}_{l\in\mathbb{N}}$ from $\{\mathcal{E}_{k_1(l)}(2)\}_{l\in\mathbb{N}}$, converging to some set $\mathcal{E}(2)$. It holds, as $l \to \infty$,

$$\mathcal{E}_{k_2(l)}(1) \to \mathcal{E}(1)$$
$$\mathcal{E}_{k_2(l)}(2) \to \mathcal{E}(2)$$

Repeating this proceeding until $h = N$, we find the subsequence $\{\mathcal{E}_{k(l)}\}_{l\in\mathbb{N}}$ setting $k(l) = k_N(l)$, $l \in \mathbb{N}$.

Now let's prove that $\{\mathcal{E}(h)\}_{h=1,\dots,N}$ is an N-cluster.

- For each $h = 1, \dots, N$, $\mathcal{E}(h)$ is a set of finite perimeter in $\mathbb{R}^n$ with $|\mathcal{E}(h)| < +\infty$. This easily follows from (1.15).
- It holds $|\mathcal{E}(h)| > 0$. Indeed, by hypothesis (1.13), we know that for every $h = 1, \dots, N$ $\inf_{k\in\mathbb{N}} |\mathcal{E}_k(h)| > 0$. Since for each $h = 1, \dots, N$ we have $\left|\mathcal{E}_{k(l)}(h)\right| \stackrel{l\to\infty}{\longrightarrow} |\mathcal{E}(h)|$, the conclusion is deduced immediately.
- Finally $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$ for every distinct $h, k = 1, \dots, N$. In fact we have
$$\mathcal{E}_{k(l)}(h) \cap \mathcal{E}_{k(l)}(k) \to \mathcal{E}(h) \cap \mathcal{E}(k), \quad l \to \infty$$
Hence, since $\left|\mathcal{E}_{k(l)}(h) \cap \mathcal{E}_{k(l)}(k)\right| = 0$ for any $l \in \mathbb{N}$, we conclude that $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$.

$\square$

Fixed $\mathbf{m} \in \mathbb{R}^N_+$, consider the partitioning problem

$$\gamma = \inf\{P(\mathcal{E}) : \mathbf{m}(\mathcal{E}) = \mathbf{m}\}$$

and a minimizing sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$. This means that for every $k \in \mathbb{N}$, $\mathcal{E}_k$ is an N-cluster with $\mathbf{m}(\mathcal{E}_k) = \mathbf{m}$ and that

$$\lim_{k\to\infty} P(\mathcal{E}_k) = \gamma$$

Clearly $\gamma < \infty$. In fact we can take the N-cluster $\mathcal{E}'$ given by N disjoint balls with radii $\{r_h\}_{h=1}^N$ such that $\omega_h r_h^n = \mathbf{m}(h)$. In this way

$$\gamma \leq P(\mathcal{E}') < \infty$$

Then we can assume that the minimizing sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ satisfies $\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < P(\mathcal{E}') < \infty$. Moreover

$$\inf_{k\in\mathbb{N}} \min_{h=1,\dots,N} \left|\mathcal{E}_k(h)\right| = \inf_{k\in\mathbb{N}} \min_{h=1,\dots,N} \mathbf{m}(h) > 0$$

Thus two of the assumptions in proposition 1.8 are satisfied. It is not obvious that, up to extracting subsequences, it holds $\mathcal{E}_k(h) \subseteq B_R$ for some $R > 0$, for every $k \in \mathbb{N}$, $h = 1, \dots, N$. In fact, even if $\mathcal{E}$ is the minimizer, the sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}} = \{x_k + \mathcal{E}\}_{k\in\mathbb{N}}$ with $x_k \overset{k\to\infty}{\longrightarrow} \infty$ is a minimizing sequence which clearly do not satisfy the second hypothesis of 1.8. The following statement provides a sufficient condition which guarantees that we can suppose the minimizing sequence uniformly bounded.

Proposition 1.9. *Let $R > 0$, $L \in \mathbb{N}$, $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ N-clusters and $\{\Omega_k\}_{k\in\mathbb{N}}$ a sequence of finite sets such that:*

$$\begin{aligned} &\mathcal{E}_k(h) \subseteq \bigcup_{x\in\Omega_k} B(x,R), \quad k \in \mathbb{N},\ h = 1,\dots,N \\ &\mathcal{H}^0(\Omega_k) \leq L, \quad k \in \mathbb{N} \end{aligned}$$

Then there exists a sequence $\{\mathcal{E}'_k\}_{k\in\mathbb{N}}$ of N-clusters such that

$$\begin{aligned} &P(\mathcal{E}'_k) = P(\mathcal{E}_k), \quad \mathbf{m}(\mathcal{E}'_k) = \mathbf{m}(\mathcal{E}_k), \quad k \in \mathbb{N} \\ &\mathcal{E}'_k(h) \subseteq B_{13L^2R}, \quad k \in \mathbb{N},\ h = 1,\dots,N \end{aligned} \tag{1.16}$$

Proof. First of all, for every $k \in \mathbb{N}$ we define the sets $\{F_{k,i}\}_{i=1}^{L(k)}$ as the connected components of $\bigcup_{x\in\Omega_k} \overline{B(x,R)}$. In particular for every $k \in \mathbb{N}$, $F_{k,i}$ is an union of closed balls with non empty intersection and $L(k) \leq L$ since $\mathcal{H}^0(\Omega_k) \leq L$. Moreover, if $F_{k,i} = \bigcup_{j=1}^{n(k,i)} \overline{B(x_j,R)}$, we get

$$\operatorname{diam}(F_{k,i}) \leq 2Rn(k,i) \leq 2LR$$

We claim that

$$F_{k,i} \subseteq B(z_{k,i}, 4RL)$$

for some $z_{k,i}$. In fact, if $\bar{x}, \bar{y} \in F_{k,i}$ are such that $|\bar{x} - \bar{y}| = \operatorname{diam}(F_{k,i})$, consider the point $z_{k,i} = \frac{\bar{x}+\bar{y}}{2} \in C(F_{k,i})$ (here $C(F_{k,i})$ denotes the convex hull of $F_{k,i}$). Since $\operatorname{diam}(C(F_{k,i})) = \operatorname{diam}(F_{k,i}) \leq 2RL$, then

$$\max\{d(z, z_{k,i}) : z \in F_{k,i}\} \leq 2RL$$

and so for sure $F_{k,i} \subseteq B(z_{k,i}, 4RL)$. Thus we define $x_{k,i} = -z_{k,i} + 9RLie_n$; translating the set $F_{k,i}$ using $x_{k,i}$ we have

$$x_{k,i} + F_{k,i} \subseteq B(9RLie_n, 4RL)$$

Notice that we have chosen $x_{k,i}$ in such a way that $\{B(9RLie_n, 4RL)\}_{i=1}^{L(k)}$ are disjoint, and consequently the sets $\{x_{k,i} + F_{k,i}\}_i$ are too. Then for every $k \in \mathbb{N}$ we define the map

$$\begin{aligned} f_k \colon \bigcup_{x \in \Omega_k} B(x, R) &\to \mathbb{R}^n \\ x &\mapsto x + x_{k,i}, \quad \text{if } x \in F_{k,i} \end{aligned}$$

and the new clusters $\{\mathcal{E}'_k\}_{k \in \mathbb{N}}$ with chambers

$$\mathcal{E}'_k(h) = f_k\left(\mathcal{E}_k(h)\right), \quad h = 1, \dots, N$$

Hence

$$\begin{aligned} \mathcal{E}'_k(h) &= f_k\left(\bigcup_{i=1}^{L(k)} (\mathcal{E}_k(h) \cap F_{k,i})\right) \\ &= \bigcup_{i=1}^{L(k)} x_{k,i} + \left(\mathcal{E}_k(h) \cap F_{k,i}\right) \\ &= \bigcup_{i=1}^{L(k)} \mathcal{E}'_{k,i}(h) \end{aligned}$$

with $\mathcal{E}_{k,i}(h) = \mathcal{E}_k(h) \cap F_{k,i}, \mathcal{E}'_{k,i}(h) = x_{k,i} + \mathcal{E}_{k,i}(h)$. Furthermore $\mathcal{E}'_{k,i}(h) \subseteq B(9RLie_n, 4RL)$ for every $k \in N, i = 1, \dots, L(k)$ and we notice that it holds

$$\bigcup_{i=1}^{L} B(9RLie_n, 4RL) \subseteq \overline{B_{13RL^2}}$$

Now we have to prove that $\{\mathcal{E}'_k\}_{k \in \mathbb{N}}$ are clusters satisfying the required conditions (1.16). Clearly $\left|\mathcal{E}'_k(h)\right| = \left|\mathcal{E}_k(h)\right|$. Indeed

$$\left|\mathcal{E}'_k(h)\right| = \sum_{i=1}^{L(k)} |\mathcal{E}'_{k,i}(h)| = \sum_{i=1}^{L(k)} \left|\mathcal{E}_{k,i}(h)\right| = \left|\mathcal{E}_k(h)\right|$$

The equality of the perimeters follows from this fact: if $E \subseteq F_1 \cup F_2$ is a set of finite perimeter and $\operatorname{dist}(F_1, F_2) > 0$ then $P(E) = P(E \cap F_1) + P(E \cap F_2)$. Indeed, as $\mathcal{E}_k(h) = \bigcup_{i=1}^{L(k)} \mathcal{E}_{k,i}$, we have

$$P(\mathcal{E}_k(h)) = \sum_{i=1}^{L(k)} P(\mathcal{E}_{k,i}) = \sum_{i=1}^{L(k)} P(\mathcal{E}'_{k,i}) = P(\mathcal{E}'_k(h))$$

and thus finally

$$P(\mathcal{E}'_k) = \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}'_k(h)) = \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}_k(h)) = P(\mathcal{E}_k)$$

The rest of the proof shows that $\mathcal{E}'_k$ is an N-cluster. First of all, $\mathcal{E}'_k(h)$ is clearly a set of finite perimeter for every $h = 1, \dots, N$, by its definition. Since $\left|\mathcal{E}'_k(h)\right| = \sum_{i=1}^{L(k)} \left|\mathcal{E}'_{k,i}(h)\right|$, then for every $h = 1, \dots, N$ it holds:

- $\left|\mathcal{E}'_k(h)\right| > 0$ because $\left|\mathcal{E}'_{k,i}(h)\right| = \left|\mathcal{E}_k(h) \cap F_{k,i}\right| > 0$ for at least one $i = 1, \dots, L(k)$;
- $\left|\mathcal{E}'_k(h)\right| < +\infty$ because $\left|\mathcal{E}'_{k,i}(h)\right| < +\infty$ for every $i = 1, \dots, L(k)$.

Lastly

$$\left|\mathcal{E}'_k(h) \cap \mathcal{E}'_k(l)\right| = \sum_{i=1}^{L(k)} \left|\mathcal{E}'_{k,i}(h) \cap \mathcal{E}'_{k,i}(l)\right| = 0$$

as $\left|\mathcal{E}'_{k,i}(h) \cap \mathcal{E}'_{k,i}(l)\right| = \left|\mathcal{E}_{k,i}(h) \cap \mathcal{E}_{k,i}(l)\right| \leq \left|\mathcal{E}_k(h) \cap \mathcal{E}_k(l)\right| = 0$. This concludes the proof of the proposition. □

Using the confinement proposition 1.9 in order to have a family of clusters uniformly bounded and the compactness criterion 1.8, we get the following corollary.

COROLLARY 1.10. *Let $R > 0$, $L \in \mathbb{N}$ and $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ satisfying the hypotheses of proposition 1.9. Moreover let us assume that*

$$\sup_{k\in\mathbb{N}} P(\mathcal{E}_k) < \infty, \qquad \inf_{k\in\mathbb{N}} \min_{h=1,\dots,N} \left|\mathcal{E}_k(h)\right| > 0$$

Then there exist $\{\mathcal{E}'_k(h)\}_{k\in\mathbb{N}}$ a sequence of N-clusters, and $\mathcal{E}$ N-cluster such that

$$\begin{aligned}
&P(\mathcal{E}'_k) = P(\mathcal{E}_k),\ \mathbf{m}(\mathcal{E}'_k) = \mathbf{m}(\mathcal{E}_k), && k \in \mathbb{N}\\
&\mathcal{E}'_k(h) \subseteq B_{13L^2R}, && k \in \mathbb{N},\ h = 1, \dots, N\\
&\mathcal{E}'_k \overset{k\to\infty}{\longrightarrow} \mathcal{E}
\end{aligned}$$

In particular, if $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *is a minimizing sequence for the partitioning problem*

$$\inf\{P(\mathcal{E}) : \mathbf{m}(\mathcal{E}) = m\}$$

then $\{\mathcal{E}'_k\}_{k\in\mathbb{N}}$ *is a minimizing sequence for the same problem too.*

Proof. The corollary is an immediate consequence of the two previous propositions. In fact using 1.9 we can construct a sequence of clusters $\{\mathcal{E}'_k\}_{k\in\mathbb{N}}$ which satisfies the hypotheses of 1.8. Then, up to extracting a subsequence, $\mathcal{E}'_k \to \mathcal{E}$ for some N-cluster $\mathcal{E}$. □

Now we are going to state two important lemmas. The first one is the Nucleation Lemma and it guarantees that, given a set of finite perimeter E there exists a discrete set I of points such that the union of the balls with center in I and radius 2 covers almost entirely E, that is up to a certain error ε. Moreover the volumes of the balls centered at a point of I with radius 1 can be uniformly bounded from below in ε.

Lemma 1.11 (Nucleation lemma). *There is a constant* $c(n) > 0$ *with the following property. Consider a set of finite perimeter* E *with* $|E| \in (0,\infty)$ *and*

$$\varepsilon \le \min\left\{|E|, \frac{P(E)}{2nc(n)}\right\}$$

Then we can find a finite set of points $I \subseteq \mathbb{R}^n$ *such that*

$$\left|E \setminus \bigcup_{x\in I} B(x,2)\right| < \varepsilon \tag{1.17}$$

$$\left|E \cap B(x,1)\right| \ge \left(c(n)\frac{\varepsilon}{P(E)}\right)^n, \quad x \in I \tag{1.18}$$

Moreover the balls $\{B(x,1)\}_{x\in I}$ *are disjoint and the cardinality of* I *is bounded by a constant depending on* ε*, namely*

$$\#I \le |E|\left(\frac{P(E)}{c(n)\varepsilon}\right)^n \tag{1.19}$$

Proof. First step. We claim that there exists a constant $c(n) > 0$ such that if F is a closed set of $\mathbb{R}^n$ with $\left|\{x \in E : \operatorname{dist}(x,F) > 1\}\right| \ge \varepsilon$ then there is $x \in E^{(1)}$ with

$$\operatorname{dist}(x,F) > 1$$

$$\left|E \cap B(x,1)\right| \ge \left(c(n)\frac{\varepsilon}{P(E)}\right)^n$$

Assuming the claim true, we construct the set I in this way. Applying the claim to $F = \emptyset$, since $\operatorname{dist}(x,\emptyset) = +\infty$, the hypothesis is clearly satisfied and so we can find $x_1 \in E^{(1)}$ such that $\left|E \cap B(x_1,1)\right| \ge \left(c(n)\frac{\varepsilon}{P(E)}\right)^n$. We set $I = \{x_1\}$. Suppose that, iterating the proceeding, we have determined $I = \{x_i\}_{i=1}^s$ with $\left|x_i - x_j\right| > 2$ for $i \ne j$ and with (1.18) holding true. Then:

- if $\left|E \setminus \bigcup_{x\in I} B(x,2)\right| < \varepsilon$ we can stop and thus the lemma is proved;
- otherwise, if $\left|E \setminus \bigcup_{x=1}^{s} B(x_i,2)\right| \geq \varepsilon$, setting $F := \bigcup_{x=1}^{s} \overline{B(x_i,1)}$ it holds

$$\left|\{x \in E:\ \operatorname{dist}(x,F) > 1\}\right| \geq \varepsilon \tag{1.20}$$

because

$$E \setminus \bigcup_{x=1}^{s} B(x_i,2) \subseteq \{x \in E:\ \operatorname{dist}(x,F) > 1\}$$

Indeed if $x \in E \setminus \bigcup_{x=1}^{s} B(x_i,2)$ then $x \in E$ and, for every $i = 1,\dots,s$, $|x - x_i| > 2$. In particular, by the last inequality, we have that $\operatorname{dist}(x,\overline{B(x_i,1)}) > 1$ for any $i = 1,\dots,s$ and then

$$\operatorname{dist}(x,F) = \min_{i=1,\dots,s} \operatorname{dist}(x,\overline{B(x_i,1)}) > 1$$

By (1.20), we can apply the claim, which provides the existence of a point $x_{s+1} \in E^{(1)}$ such that $\operatorname{dist}(x_{s+1}.F) > 1$ and

$$\left|E \cap B(x_{s+1},1)\right| \geq \left(c(n)\frac{\varepsilon}{P(E)}\right)^n$$

Since $\operatorname{dist}(x_{s+1},F) > 1$, it holds $|x_{s+1} - x_i| > 2$ for every $i = 1,\dots,s$. In fact, if y_i is the intersection of the line segment $[x_{s+1},x_i]$ and $\overline{B(x_i,1)}$ then

$$|x_{s+1} - x_i| = |x_{s+1} - y_i| + |y_i - x_i| > 2$$

Finally we redefine $I = \{x_i\}_{i=1}^{s+1}$.

We iterate this proceeding up to (1.17) holds true. It ends in a finite number of steps because $|E| < \infty$ and the balls $\{B(x_i,1)\}_{i=1}^{s}$ are disjoint. Finally also (1.19) holds true. In fact

$$\begin{aligned}
|E| &= \sum_{i=1}^{\#I} \left|E \cap B(x_i,1)\right| + \left|E \setminus \bigcup_{i=1}^{\#I} B(x_i,1)\right| \\
&\geq \sum_{i=1}^{\#I} \left|E \cap B(x_i,1)\right| \\
&\geq \#I \left|E \cap B(x_j,1)\right| \qquad \text{with } j \text{ realizing the minimum} \\
&\geq \#I \left(c(n)\frac{\varepsilon}{P(E)}\right)^n
\end{aligned}$$

Thus (1.19) follows.

Second step. In order to prove the claim, let's show the following statement. *Statement:* if $\alpha \geq n, x \in E^{(1)}$, and

$$\left|E \cap B(x,1)\right| < \left(\frac{1}{2\alpha}\right)^n \tag{1.21}$$

then there exists $r_x \in (0,1)$ such that $P(E; B(x,r_x)) > \alpha \left|E \cap B(x,r_x)\right|$.

Let $m(r) = \left|E \cap B(x,r)\right|$ for $r > 0$ and assume by contradiction that

$$P(E; B(x,r)) \leq \alpha m(r), \quad \forall r \in (0,1)$$

Since for almost every $r \in (0,1)$ it holds $m'(r) = \mathcal{H}^{n-1}\left(E \cap \partial B(x,r)\right)$, we have

$$P(E \cap B(x,r)) = P(E; B(x,r)) + \mathcal{H}^{n-1}\left(E \cap \partial B(x,r)\right) \leq \alpha m(r) + m'(r)$$

for a.e. $r \in (0,1)$. Then, reminding the non-sharp isoperimetric inequality $P(F) \geq |F|^{(n-1)/n}$, we get for a.e. $r \in (0,1)$

$$\alpha m(r) + m'(r) \geq P(E \cap B(x,r)) \geq m(r)^{(n-1)/n} \tag{1.22}$$

Now we estimate $\alpha m(r)$. As m is increasing, $m(r) \leq m(1) < \left(\frac{1}{2\alpha}\right)^n$ for every $r \in (0,1)$; then $\alpha m(r)^{1/n} < \frac{1}{2}$ and

$$\alpha m(r) \leq \frac{m(r)^{(n-1)/n}}{2}$$

Thus, taking (1.22) into account, we find out that

$$\frac{m(r)^{(n-1)/n}}{2} \leq m'(r) \quad \text{for a.e. } r \in (0,1)$$

Since $x \in E^{(1)}$, we can divide by $m(r)^{(n-1)/n} > 0$. Therefore the last inequality becomes

$$m'(r)m(r)^{1/n-1} \geq \frac{1}{2}, \qquad \text{for a.e. } r \in (0,1)$$

that is $n\left(m(r)^{1/n}\right)' \geq \frac{1}{2}$. Integrating this in $(0,r)$, we get

$$m(r) \geq \left(\frac{r}{2n}\right)^n, \qquad \text{for } r \in (0,1)$$

As $r \to 1^-$, we have $m(1) \geq \left(\frac{1}{2n}\right)^n$, which implies, by (1.21),

$$\frac{1}{2\alpha} > \frac{1}{2n}$$

This is a contradiction, since we have assumed $\alpha \geq n$.

Third step. Let's finally prove the claim. Assume by contradiction that for every positive constant $c(n)$ there exists a closed set F in $\mathbb{R}^n$ with $\big|\{x \in E:\ \mathrm{dist}(x,F) > 1\}\big| \geq \varepsilon$ and such that, however we choose $x \in E^{(1)}$ with $\mathrm{dist}(x,F) > 1$, it holds

$$\big|E \cap B(x,1)\big| < \left(c(n)\frac{\varepsilon}{P(E)}\right)^n$$

Define $\alpha \in \mathbb{R}$ such that $\frac{c(n)\varepsilon}{P(E)} = \frac{1}{2\alpha}$, that is $\alpha = \frac{1}{2c(n)}\frac{P(E)}{\varepsilon}$. As by assumption $\varepsilon \leq \frac{P(E)}{2nc(n)}$, we get

$$\frac{1}{2\alpha} = \frac{c(n)\varepsilon}{P(E)} \leq \frac{1}{2n}$$

and so $\alpha \geq n$. If $x \in E^{(1)}$ and $\mathrm{dist}(x,F) > 1$, then $\big|E \cap B(x,1)\big| < \left(\frac{1}{2\alpha}\right)^n$ and so, from what we have just seen in the second step, there exists $r_x \in (0,1)$ such that

$$P(E; B(x,r_x)) > \alpha\,\big|E \cap B(x,r_x)\big| \tag{1.23}$$

Set $\mathcal{F} = \{\overline{B(x,r_x)} \,|\, x \in E^{(1)}, \mathrm{dist}(x,F) > 1\}$. As

$$\sup\{\mathrm{diam}(\bar{B}) | \bar{B} \in \mathcal{F}\} \leq 1$$

we can apply Besicovitch theorem to $\mathcal{F}$: there exist $\xi(n)$ (depending only on n) and subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_{\xi(n)}$ such that each $\mathcal{F}_i$ is disjoint, at most countable and

$$C := \{x \in E^{(1)},\ \mathrm{dist}(x,F) > 1\} \subseteq \bigcup_{i=1}^{\xi(n)} \bigcup_{\bar{B} \in \mathcal{F}_i} \bar{B}$$

Hence we deduce that, for some $\mathcal{F}' \in \{\mathcal{F}_1, \ldots, \mathcal{F}_{\xi(n)}\}$, we have

$$\begin{aligned}
|C| &= \big|\{x \in E,\ \mathrm{dist}(x,F) > 1\}\big| \\
&\leq \xi(n) \sum_{\bar{B}(x,r_x) \in \mathcal{F}'} \big|E \cap \bar{B}(x,r_x)\big| \\
&< \frac{\xi(n)}{\alpha} \sum_{\bar{B}(x,r_x) \in \mathcal{F}'} P(E; \bar{B}(x,r_x)) \\
&\leq \frac{\xi(n)P(E)}{\alpha} \\
&= \xi(n)P(E)\frac{2c(n)\varepsilon}{P(E)} = 2\varepsilon\xi(n)c(n)
\end{aligned}$$

Choosing $c(n) = \frac{1}{2\xi(n)}$, we find

$$\left|\{x \in E^{(1)} | \operatorname{dist}(x, F) > 1\}\right| < \varepsilon$$

which contradicts the hypothesis on F. Thus the claim is proved.

$\square$

REMARK 1.12. From the above lemma and in particular applying the claim to $F = \emptyset$, we find that, if E is a set of finite perimeter with $0 < |E| < \infty$ and $\varepsilon = \min\left\{|E|, \frac{P(E)}{2nc(n)}\right\}$, then there exists $x \in \mathbb{R}^n$ such that

$$\left|E \cap B(x, 1)\right| \geq \min\left\{c(n)\frac{|E|}{P(E)}, \frac{1}{2n}\right\}^n$$

In fact, if ε satisfies $\varepsilon = |E|$ then there exists $x \in \mathbb{R}^n$ such that

$$\left|E \cap B(x, 1)\right| \geq \left(\frac{c(n)\varepsilon}{P(E)}\right)^n = \left(\frac{c(n)\,|E|}{P(E)}\right)^n \geq \min\left\{\frac{c(n)\,|E|}{P(E)}, \frac{1}{2n}\right\}^n$$

Instead, if $\varepsilon = \frac{P(E)}{2nc(n)}$ then

$$\left|E \cap B(x, 1)\right| \geq \left(\frac{c(n)\varepsilon}{P(E)}\right)^n = \left(\frac{1}{2n}\right)^n \geq \min\left\{\frac{c(n)\,|E|}{P(E)}, \frac{1}{2n}\right\}^n$$

The following lemma provides a way to redefine a new N-cluster $\mathcal{E}'$ which reduces the perimeter with a very precise estimation.

LEMMA 1.13 (TRUNCATION LEMMA). *Let $F \subseteq \mathbb{R}^n$ be a closed set, $\mathcal{E}$ an N-cluster in $\mathbb{R}^n$, $u(x) = \operatorname{dist}(x, F)$ and $\alpha > 0$ such that*

$$\sum_{h=1}^{N} \left|\mathcal{E}(h) \setminus F\right| \leq \alpha$$

Then there exists $r_0 \in [0, 7n\alpha^{1/n}]$ such that the new N-cluster $\mathcal{E}'$ defined as

$$\mathcal{E}'(h) = \mathcal{E}(h) \cap \{x \in \mathbb{R}^n : u(x) \leq r_0\}, \quad h = 1, \ldots, N$$

satisfies the following estimation on the perimter:

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{d(\mathcal{E}, \mathcal{E}')}{4\alpha^{1/n}}$$

Proof. By simplicity we write $\{u < r\}$ for $\{x \in \mathbb{R}^n : u(x) < r\}$.

If $\sum_{h=1}^{N} \left|\mathcal{E}(h) \setminus F\right| = 0$, we set $r_0 = 0$. In fact in this case each $\mathcal{E}(h)$ is "contained in measure" in F. Then, setting $r_0 = 0$ and consequently $\mathcal{E}'(h) = \mathcal{E}(h) \cap \{u \leq 0\} = \mathcal{E}(h) \cap F = \mathcal{E}(h)$, we have $d(\mathcal{E}, \mathcal{E}') = 0$ and $P(\mathcal{E}) = P(\mathcal{E}')$.

Then we can assume that $\sum_{h=1}^{N} \left|\mathcal{E}(h) \setminus F\right| > 0$. We define the function $m\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $m(r) = \sum_{h=1}^{N} \left|\mathcal{E}(h) \cap \{u > r\}\right|$. Notice that $m(r) \leq \alpha$ because $m(r) \leq \sum_{h=1}^{N} \left|\mathcal{E}(h) \cap F^c\right| \leq \alpha$. Since u is a Lipschitz function and $|\nabla u| = 1$ almost everywhere on $\mathbb{R}^n$, then, thanks to the generalized coarea formula applied to u, we have

$$\begin{aligned}
\left|\mathcal{E}(h) \cap \{u > r\}\right| &= \int_{\mathcal{E}(h)\cap\{u>r\}} |\nabla u| \, \mathrm{d}\mathcal{L}^n \\
&= \int_{\mathbb{R}} P\left(\{u > t\}; \mathcal{E}(h) \cap \{u > r\}\right) \mathrm{d}t \\
&= \int_{\mathbb{R}} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \{u > r\} \cap \{u = t\}\right) \mathrm{d}t \\
&= \int_{r}^{+\infty} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \{u = t\}\right) \mathrm{d}t
\end{aligned}$$

Hence $m(r) = \sum_{h=1}^{N} \int_r^{+\infty} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \{u = t\}\right) \mathrm{d}t$ and, for almost every $r > 0$,

$$m'(r) = -\sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \{u = r\}\right) \tag{1.24}$$

As $m(0) > 0$, we have $\mathrm{spt}(m) = [0, r_1]$ for some $r_1 > 0$. We claim that either $\mathcal{E}(h) \subseteq \{u \leq 7n\alpha^{1/n}\}$ for each $h = 1, \ldots, N$, or there exists some $r_0 < 7n\alpha^{1/n}$ such that the cluster $\mathcal{E}'$ defined above satisfies

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \frac{m(r_0)}{4\alpha^{1/n}}$$

If the claim is true then:

- in the first case setting $r_0 = 7n\alpha^{1/n}$ we have $\mathcal{E}'(h) = \mathcal{E}(h), h = 1, \ldots, N$ and so $P(\mathcal{E}') = P(\mathcal{E})$;
- in the second case, observing that

$$\begin{aligned}
d(\mathcal{E}, \mathcal{E}') &= \sum_{h=1}^{N} \left|\mathcal{E}(h) \Delta \mathcal{E}'(h)\right| \\
&= \sum_{h=1}^{N} \left|\mathcal{E}(h) \setminus \mathcal{E}'(h)\right| \\
&= \sum_{h=1}^{N} \left|\mathcal{E}(h) \cap \{u > r_0\}\right| = m(r_0)
\end{aligned}$$

  the claim provides to distance r_0 which we were looking for.

Thus, proving the claim, we end the proof of the lemma.

In order to demonstrate the claim, it is enough to show that either $r_1 < 7n\alpha^{1/n}$ or $P(\mathcal{E}) \geq P(\mathcal{E}^{r_0}) + \frac{m(r_0)}{4\alpha^{1/n}}$ for some $r_0 < 7n\alpha^{1/n}$ (here $\mathcal{E}^r(h) = \mathcal{E}(h) \cap \{u \leq r\}$). In fact, if $r_1 < 7n\alpha^{1/n}$, then $0 = \sum_{h=1}^{M} \left|\mathcal{E}(h) \cap \{u > 7n\alpha^{1/n}\}\right|$ and so, up to a set of null $\mathcal{L}^n$-measure, $\mathcal{E}(h) \subseteq \{u \leq 7n\alpha^{1/n}\}, h = 1, \ldots, N$.

Let's assume by contradiction the following inequalities true:

$$r_1 \geq 7n\alpha^{1/n} \tag{1.25}$$

$$P(\mathcal{E}) < P(\mathcal{E}^r) + \frac{m(r)}{4\alpha^{1/n}}, \quad \forall r < 7n\alpha^{1/n} \tag{1.26}$$

We rewrite the quantities $P(\mathcal{E}), P(\mathcal{E}^r)$, for almost every $r > 0$, as

$$P(\mathcal{E}) = P(\mathcal{E}; \{u < r\}) + P(\mathcal{E}; \{u > r\})$$

and

$$\begin{aligned}
P(\mathcal{E}^r) &= \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{E}^r(h)) \\
&= \frac{1}{2}\sum_{h=1}^{N} P(\mathcal{E}(h) \cap \{u < r\}) + \frac{1}{2}P(\mathcal{E}^r(0)) \\
&= P(\mathcal{E}; \{u < r\}) + \sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \{u = r\}\right)
\end{aligned}$$

In fact for $h = 1, \ldots, N$ and for a.e. $r > 0$ it holds

$$P(\mathcal{E}(h) \cap \{u < r\}) = P(\mathcal{E}(h); \{u < r\}) + \mathcal{H}^{n-1}(\mathcal{E}(h) \cap \{u = r\})$$

$$P(\mathcal{E}^r(0)) = P(\mathcal{E}(0); \{u < r\}) + \sum_{h=1}^{N} \mathcal{H}^{n-1}(\mathcal{E}(h) \cap \{u = r\})$$

Then, taking into account also (1.26), we find

$$P(\mathcal{E}; \{u > r\}) < \sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\{u = r\} \cap \mathcal{E}(h)\right) + \frac{m(r)}{4\alpha^{1/n}}, \qquad \text{for a.e. } r < 7n\alpha^{1/n} \tag{1.27}$$

Adding $\frac{1}{2}\sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\{u = r\} \cap \mathcal{E}(h)\right)$ to both the side of (1.27), we have, for a.e. $r < 7n\alpha^{1/n}$,

$$\frac{1}{2}\sum_{h=1}^{N} P(\mathcal{E}(h) \cap \{u > r\}) < \frac{3}{2}\sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\{u = r\} \cap \mathcal{E}(h)\right) + \frac{m(r)}{4\alpha^{1/n}}$$

Using the non-sharp isoperimeteric inequality, we find that

$$\begin{aligned}\frac{1}{2}\sum_{h=1}^{N} P(\mathcal{E}(h)\cap\{u>r\}) &\geq \frac{1}{2}P\left(\bigcup_{h=1}^{N}\mathcal{E}(h)\cap\{u>r\}\right)\\ &\geq \frac{1}{2}\left(\sum_{h=1}^{N}\left|\mathcal{E}(h)\cap\{u>r\}\right|\right)^{(n-1)/n}\\ &= \frac{1}{2}m(r)^{(n-1)/n}\end{aligned}$$

Finally, since m is not increasing, $m(r)\leq m(0)^{1/n}m(r)^{(n-1)/n}\leq \alpha^{1/n}m(r)^{(n-1)/n}$ and so

$$\frac{m(r)}{4\alpha^{1/n}}\leq\frac{m(r)^{(n-1)/n}}{4}$$

Then, taking into account (1.24) and the previous inequalities, we get

$$\frac{m(r)^{(n-1)/n}}{2}<-\frac{3}{2}m'(r)+\frac{m(r)^{(n-1)/n}}{4}$$

and so

$$m(r)^{(n-1)/n}<-6m'(r)$$

for almost every $r<7n\alpha^{1/n}$. As $r_1\geq 7n\alpha^{1/n}$, $m(r)>0$ if $r<7n\alpha^{1/n}$ and then, dividing by $m(r)>0$ and recognising a derivative, we find

$$n\left(m(r)^{1/n}\right)'<-\frac{1}{6}$$

Taking the integral in $(0,7n\alpha^{1/n})$ of the last inequality we have

$$m(7n\alpha^{1/n})^{1/n}-m(0)^{1/n}<-\frac{1}{6n}(7n\alpha^{1/n}-0)$$

and thus

$$\frac{7}{6}\alpha^{1/n}<m(0)^{1/n}-m(7n\alpha^{1/n})\leq m(0)^{1/n}$$

This contradicts the initial hypothesis that $m(0)\leq\alpha$ and it ends the proof of the lemma.

□

1.3 Volume restoration diffeomorphisms

In this section we are going to prove two important statements. The first one is a lemma that ensures the existence of a family of diffeomorphisms which locally modify $\mathcal{E}$ around an interface point $z\in\mathcal{E}(h,k)$ and with a controlled bound on the variation of the perimeter. Moreover it is provided an estimate of the first order measure variations of the chambers $\{\mathcal{E}'(i)\}_{i=1}^{N}$ inside the ball $B(z,\varepsilon)$.

LEMMA 1.14. *Let $\delta > 0$, $\mathcal{E}$ an N-cluster in $\mathbb{R}^n$, $0 \le h < k \le N$, and $z \in \mathcal{E}(h,k)$ point of null density for $\partial^*\mathcal{E}(j)$ (that is $\theta_{n-1}(\partial^*\mathcal{E}(j))(z) = 0$) for every $j \ne h,k$. Then there exist $\varepsilon(\mathcal{E},z,\delta) > 0$, $\varepsilon_1(\mathcal{E},z,\delta) > 0$, $\varepsilon_2(\mathcal{E},n,\varepsilon_1) > 0$, $C_0(n,\varepsilon_1) > 0$, and a family of diffeomorphisms $\{f_t\}_{|t|<\varepsilon_1}$ such that:*

(i) for $|t| < \varepsilon_1$, $\{x \in \mathbb{R}^n \,|\, x \ne f_t(x)\} \subset\subset B(z,\varepsilon)$;

(ii) if $\mathcal{E}'$ is another N-cluster with $d(\mathcal{E},\mathcal{E}') < \varepsilon_2$, $|t| < \varepsilon_1$, $j \ne h,k$, and $i = 1,\dots,N$, then

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(h)) \cap B(z,\varepsilon)\right| - 1\right| < \delta \tag{1.28}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(k)) \cap B(z,\varepsilon)\right| + 1\right| < \delta \tag{1.29}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(j)) \cap B(z,\varepsilon)\right|\right| < \delta \tag{1.30}$$

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left|f_t(\mathcal{E}'(i)) \cap B(z,\varepsilon)\right|\right| < C_0 \tag{1.31}$$

(iii) given a $\mathcal{H}^{n-1}$-rectifiable set Σ in $\mathbb{R}^n$, it holds

$$\left|\mathcal{H}^{n-1}\left(f_t(\Sigma)\right) - \mathcal{H}^{n-1}\left(\Sigma\right)\right| \le C_0 \mathcal{H}^{n-1}\left(\Sigma\right)|t|, \quad |t| < \varepsilon_1 \tag{1.32}$$

Proof. First step. In this first step we are going to construct the diffeomorphisms $f_t, |t| < \varepsilon_1$. Let's consider $\varepsilon > 0$, $\nu \in S^{n-1}$, $u \in C_c^\infty((-n^{-1/2}, n^{-1/2}))$ with $u \ge 0, u(0) > 0$. We define $v(x) = c\prod_{i=1}^N u(x_i)$ with $c \in \mathbb{R}^n$ such that $\int_{\mathbb{R}^{n-1}} v(x',0)\mathrm{d}x' = 1$; by continuity we can always find such constant c. Notice that $v \in C_c^\infty(B_1)$. Then we define $v_\varepsilon \in C_c^\infty(B_\varepsilon)$ as

$$v_\varepsilon(x) = \frac{1}{\varepsilon^{n-1}} v\left(\frac{x}{\varepsilon}\right)$$

It holds

$$\int_{\mathbb{R}^{n-1}} v_\varepsilon(x',0)\mathrm{d}x' = \frac{1}{\varepsilon^{n-1}}\int_{\mathbb{R}^{n-1}} v\left(\frac{x'}{\varepsilon},0\right)\mathrm{d}x' = \int_{\mathbb{R}^{n-1}} v(y',0)\mathrm{d}y' = 1$$

and, for some $C \in \mathbb{R}$,

$$\nabla v_\varepsilon(x) = \frac{1}{\varepsilon^n}(\nabla v)\left(\frac{x}{\varepsilon}\right)$$

$$\left|\nabla v_\varepsilon(x)\right| \le \frac{C}{\varepsilon^n}, \quad x \in \mathbb{R}^n$$

since $v \in C_c^\infty(B_1)$. Now we choose Q_ν, orthogonal $n \times n$ matrix such that $Q_\nu(\nu) = e_n$ and we define $T \in C_c^\infty(B(z,\varepsilon),\mathbb{R}^n)$

$$T(x) \coloneqq T[\varepsilon,z,\nu](x) = v_\varepsilon(Q_\nu(x-z))\nu$$

Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\nabla T(x) &= \nu D(v_\varepsilon(Q_\nu(x-z))) = \nu(Dv_\varepsilon)(Q_\nu(x-z))Q_\nu \\
\left|\nabla T(x)\right| &= \sqrt{\operatorname{trace}(Q_\nu^T\, Dv_\varepsilon(Q_\nu(x-z))^T \nu^T \nu Dv_\varepsilon(Q_\nu(x-z))Q_\nu)} \\
&= \left|Dv_\varepsilon(Q_\nu(x-z))\right| \leq \frac{C}{\varepsilon^n}
\end{aligned}$$

Finally we set

$$f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n, \quad f(t,x) = f_t(x) = x + tT(x)$$

Notice that, if $|t| < \varepsilon_1$ for ε_1 sufficiently small, then $\{f_t\}_{|t|<\varepsilon_1}$ are diffeomorphisms with $\{x \in \mathbb{R}^n \ \ x \neq f_t(x)\} \subset\subset B(z,\varepsilon)$. In fact, since f_t are proper maps (i.e. for every $K \in \mathbb{R}^n$ compact, $f_t^{-1}(K)$ is compact), setting

$$\Sigma_t = \{x \,:\, \det(Df_t(x)) = 0\}$$

it is enough to show that $\Sigma_t = \emptyset$ for $|t| < \varepsilon_1$. As $Df_t(x) = \operatorname{Id} + tDT(x)$, $\lambda = 0$ is an eigenvalue of $Df_t(x)$ if and only if $tDT(x)v = -v$, for some $v \neq 0$. Since $\left|DT(x)\right| \leq \frac{C}{\varepsilon^n}$, every eigenvalue λ of $DT(x)$ satisfies $|\lambda| \leq \frac{C}{\varepsilon^n}$. Then we can determine ε_1 small enough and independent on x, such that $tDT(x)$ does not have -1 as eigenvalue, for every $|t| < \varepsilon_1$. Thus $\Sigma_t = \emptyset$. Finally, it is easy to see that $\{x \in \mathbb{R}^n : x \neq f_t(x)\} \subset\subset B(z,\varepsilon)$ because

$$\{x \in \mathbb{R}^n : x \neq f_t(x)\} = \{x \in \mathbb{R}^n : T(x) \neq 0\}$$

which is compactly contained in $B(z,\varepsilon)$.

Second step. Now choose $z \in \mathcal{E}(h,k)$ for $0 \leq h < k \leq N$ in such a way that (1.10) holds true. Then define $\nu = \nu_{\mathcal{E}(h)}$ and T, f_t exactly as in the previous part. First of all, we notice that if E is a set of finite perimter and Σ is a $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ then we have:

$$\begin{aligned}
\left|f_t(E)\cap B(z,\varepsilon)\right| &= \left|f_t(E\cap B(z,\varepsilon))\right| = \int_{E\cap B(z,\varepsilon)} Jf_t(x)\mathrm{d}x \\
\mathcal{H}^{n-1}\left(f_t(\Sigma)\right) &= \int_\Sigma J^\Sigma f_t(x)\mathrm{d}\mathcal{H}^{n-1}(x)
\end{aligned}$$

Moreover $Jf_t, J^\Sigma f_t$ are smooth function in t. Now we claim that, if V is an hyperplane in $\mathbb{R}^n$ and S is an $n\times n$ matrix, the map

$$S \mapsto J^V(\operatorname{Id}+S) = \sqrt{\det\left((\operatorname{Id}+S)_V^T(\operatorname{Id}+S)_V\right)}$$

is locally Lipschitz, uniformly on V. We will need a weaker statement and now we prove it. Since $S \mapsto \Phi(S) = \det\Big((\operatorname{Id}+S)_V^T(\operatorname{Id}+S)_V\Big)$ is a polynomial

with respect to the entries of S, it is clearly a C^∞ function of S. Then, as $\Phi(0) = 1$, there exists δ' such that $\left|\Phi(S) - 1\right| < 1/2$ for every $|S| < \delta'$. This means that if the norm of S is sufficiently small, the value of $\Phi(S)$ is close to 1. Moreover, since $x \mapsto \sqrt{x}$ is C^∞ in $(0, +\infty)$, then the function $J^V(\mathrm{Id} + S)$ is C^∞ for $|S| < \delta'$, and so it is Lipschitz, with Lipschitz constant L_V.
Now we prove that we can find a constant L greater than every Lipschitz constant L_V. In fact let Q_V the orthonormal matrix that moves the basis $\{e_1, \dots, e_{n-1}\}$ into the orthonormal basis $\{v_1, \dots, v_{n-1}\}$ of V, hyperplane of $\mathbb{R}^n$. Then

$$(\mathrm{Id} + S)_V = (\mathrm{Id} + S)Q_V$$

and

$$(\mathrm{Id} + S)_V^T(\mathrm{Id} + S)_V = Q_V^T(\mathrm{Id} + S)^T(\mathrm{Id} + S)Q_V$$

As $\Phi_Q(S) = \det\Big((\mathrm{Id} + S)_V^T(\mathrm{Id} + S)_V\Big)$ is a polynomial with respect to the entries of Q, which staisfies $\left|q_{ij}\right| \le 1$, $Q = (q_{ij})$, then we can find a majorant of $\Phi_Q'(S)$ independent on Q. Thus the constant L is determined. By the claim we get

$$\begin{aligned}
\left|J^\Sigma f_t(x) - 1\right| &= \left|J^\Sigma(\mathrm{Id} + tT(x)) - J^\Sigma(\mathrm{Id})\right| \\
&= \left|J^{T_x\Sigma}(\mathrm{Id} + tT(x)) - J^{T_x\Sigma}(\mathrm{Id})\right| \\
&\le L\left|tT(x)\right| \le L\frac{C}{\varepsilon^n}\,|t| =: C_0\,|t|
\end{aligned}$$

Then we can conclude that

$$\begin{aligned}
\left|\mathcal{H}^{n-1}\left(f_t(\Sigma)\right) - \mathcal{H}^{n-1}(\Sigma)\right| &\le \int_\Sigma \left|J^\Sigma f_t(x) - 1\right| \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\le C_0\mathcal{H}^{n-1}(\Sigma)\,|t|
\end{aligned}$$

which proves (1.32).

Now let's prove *(i)*. If $|t| < \varepsilon_1$, f_t is a diffeomorphism with $f_t(x) = x$ if $x \notin B(z, \varepsilon)$. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}Jf_t = \frac{\mathrm{d}^2}{\mathrm{d}t^2}Jf_t = 0, \quad \text{on } \mathbb{R}^n \setminus B(z, \varepsilon) \\
&\sup_{x\in\mathbb{R}^n}\left|\frac{\mathrm{d}}{\mathrm{d}t}Jf_t\right| + \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}Jf_t\right| \le C', \quad \text{for some } C' > 0
\end{aligned}$$

Thus we find that

$$\begin{aligned}
\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left|f_t(\mathcal{E}'(i))\cap B(z,\varepsilon)\right|\right| &= \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left|f_t(\mathcal{E}'(i)\cap B(z,\varepsilon))\right|\right| \\
&= \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathcal{E}'(i)\cap B(z,\varepsilon)} Jf_t(x)\mathrm{d}x\right| \\
&= \left|\int_{\mathcal{E}'(i)\cap B(z,\varepsilon)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} Jf_t(x)\mathrm{d}x\right| \\
&\le C'\left|\mathcal{E}'(i)\cap B(z,\varepsilon)\right| \le C'\left|B(z,\varepsilon)\right| \le C_0
\end{aligned}$$

up to increasing the previous value of $C_0 = C_0(n,\varepsilon)$. This proves (1.31).

Now we are going to do a simplification. In fact we notice that

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(i))\cap B(z,\varepsilon)\right| - \frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(i))\cap B(z,\varepsilon)\right|\right| \\
&= \left|\int_{\mathcal{E}'(i)\cap B(z,\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}t} Jf_t(x)\mathrm{d}x - \int_{\mathcal{E}(i)\cap B(z,\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}t} Jf_t(z)\mathrm{d}x\right| \\
&= \left|\int_{(\mathcal{E}'(i)\setminus\mathcal{E}(i))\cap B(z,\varepsilon)} Jf_t(x)\mathrm{d}x - \int_{(\mathcal{E}(i)\setminus\mathcal{E}'(i))\cap B(z,\varepsilon)} Jf_t(x)\mathrm{d}x\right| \\
&\le C'\left(\left|(\mathcal{E}'(i)\setminus\mathcal{E}(i))\cap B(z,\varepsilon)\right| + \left|(\mathcal{E}(i)\setminus\mathcal{E}'(i))\cap B(z,\varepsilon)\right|\right) \\
&\le C'\left|\mathcal{E}(i)\Delta\mathcal{E}'(i)\right| \le C' d(\mathcal{E},\mathcal{E}')
\end{aligned}$$

Then, it is enough to prove estimations (1.28)-(1.30) with $\mathcal{E} = \mathcal{E}'$. In fact if these inequalities are proved for $\mathcal{E} = \mathcal{E}'$, by the triangular inequality and decreasing the value of ε_2, we can see that also the others are true. For example, if

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(h))\cap B(z,\varepsilon)\right| - 1\right| = s\delta < \delta, \quad s\in(0,1)$$

then

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(h))\cap B(z,\varepsilon)\right| - 1\right| \le \\
&\le \left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(h))\cap B_z(\varepsilon)\right| - \frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(h))\cap B_z(\varepsilon)\right|\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(h))\cap B_z(\varepsilon)\right| - 1\right| \\
&\le C' d(\mathcal{E},\mathcal{E}') + s\delta
\end{aligned}$$

So, as $d(\mathcal{E},\mathcal{E}') < \varepsilon_2$, up to decreasing the value of ε_2, we can assume that $C'd(\mathcal{E},\mathcal{E}') + s\delta < \delta$. Moreover, we will prove the inequalities just for $t = 0$.

Indeed, by (1.31) we know that $\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}'(i)) \cap B(z,\varepsilon)\right|$ is Lipschitz in t, for every $i = 1,\dots,N$. Therefore, if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(h)) \cap B(z,\varepsilon)\right|_{|t=0} \in (1-\delta, 1+\delta)$$

then, up to taking ε_1 sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(h)) \cap B(z,\varepsilon)\right| \in (1-\delta, 1+\delta), \qquad \text{for every } |t| < \varepsilon_1$$

Resuming, we prove the estimations of (ii) with $\mathcal{E} = \mathcal{E}', t = 0$. Set $Q_i = Q_{\nu_{\mathcal{E}(i)}}$. Notice that, since $\mathrm{spt}T \subset\subset B(z,\varepsilon)$ and $f_t(E) \cap B(z,\varepsilon)^c = E \cap B(z,\varepsilon)^c$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(E) \cap B(z,\varepsilon)\right|_{|t=0} = \int_{\partial^* E \cap B(z,\varepsilon)} T \cdot \nu_E \,\mathrm{d}\mathcal{H}^{n-1}$$

Then, by the change of variables $x = g(z) \coloneqq z + \varepsilon y$, we have

$$\begin{aligned} T(x) &= v_\varepsilon(Q_h(x-z))\nu_{\mathcal{E}(h)}(z) \\ &= v_\varepsilon(Q_h(\varepsilon y))\nu_{\mathcal{E}(h)}(z) \\ &= \frac{1}{\varepsilon^{n-1}} v(Q_h y)\nu_{\mathcal{E}(h)}(z) \end{aligned}$$

and so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(i)) \cap B(z,\varepsilon)\right|_{|t=0} &= \int_{\partial^*\mathcal{E}(i) \cap B(z,\varepsilon)} T(x) \cdot \nu_{\mathcal{E}(i)}(x) \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \nu_{\mathcal{E}(h)}(x) \cdot \int_{\frac{\partial^*\mathcal{E}(i) - z}{\varepsilon} \cap B} v(Q_h y)\, \nu_{\mathcal{E}(i)}(z + \varepsilon y) \mathrm{d}\mathcal{H}^{n-1}(y) \end{aligned}$$

If $j \neq h, k$, we find

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t(\mathcal{E}(j)) \cap B(z,\varepsilon)\right|_{|t=0}\right| &\le 1 \cdot \left|\int_{\frac{\partial^*\mathcal{E}(j) - z}{\varepsilon} \cap B} v(Q_h y)\, \nu_{\mathcal{E}(j)}(z + \varepsilon y) \mathrm{d}\mathcal{H}^{n-1}(y)\right| \\ &\le \sup_{x \in \mathbb{R}^n} |v(x)| \cdot \int_{\frac{\partial^*\mathcal{E}(j) - z}{\varepsilon} \cap B} \mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \sup_{\mathbb{R}^n} |v|\, \mathcal{H}^{n-1}\left(\frac{\partial^*\mathcal{E}(j) - z}{\varepsilon} \cap B(0,1)\right) \\ &= \sup_{\mathbb{R}^n} |v|\, \mathcal{H}^{n-1}\left(\frac{\partial^*\mathcal{E}(j) \cap B(z,\varepsilon)}{\varepsilon} - z\right) \\ &= \sup_{\mathbb{R}^n} |v|\, \frac{\mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(j) \cap B(z,\varepsilon)\right)}{\varepsilon^{n-1}} \end{aligned}$$

As we have choosen $z \in \mathcal{E}(h,k)$ such that (1.10) holds true, then

$$\frac{\mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(j) \cap B(z,\varepsilon)\right)}{\varepsilon^{n-1}} \to 0, \quad \varepsilon \to 0^+$$

and thus, up to take ε sufficiently small, we have

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \left| f_t(\mathcal{E}(j)) \cap B(z,\varepsilon) \right|_{|t=0} \right| < \delta$$

which proves (1.30). Now let's prove the case $i = h$. We notice that, if $y \in \nu_{\mathcal{E}(h)}(z)^\perp$, then $0 = Q_h(y) \cdot Q_h(\nu_{\mathcal{E}(h)}(z)) = Q_h(y) \cdot e_n$, i.e. $Q_h(y) = (x', 0)$ for some $x' \in \mathbb{R}^{n-1}$. Thus, by this fact and the convergence of the blow-ups, we get:

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} \left| f_t(\mathcal{E}(h)) \cap B(z,\varepsilon) \right|_{|t=0} = \\
&= \lim_{\varepsilon \to 0^+} \nu_{\mathcal{E}(h)}(z) \cdot \int_{\frac{\partial^* \mathcal{E}(h) - z}{\varepsilon} \cap B} v(Q_h y) \nu_{\mathcal{E}(h)}(z + \varepsilon y) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \nu_{\mathcal{E}(h)}(z) \cdot \int_{\nu_{\mathcal{E}(h)}(z)^\perp \cap B} v(Q_h y) \nu_{\mathcal{E}(h)}(z) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \int_{\nu_{\mathcal{E}(h)}(z)^\perp \cap B} v(Q_h y) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \int_{\mathbb{R}^{n-1}} v(x', 0) \mathrm{d}x' = 1
\end{aligned}$$

and so also (1.28) is proved. Finally, recalling that $\nu_{\mathcal{E}(h)}(z) = -\nu_{\mathcal{E}(k)}(z)$ and $\nu_{\mathcal{E}(h)}(z)^\perp = \nu_{\mathcal{E}(k)}(z)^\perp$, we find

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} \left| f_t(\mathcal{E}(k)) \cap B(z,\varepsilon) \right|_{|t=0} = \\
&= \lim_{\varepsilon \to 0^+} \nu_{\mathcal{E}(h)}(z) \cdot \int_{\frac{\partial^* \mathcal{E}(k) - z}{\varepsilon} \cap B} v(Q_h(y)) \nu_{\mathcal{E}(k)}(z + \varepsilon y) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \nu_{\mathcal{E}(h)}(z) \cdot \int_{\nu_{\mathcal{E}(k)}(z)^\perp \cap B} v(Q_h(y)) \nu_{\mathcal{E}(k)}(z) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&= - \int_{\mathbb{R}^{n-1}} v(x', 0) \mathrm{d}x' = -1
\end{aligned}$$

□

Now we are going to state and prove another important theorem. We use the previous lemma in order to find a family of diffeomorphisms with local variations around some interfaces points and indexed on small volume changes. This means that we can modify our clusters, through these diffeomoriphisms which can restore the desired measures. Moreover, it provides a very important bound on the perimeter variation in terms of the relative measures change.

The basic idea of the proof is the following. First of all, we are going to construct the set of interfaces points with a certain procedure. Then, thanks to lemma 1.14, we will define diffeomorphisms $\Psi(\cdot, \cdot)$ indexed on a generic variable t. The key idea will be to define some maps $\psi_h(t)$ as the difference between the measures of $\Psi(t, \mathcal{E}'(h))$ and $\mathcal{E}'(h)$ and to provide the inverse maps φ of these functions. Then we will place $t = \varphi(a), a \in \mathbb{R}^{N+1}$.

Define by convenience $V = \{a \in \mathbb{R}^{N+1} : \sum_{h=0}^{N} a(h) = 0\}$

THEOREM 1.15. *Let $\mathcal{E}$ be an N-cluster in $\mathbb{R}^n$. There exist $\eta, \varepsilon_1, \varepsilon_2, C_1, R > 0$ such that for every N-cluster $\mathcal{E}'$ in $\mathbb{R}^n$ with $d(\mathcal{E}, \mathcal{E}') < \varepsilon_2$ the following property holds. There exists a C^1 map $\Phi\colon ((-\eta, \eta)^{N+1} \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$ such that:*

i) for every $a \in (-\eta, \eta)^{N+1} \cap V$, the diffeomorphism $\Phi(a, \cdot)$ is "supported" in an union of disjoint balls centered in intereface points $\{z_\alpha\}_{\alpha=1}^{M}$ of $\mathcal{E}$: namely

$$\{x \in \mathbb{R}^n : x \neq \Phi(a, x)\} \subseteq \bigcup_{\alpha=1}^{M} B(z_\alpha, \varepsilon_1) \subseteq B_R$$

with $\left|z_\alpha - z_\beta\right| > \varepsilon_1$ for each $1 \leq \alpha < \beta \leq M$, $N \leq M \leq 2N^2$;

ii) for every $a \in (-\eta, \eta)^{N+1} \cap V$, $\left|\Phi(a, \mathcal{E}'(h)) \cap B_R\right| = \left|\mathcal{E}'(h) \cap B_R\right| + a(h)$;

iii) for every $a \in (-\eta, \eta)^{N+1} \cap V$ and $\mathcal{H}^{n-1}$-rectifiable set Σ,

$$\left|\mathcal{H}^{n-1}\left(\Phi(a, \Sigma)\right) - \mathcal{H}^{n-1}\left(\Sigma\right)\right| \leq C_1 \mathcal{H}^{n-1}\left(\Sigma\right) \sum_{h=0}^{N} \left|a(h)\right|$$

iv) choosen another family of interface points $\{y_\alpha\}_{\alpha=1}^{M}$ with y_α, z_α belonging to the same interface of $\mathcal{E}$ and $\left|y_\alpha - y_\beta\right| > \varepsilon_1$, then there exist positive constants $\eta', \varepsilon'_1, \varepsilon'_2, C'_1, R'$ such that all the previous statements hold true with $\eta', \varepsilon'_1, \varepsilon'_2, C'_1, R', \{y_\alpha\}$ in place of $\eta, \varepsilon_1, \varepsilon_2, C_1, R, \{z_\alpha\}$.

Proof. Step one. We start the proof giving some useful definitions. We say that $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are *neighboring chambers* if $\mathcal{H}^{n-1}\left(\mathcal{E}(h, k)\right) > 0$. We say that $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are *linked chambers* if there is a sequence of neighboring chambers that starts with $\mathcal{E}(h)$ and ends with $\mathcal{E}(k)$. We call it linking sequence. If $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are linked chambers, their *order of link* is the minimum cardinality of a linking sequence.

Let us prove that every chamber is linked with $\mathcal{E}(0)$. First of all, there is at least one chamber neighboring with $\mathcal{E}(0)$. Suppose by contradiction that, for every $h = 1, \ldots, N$, we have $\mathcal{H}^{n-1}\left(\mathcal{E}(h, 0)\right) = 0$. Then, since $\partial^* \mathcal{E}(0) \approx \cup_{h=1}^{N} \mathcal{E}(h, 0)$, we get

$$P\left(\bigcup_{h=1}^{N} \mathcal{E}(h)\right) = P(\mathcal{E}(0)) = \sum_{h=1}^{N} \mathcal{H}^{n-1}\left(\mathcal{E}(h, 0)\right) = 0$$

and so $\left|\bigcup_{h=1}^{N}\mathcal{E}(h)\right| = 0$, which is clearly a contradiction. With a similar argument we prove that every chamber is linked with $\mathcal{E}(0)$. Indeed, let $\Lambda \subset \{1, \ldots, N\}$ the set of the indices k such that $\mathcal{E}(k)$ is linked to $\mathcal{E}(0)$. Since $\mathcal{H}^{n-1}\left(\mathcal{E}(h,k)\right) = 0$ whenever $h \in \Lambda, k \notin \Lambda$ (because $\mathcal{E}(h)$ is linked to $\mathcal{E}(0)$ while $\mathcal{E}(k)$ is not) and

$$\partial^*\left(\bigcup_{h\in\Lambda}\mathcal{E}(h)\right) \approx \bigcup_{h\in\Lambda}\bigcup_{k\notin\Lambda}\mathcal{E}(h,k)$$

then

$$P\left(\bigcup_{h\in\Lambda}\mathcal{E}(h)\right) = 0$$

This implies that $\left|\bigcup_{h\in\Lambda}\mathcal{E}(h)\right| = 0$ and so $\left|\mathcal{E}(h)\right| = 0$ for at least one $h = 1, \ldots, N$: contradiction.
Thus for every $h, k = 1, \ldots, N$, $\mathcal{E}(h)$ is linked to $\mathcal{E}(0)$ and $\mathcal{E}(h)$ is linked to $\mathcal{E}(k)$.

Now we are going to construct the sequence of interface points $\{z_\alpha\}_{\alpha=1}^{M}$. As we know that $\mathcal{E}(0)$ and $\mathcal{E}(1)$ are linked, there is a sequence of pairwise neighboring chambers $\mathcal{E}(h_0), \mathcal{E}(h_1), \ldots, \mathcal{E}(h_{M_1})$ with $h_0 = 0, h_{M_1} = 1, M_1 \leq N$. We can choose points $z_1 \in \mathcal{E}(h_0, h_1)$, $z_2 \in \mathcal{E}(h_1, h_2), \ldots,$ $z_{M_1} \in \mathcal{E}(h_{M_1-1}, h_{M_1})$. Then we determine other points $z_{M_1+1}, z_{M_1+2}, \ldots, z_{2M_1}$ starting from $\mathcal{E}(1)$ and arriving in $\mathcal{E}(0)$ with an analogous procedure. In this way we have constructed the set $\{z_\alpha\}_{\alpha=1}^{2M_1}$. We iterate this procedure with $\mathcal{E}(0), \mathcal{E}(2)$ and find the set of points $\{z_\alpha\}_{\alpha=2M_1+1}^{2M_2}$, then with $\mathcal{E}(0)$ and $\mathcal{E}(3)$, $\mathcal{E}(0)$ and $\mathcal{E}(4)$ up to $\mathcal{E}(0)$ and $\mathcal{E}(N)$. In this way, we have determined the set $\{z_\alpha\}_{\alpha=1}^{M}$, with $N \leq M \leq 2N^2$.

Now, we are going to define the matrix L. Set, for every $\alpha = 1, \ldots, M$, the indices $h(\alpha), k(\alpha) \in \{0, \ldots, N\}$ as the ones for which

- $z_\alpha \in \mathcal{E}(h(\alpha), k(\alpha))$;
- $h(\alpha + 1) = k(\alpha)$ for $1 \leq \alpha \leq M - 1$;
- $h(1) = k(M) = 0$.

In other words, z_α is to be found when we move from $\mathcal{E}(h(\alpha))$ to $\mathcal{E}(k(\alpha))$. Then define

$$L_{i\alpha} = \begin{cases} 1, & \text{if } i = h(\alpha) \\ -1, & \text{if } i = k(\alpha) \\ 0, & \text{otherwise} \end{cases}$$

for $i = 0, \ldots, N$ and $\alpha = 1, \ldots, M$. This means that $L_{i\alpha} = 1$ if and only if $i = h(\alpha)$, that is z_α leaves $\mathcal{E}(i)$.

We claim that the matrix $L = (L_{i\alpha})$ has rank equal to N. We can assume, without loss of generality, that the chambers are labeled in such a way that their orders of link with respect to $\mathcal{E}(0)$ are increasing. In particular the order of $\mathcal{E}(1)$ is 1. Setting, for every $i \in \{1, \dots, N\}$, $\beta(i) \in \{1, \dots, M\}$ as the first column index for which $L_{i\beta(i)} \neq 0$, we state that

$$\begin{array}{ll} L_{i\beta(i)} = -1, & i = 1, \dots, N \\ L_{j\beta(i)} = 0, & \text{if } i+1 \leq j \leq N,\, i \leq N-1 \end{array} \tag{1.33}$$

Notice that $\beta(i)$ represents the first point index $\alpha \in \{1, \dots, M\}$ such that $i = h(\alpha)$ or $i = k(\alpha)$, that is $z_{\beta(i)} = z_\alpha$ enters or exits $\mathcal{E}(i)$. We give a quick proof of (1.33). Consider $i = 1$. Clearly z_1 is the first point which enters $\mathcal{E}(1)$ and so $\beta(1) = 1$. Then we have $L_{1\beta(1)} = -1$ and

$$L = \begin{pmatrix} 1 & -1 & \cdots\cdots \\ -1 & 1 & \cdots\cdots \\ 0 & 0 & \cdots\cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Moreover $L_{i\beta(1)} = 0$ for every $i \geq 2$ because each column has just two non-zero entries. Similar arguments hold true for the other chambers of order 1 $\mathcal{E}(2), \dots, \mathcal{E}(m_1)$.

Now consider $i = m_1 + 1$ and suppose, by convenience, that $\mathcal{E}(m_1 + 1)$ has order 2. For some $l = 1, \dots, m_1$, $\mathcal{E}(l)$ and $\mathcal{E}(m_1 + 1)$ are neighboring chambers. Then, we determine $z_{2m_1+1} \in \mathcal{E}(0, l)$ and $z_{2m_1+2} \in \mathcal{E}(l, m_1 + 1)$. Clearly $\beta(m_1 + 1) = 2m_1 + 2$ and

$$\begin{array}{ll} L_{0,2m_1+1} = 1 & \qquad L_{l,2m_1+2} = 1 \\ L_{l,2m_1+1} = -1 & \qquad L_{m_1+1,2m_1+2} = -1 \end{array}$$

This and the arguments of the previous case prove (1.33).

Now we are going to use (1.33) to prove that $\mathrm{rank} L = N$. For every $h = 1, \dots, N$, we define v_h as $L_{\cdot,\beta(h)} \in \mathbb{R}^{N+1}$, that is the $\beta(h)$-column of L. Then $\{v_h\}_{h=1,\dots,N}$ is a subset of the columns of L. We notice that they are N linearly independent vectors. In fact

$$\{v_h\}_{h=1,\dots,N} = \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ -1 & \cdots & \cdots & \cdots \\ 0 & -1 & \cdots & \cdots \\ 0 & 0 & -1 & \cdots \\ \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

because $L_{i\beta(i)} = -1$. Moreover the columns of L belong to V, because each of them has exactly one component equal to 1 and another equal to -1, and V is an N-dimensional vector space. By this, we conclude that $\mathrm{rank} L = N$.

Step two. In this step we are just going to state a kind of inversion theorem. If $\varepsilon, k, C > 0$ we can find $\eta = \eta(\varepsilon, k, C) > 0$ such that the following property holds. Given $\psi\colon (-\varepsilon,\varepsilon)^M \to V$ with $\psi(0) = 0$, $\nabla\psi(0)$ of rank N and

$$\begin{aligned} \left|\nabla\psi(0)w\right| &\geq k\,|w|\,, && \forall w \in W \\ \left|\nabla^2\psi(t)\right| &\leq C, && \forall t \in (-\varepsilon,\varepsilon)^M \end{aligned}$$

where $W = (\ker\nabla\psi(0))^\perp$, then there exists $\varphi : (-\eta,\eta)^{N+1} \cap V \to W$ with

$$\varphi(0) = 0, \qquad \psi(\varphi(a)) = a, \qquad \left|\varphi(a)\right| \leq \frac{2}{k}\,|a|$$

This proposition states that it is possible to have, under suitable condition on the function ψ, a (local) right-side inverse function, defined on the set $(-\eta,\eta)^M \cap V$ which does not depend on the function ψ. Moreover this inverse φ is a Lipschitz continuous function.

Step three. Let L be the matrix defined in *Step one*. We know that $\mathrm{rank}L = N$ and that $\mathrm{Im}L = V$. There exists a vector subspace $W \leq \mathbb{R}^M$ where $L_{|W}\colon W \to V$ is an isomorphism. We state that, for every $\delta > 0$ small enough, there exists $k > 0$ such that, if L' is a matrix with $\mathrm{Im}L' = V$ and

$$\left|L_{i\alpha} - L'_{i\alpha}\right| < \delta, \quad i = 1,\ldots,N,\ \alpha = 1,\ldots,M$$

then, setting $W' = (\ker L')^\perp$, $L'_{|W'} : W' \to V$ is an isomorphism and

$$\left|L'w\right| \geq k\,|w|\,, \quad w \in W' \tag{1.34}$$

It is easy to see that $W' = \langle w_1,\ldots,w_N\rangle$ is an N-dimensional vector subspace of $\mathbb{R}^M$. Thus in order to prove that $L'_{|W'}$ is an isomorphism, it is enough to show that $\{L'w_1,\ldots,L'w_N\}$ are linearly independent. Finally, condition (1.34) is equivalent to

$$\left|L'u\right| \geq k, \quad u \in W', |u| = 1$$

that is $\left|L'\cdot\right|$ has a minimum $k > 0$ independent on δ. This is true because we can bound, with a simple computation, the difference of the minima $m - m'$, respectively of $|L\cdot|$ and $\left|L'\cdot\right|$, in terms of δ. Since $m > 0$, then also $m' > 0$ for δ sufficiently close to zero.

Step four. Now we want to construct the diffeomorphism Φ. Applying the previous lemma to each point z_α, we determine positive constants $\varepsilon, \varepsilon_1, \varepsilon_2, C, R$ and diffeomorphisms $\{f_t^\alpha\}_{|t|<\varepsilon}$ such that

$$\{x \in \mathbb{R}^n : x \neq f_t^\alpha(x)\} \subset\subset B(z_\alpha, \varepsilon_1) \subseteq B_R \quad |t| < \varepsilon$$

and, for every N-cluster $\mathcal{E}'$ with $d(\mathcal{E},\mathcal{E}') < \varepsilon_2$, it holds

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t^\alpha(\mathcal{E}'(h(\alpha))) \cap B(z_\alpha,\varepsilon)\right| - 1\right| < \delta \\
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t^\alpha(\mathcal{E}'(k(\alpha))) \cap B(z_\alpha,\varepsilon)\right| + 1\right| < \delta \\
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left|f_t^\alpha(\mathcal{E}'(j)) \cap B(z_\alpha,\varepsilon)\right|\right| < \delta, \quad j \neq h(\alpha), k(\alpha) \\
&\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left|f_t^\alpha(\mathcal{E}'(i)) \cap B(z_\alpha,\varepsilon)\right|\right| < C, \quad i = 0,\dots,N
\end{aligned} \tag{1.35}$$

for $\alpha = 1,\dots,M$ and $|t| < \varepsilon$. Moreover

$$\left|\mathcal{H}^{n-1}\left(f_t^\alpha(\Sigma)\right) - \mathcal{H}^{n-1}(\Sigma)\right| < C\mathcal{H}^{n-1}(\Sigma)\,|t|$$

whenever Σ is an $\mathcal{H}^{n-1}$-rectifiable set of $\mathbb{R}^n$ and $|t| < \varepsilon$. Up to decreasing the value of ε_1, we can assume that $\left|z_\alpha - z_\beta\right| > \varepsilon_1$ for $\alpha \neq \beta$; thus $\{B(z_\alpha,\varepsilon_1)\}_\alpha$ are disjoint balls lying at mutually positive distance.

Let us define $\Psi\colon (-\varepsilon,\varepsilon)^M \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Psi(t,x) = f_{t_1}^1 \circ f_{t_2}^2 \circ \dots \circ f_{t_M}^M(x), \quad t = (t_1,\dots,t_M)$$

Since variations of the $\{f_t^\alpha\}_\alpha$ are on disjoint balls, it holds

$$\Psi(t,x) = f_{t_\alpha}^\alpha(x), \quad \text{if } x \in B(z_\alpha,\varepsilon_1)$$

Moreover notice that $\{x \in \mathbb{R}^n : x \neq \Psi(t,x)\} \subseteq \bigcup_{\alpha=1}^M B(z_\alpha,\varepsilon_1)$.

Let us fix a cluster $\mathcal{E}'$. We define the map $\psi\colon (-\varepsilon,\varepsilon)^M \to V$ as

$$\begin{aligned}
\psi_h(t) &= \left|\Psi(t,\mathcal{E}'(h)) \cap B_R\right| - \left|\mathcal{E}'(h) \cap B_R\right| \\
&= \sum_{\alpha=1}^M \left(\left|f_{t_\alpha}^\alpha(\mathcal{E}'(h)) \cap B(z_\alpha,\varepsilon_1)\right| - \left|\mathcal{E}'(h) \cap B(z_\alpha,\varepsilon_1)\right|\right)
\end{aligned}$$

for $h = 0,\dots,N$. What follows holds:

- $\psi(0) = 0$.
- $\left|\nabla^2\psi(t)\right| \leq C$ for every $t \in (-\varepsilon,\varepsilon)^M$. In fact

$$\begin{aligned}
\frac{\partial\psi_h(t)}{\partial t_\alpha} &= \frac{\partial}{\partial t_\alpha}\left|f_{t_\alpha}^\alpha(\mathcal{E}'(h)) \cap B(z_\alpha,\varepsilon_1)\right| \\
\frac{\partial^2\psi_h(t)}{\partial t_\beta \partial t_\alpha} &= \begin{cases} \dfrac{\partial^2}{\partial t_\alpha^2}\left|f_{t_\alpha}^\alpha(\mathcal{E}'(h)) \cap B(z_\alpha,\varepsilon_1)\right| \leq C, & \text{if } \beta = \alpha \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

- Since

$$D\psi(0) = \left(\frac{\partial \psi_h(0)}{\partial t_\alpha} \right)_{h,\alpha}$$

and by (1.35) we know that $L' = D\psi(0)$ satisfies

$$\left| L_{i\alpha} - L'_{i\alpha} \right| < \delta, \quad i = 0, \ldots, N,\ \alpha = 1, \ldots, M$$

Moreover $\mathrm{Im} L' = V$.

- Then, by *Step three* we conclude that $\left| L'w \right| \geq k \left| w \right|$ for every $w \in W'$ and for some positive constant k.

These four points state the validity of the hypotheses of *Step two* proposition. Thus there exist $\eta > 0$ and $\varphi \colon (-\eta, \eta)^{N+1} \cap V \to \mathbb{R}^M$ such that $\varphi(0) = 0$ and

$$\psi(\varphi(a)) = a, \qquad \left| \varphi(a) \right| \leq \frac{2}{k} \left| a \right|$$

for every $a \in (-\eta, \eta)^{N+1} \cap V$. Finally define $\Phi \colon ((-\eta, \eta)^M \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi(a, x) = \Psi(\varphi(a), x)$$

We deduce the following assertions.

i) Clearly $\{x \neq \Phi(a, x)\} \subseteq \bigcup_{\alpha=1}^M B(z_\alpha, \varepsilon_1) \subset\subset B_R$ for some $\varepsilon_1 > 0$ and $\{z_\alpha\}_{\alpha=1}^M$ with $\left| z_\alpha - z_\beta \right| > \varepsilon_1$ if $\alpha \neq \beta$.

ii) Since $\psi_h(t) = \left| \Psi(t, \mathcal{E}'(h)) \cap B_R \right| - \left| \mathcal{E}'(h) \cap B_R \right|$, then we have

$$\begin{aligned} \left| \Phi(a, \mathcal{E}'(h)) \cap B_R \right| &= \psi_h(\varphi(a)) + \left| \mathcal{E}'(h) \cap B_R \right| \\ &= a(h) + \left| \mathcal{E}'(h) \cap B_R \right| \end{aligned}$$

for every $a \in (-\eta, \eta)^{N+1} \cap V$.

iii) If Σ is a $\mathcal{H}^{n-1}$-rectifiable set of $\mathbb{R}^n$ then

$$\begin{aligned}
&\left|\mathcal{H}^{n-1}\left(\Phi(a,\Sigma)\right)-\mathcal{H}^{n-1}\left(\Sigma\right)\right| = \left|\mathcal{H}^{n-1}\left(\Psi(\varphi(a),\Sigma)\right)-\mathcal{H}^{n-1}\left(\Sigma\right)\right| \\
&= \left|\mathcal{H}^{n-1}\left(\bigcup_{\alpha=1}^{M} f^{\alpha}_{\varphi_\alpha(a)}(\Sigma)\cap B(z_\alpha,\varepsilon_1)\right) - \mathcal{H}^{n-1}\left(\bigcup_{\alpha=1}^{M}\Sigma\cap B(z_\alpha,\varepsilon_1)\right)\right| \\
&= \left|\sum_{\alpha=1}^{M}\left(\mathcal{H}^{n-1}\left(f^{\alpha}_{\varphi_\alpha(a)}\cap B(z_\alpha,\varepsilon_1)\right)-\mathcal{H}^{n-1}\left(\Sigma\cap B(z_\alpha,\varepsilon_1)\right)\right)\right| \\
&\leq C\sum_{\alpha=1}^{M}\mathcal{H}^{n-1}\left(\Sigma\cap B(z_\alpha,\varepsilon_1)\right)\left|\varphi_\alpha(a)\right| \\
&\leq C\sqrt{M}\left|\varphi(a)\right|\mathcal{H}^{n-1}\left(\Sigma\right) \\
&\leq C\sqrt{M}\frac{2}{k}\mathcal{H}^{n-1}\left(\Sigma\right)|a| \\
&\leq C_1\mathcal{H}^{n-1}\left(\Sigma\right)\sum_{h=0}^{N}\left|a(h)\right|
\end{aligned}$$

for some $C_1 > 0$.

iv) Following the contruction made in this proof it is easily seen that these conclusions hold true also for other interface points $\{y_\alpha\}_\alpha$ with other positive constant $\eta', \varepsilon_1', \varepsilon_2', C_1', R'$.

□

With the following corollary, we want to generalize the results of the previous theorem inside an open set A.

COROLLARY 1.16. *Let $\mathcal{E}$ a N-cluster and $A \subseteq \mathbb{R}^n$ an open set. Assume that for every $h = 1, \dots, N$ there exists a connected component $\tilde{A}$ of A such that*

$$\left|\tilde{A}\cap\mathcal{E}(0)\right| > 0, \qquad \left|\tilde{A}\cap\mathcal{E}(h)\right| > 0 \tag{1.36}$$

Then the conclusions of the previous theorem keep holding true with the points $\{z_\alpha\}$ satisfying

$$B(z_\alpha,\varepsilon_1) \subset\subset A$$

for each $\alpha = 1, \dots, M$.

In fact it is enough to show that (1.36) implies that $\mathcal{E}(0)$ and $\mathcal{E}(h)$ are linked in $\tilde{A}$. This means that there exists a sequence of neighboring chambers in $\tilde{A}$, i.e. $\mathcal{E}(h_0), \mathcal{E}(h_1), \dots, \mathcal{E}(h_M)$ with $h_0 = 0, h_M = h$ and

$$\mathcal{H}^{n-1}\left(\mathcal{E}(h_i,h_{i+1})\cap\tilde{A}\right) > 0, \quad i = 0,\dots,M-1$$

This is proved in lemma 1.17. The proof is based on three steps with the repetition of the same argument.

LEMMA 1.17. *Given a connected open set $A \subseteq \mathbb{R}^n$ and an N-cluster $\mathcal{E}$ such that*

$$\big|\mathcal{E}(h) \cap A\big| > 0, \quad \big|\mathcal{E}(k) \cap A\big| > 0$$

then $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are linked in A.

Proof. Without loss of generality, we can assume that $\big|\mathcal{E}(i) \cap A\big| > 0$ for every $i = 1, \dots, N$. Let $h = 0, k = 1$ and define for $i = 0, 1$, $\Lambda_i \subseteq \{0, \dots, N\}$ as the set of the indices of the chambers linked with $\mathcal{E}(i)$ in A. Then $0 \notin \Lambda_0$ and $1 \notin \Lambda_1$. Let us assume by contradiction that $\mathcal{E}(0), \mathcal{E}(1)$ are not linked in A; this implies that $0 \notin \Lambda_1$, $1 \notin \Lambda_0$. Then $\Lambda_0, \Lambda_1 \subseteq \{2, \dots, N\}$ and they are disjoint.

1. We claim that $\Lambda_0 \neq \emptyset$. If $\Lambda_0 = \emptyset$, then for every $h = 1, \dots, N$, $\mathcal{H}^{n-1}\big(A \cap \mathcal{E}(h, 0)\big) = 0$ and so

   $$P(\mathcal{E}(0); A) = \sum_{h=1}^{N} \mathcal{H}^{n-1}\big(\mathcal{E}(h, 0) \cap A\big) = 0$$

   This implies that either $\big|A \setminus \mathcal{E}(0)\big| = 0$ or $\big|\mathcal{E}(0) \cap A\big| = 0$. In both cases we have contradictions, because we would have either $\big|\mathcal{E}(1) \cap A\big| = 0$ or $\big|\mathcal{E}(0) \cap A\big| = 0$. Similarly $\Lambda_1 \neq \emptyset$.

2. Now we prove that $\Lambda_0 \cup \Lambda_1 = \{2, \dots, N\}$. In fact if $\Lambda' = \{2, \dots, N\} \setminus (\Lambda_0 \cup \Lambda_1) \neq \emptyset$ then

   $$P\Big(\bigcup_{h \in \Lambda'} \mathcal{E}(h); A\Big) = \sum_{h \in \Lambda', k \notin \Lambda'} \mathcal{H}^{n-1}\big(\mathcal{E}(h, k) \cap A\big) = 0$$

   Thus we would have either

   $$\Bigg|A \cap \bigcup_{h \in \Lambda'} \mathcal{E}(h)\Bigg| = 0, \quad \text{or} \quad \Bigg|A \setminus \bigcup_{h \in \Lambda'} \mathcal{E}(h)\Bigg| = 0$$

   which leads to contradictions. This implies that $\Lambda_0 \cup \Lambda_1 = \{2, \dots, N\}$.

3. Now consider $E = \bigcup_{h \in \{0\} \cup \Lambda_0} \mathcal{E}(h)$. We have

   $$P(E_0; A) = \sum_{h \in \{0\} \cup \Lambda_0} \sum_{\{1\} \cup \Lambda_1} \mathcal{H}^{n-1}\big(\mathcal{E}(h, k) \cap A\big) = 0$$

   and so either

   $$|A \cap E_0| = 0, \quad \text{or} \quad \big|A \setminus E_0\big| = 0$$

   In the first case we would have $\big|A \cap \mathcal{E}(0)\big| = 0$, while in the second one $\big|A \cap \mathcal{E}(1)\big| = 0$, which are contradictions.

This ends the proof of the lemma.

□

The following is the last statement necessary to prove the existence of the minimizing clusters. It is a corollary of the previous theorem and it asserts that, if $\mathcal{E}'$ and $\mathcal{F}$ are clusters which differ in a small ball centered at an arbitrary point x and $\mathcal{E}'$ is sufficiently close to $\mathcal{E}$ in measure, then it is possible to determine another cluster $\mathcal{F}'$ with the same enclosed volumes of $\mathcal{E}'$ and with controlled variation of the perimeter w.r.t. $\mathcal{F}'$.

COROLLARY 1.18. *Let $\mathcal{E}$ an N-cluster in $\mathbb{R}^n$. Then there exist $r, \varepsilon, C > 0$ such that the following property holds. For every $\mathcal{E}', \mathcal{F}$ N-clusters and for every $x \in \mathbb{R}^n$ such that*

$$\begin{aligned}
&d(\mathcal{E}, \mathcal{E}') < \varepsilon \\
&\mathcal{F}(h) \Delta \mathcal{E}'(h) \subset\subset B(x, r), \quad h = 1, \dots, N
\end{aligned}$$

there exists another N-cluster $\mathcal{F}'$ with

$$\mathcal{F}'(h) \Delta \mathcal{F}(h) \subset\subset \mathbb{R}^n \setminus \bar{B}(x, r), \quad h = 1, \dots, N \tag{1.37}$$

$$\mathbf{m}(\mathcal{F}) = \mathbf{m}(\mathcal{F}') \tag{1.38}$$

$$\left| P(\mathcal{F}) - P(\mathcal{F}') \right| \le C P(\mathcal{E}') \left| \mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E}') \right| \tag{1.39}$$

In particular, if $\mathcal{E}$ is a minimizing cluster and $\mathcal{F}(h) \Delta \mathcal{E}(h) \subset\subset B(x, r)$ for every $h = 1, \dots, N$ then

$$P(\mathcal{E}) \le P(\mathcal{F}) + C \left| \mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E}) \right|$$

Proof. The basic idea of this proof is that, thanks to the previous theorem, we can modify a cluster in such a way that the chambers volumes are fixed.

We know that there exist $\eta, \varepsilon_1, \varepsilon_2, C, \{z_\alpha\}_\alpha, \{y_\alpha\}_\alpha$ with the properties of theorem 1.15. If $\mathcal{E}'$ is another N-cluster with $d(\mathcal{E}, \mathcal{E}') < \varepsilon_2$ there exist diffeomorphisms $\Phi_1, \Phi_2 \colon ((-\eta, \eta)^{N+1} \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$ such that for every $a \in (-\eta, \eta)^{N+1} \cap V$ it holds

$$\begin{aligned}
\{x \in \mathbb{R}^n : x \neq \Phi_1(a, x)\} &\subset\subset \bigcup_{\alpha=1}^{M} B(z_\alpha, \varepsilon_1) \\
\{x \in \mathbb{R}^n : x \neq \Phi_2(a, x)\} &\subset\subset \bigcup_{\alpha=1}^{M} B(y_\alpha, \varepsilon_1)
\end{aligned}$$

Up to decreasing the value of ε_1, fixed $x \in \mathbb{R}^n$ and $r < \varepsilon_1/2$, we have either

$$B(z_\alpha, \varepsilon_1) \cap B(x, r) = \emptyset, \quad \alpha = 1, \dots, M$$

or

$$B(y_\alpha,\varepsilon_1)\cap B(x,r)=\emptyset,\quad \alpha=1,\dots,M$$

Without loss of generality, we can assume to be in the first case. Up to decreasing the value of r in such a way that $\omega_n r^n<\eta$, setting $a(h):=\big|\mathcal{E}'(h)\cap B(x,r)\big|-\big|\mathcal{F}(h)\cap B(x,r)\big|$ and $a=(a(h))_{h=0,\dots,N}$, we have

$$a\in(-\eta,\eta)^{N+1}\cap V$$

In fact, clearly $a(h)\in(-\eta,\eta)$, since $\big|B(x,r)\big|<\eta$ and $a\in V$ because

$$\begin{aligned}\sum_{h=0}^{N}a(h)&=\sum_{h=0}^{N}\big|\mathcal{E}'(h)\cap B(x,r)\big|-\sum_{h=0}^{N}\big|\mathcal{F}(h)\cap B(x,r)\big|\\&=\big|B(x,r)\big|-\big|B(x,r)\big|=0\end{aligned}$$

Then we can modify the cluster $\mathcal{F}$ through $\Phi_1(a,\cdot)$; namely we define

$$\mathcal{F}'(h)=\Phi_1(a,\mathcal{F}(h))$$

for every $h=1,\dots,N$. In this way we get what follows.

- $\mathcal{F}'(h)\Delta\mathcal{F}(h)\subset\subset\bigcup_{\alpha=1}^{N}B(z_\alpha,\varepsilon_1)\subseteq\mathbb{R}^n\setminus\bar{B}(x,r)$ and so (1.37) is proved.

- The volumes are preserved. In fact

$$\begin{aligned}\big|\mathcal{F}'(h)\big|&=\big|\Phi_1(a,\mathcal{F}(h))\setminus B(x,r)\big|+\big|\mathcal{F}(h)\cap B(x,r)\big|\\&=\big|\Phi_1(a,\mathcal{E}'(h))\setminus B(x,r)\big|+\big|\mathcal{F}(h)\cap B(x,r)\big|\\&=\big|\Phi_1(a,\mathcal{E}'(h))\big|-\big|\mathcal{E}'(h)\cap B(x,r)\big|+\big|\mathcal{F}(h)\cap B(x,r)\big|\\&=\big|\mathcal{E}'(h)\big|+a(h)-\big|\mathcal{E}'(h)\cap B(x,r)\big|+\big|\mathcal{F}(h)\cap B(x,r)\big|\\&=\big|\mathcal{E}'(h)\big|\end{aligned}$$

  We have used the fact that $\mathcal{F}(h)$ and $\mathcal{E}'(h)$ do not differ outside $B(x,r)$ and

$$\big|\Phi_1(a,\mathcal{E}'(h))\big|=a(h)+\big|\mathcal{E}'(h)\big|$$

- Finally we have to prove (1.39). It holds

$$
\begin{aligned}
P(\mathcal{F}') - P(\mathcal{F}) &= \frac{1}{2}\sum_{h=0}^{N} \mathcal{H}^{n-1}\left(\partial^*\Phi_1(a,\mathcal{F}(h))\right) - \mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\right) \\
&= \frac{1}{2}\sum_{h=0}^{N} \mathcal{H}^{n-1}\left(\Phi_1(a,\partial^*\mathcal{F}(h))\right) - \mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\right) \\
&= \frac{1}{2}\sum_{h=0}^{N}\Big(\mathcal{H}^{n-1}\left(\Phi_1(a,\partial^*\mathcal{F}(h))\cap B(x,r)\right) \\
&\qquad +\mathcal{H}^{n-1}\left(\Phi_1(a,\partial^*\mathcal{F}(h))\setminus B(x,r)\right) \\
&\qquad -\mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\cap B(x,r)\right) \\
&\qquad -\mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\setminus B(x,r)\right)\Big) \\
&= \frac{1}{2}\sum_{h=0}^{N}\Big(\mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\cap B(x,r)\right) \\
&\qquad +\mathcal{H}^{n-1}\left(\Phi_1(a,\partial^*\mathcal{E}'(h))\setminus B(x,r)\right) \\
&\qquad -\mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h)\cap B(x,r)\right) \\
&\qquad -\mathcal{H}^{n-1}\left(\partial^*\mathcal{E}'(h)\setminus B(x,r)\right)\Big) \\
&= \frac{1}{2}\sum_{h=0}^{N}\Big(\mathcal{H}^{n-1}\left(\Phi_1(a,\partial^*\mathcal{E}'(h))\right) - \mathcal{H}^{n-1}\left(\partial^*\mathcal{E}'(h)\right)\Big) \\
&= \frac{1}{2}\sum_{h=0}^{N}\sum_{k\neq h}\Big(\mathcal{H}^{n-1}\left(\Phi_1(a,\mathcal{E}'(h,k))\right) - \mathcal{H}^{n-1}\left(\mathcal{E}'(h,k)\right)\Big) \\
&= \sum_{0\le h<k\le N}\Big(\mathcal{H}^{n-1}\left(\Phi_1(a,\mathcal{E}'(h,k))\right) - \mathcal{H}^{n-1}\left(\mathcal{E}'(h,k)\right)\Big)
\end{aligned}
$$

and so we get

$$
\begin{aligned}
\left|P(\mathcal{F}) - P(\mathcal{F}')\right| &\le \sum_{0\le h<k\le N}\Big|\mathcal{H}^{n-1}\left(\Phi_1(a,\mathcal{E}'(h,k))\right) - \mathcal{H}^{n-1}\left(\mathcal{E}'(h,k)\right)\Big| \\
&\le C_0 \sum_{0\le h<k\le N}\mathcal{H}^{n-1}\left(\mathcal{E}'(h,k)\right)\sum_{j=0}^{N}\left|a(j)\right| \\
&= C_0 P(\mathcal{E}')\sum_{j=0}^{N}\left|a(j)\right|
\end{aligned}
$$

Since

$$\left|a(0)\right| = \left|-\sum_{i=1}^{N} a(i)\right| \leq \sum_{i=1}^{N} \left|\left|\mathcal{F}(i)\right| - \left|\mathcal{E}'(i)\right|\right| \leq \sqrt{N}\left|\mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E}')\right|$$

we can finally conclude that

$$\begin{aligned}\left|P(\mathcal{F}) - P(\mathcal{F}')\right| &\leq C_0 P(\mathcal{E}')\sqrt{N}\left|\mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E}')\right| \\ &= C_1 P(\mathcal{E}')\left|\mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E}')\right|\end{aligned}$$

which proves (1.39).

In particular, if $\mathcal{E}$ is a minimizing cluster, choosing $\mathcal{E}' = \mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{F}(h)\Delta\mathcal{E}(h) \subset\subset B(x,r)$, then there exists an N-cluster $\mathcal{F}'$ with

$$P(\mathcal{E}) \leq P(\mathcal{F}') \leq P(\mathcal{F}) + C_1 P(\mathcal{E})\left|\mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E})\right|$$

If $C = \max\{C_1 P(\mathcal{E}), P(\mathcal{E})\}$, then

$$P(\mathcal{E}) \leq P(\mathcal{F}) + C\left|\mathbf{m}(\mathcal{F}) - \mathbf{m}(\mathcal{E})\right|.$$

Thus the proof is completed. □

1.4 Proof of the existence

Proof. Part one. Let us fix a volume vector $\mathbf{m}$. Define the smallest and the biggest volume of $\mathbf{m}$ and perimeter as

$$\begin{aligned}\mathbf{m}_{\min} &= \min\{\mathbf{m}(h) : h = 1, \ldots, N\} \\ \mathbf{m}_{\max} &= \max\{\mathbf{m}(h) : h = 1, \ldots, N\}\end{aligned}$$

and

$$\begin{aligned}p_{\min} &= \inf\{P(\mathcal{E}_k(h)) : k \in \mathbb{N},\, h = 1, \ldots, N\} > 0 \\ p_{\max} &= \sup\{P(\mathcal{E}_k(h)) : k \in \mathbb{N},\, h = 1, \ldots, N\} < \infty\end{aligned}$$

Consider a minimizing sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ for the partitioning problem associated to $\mathbf{m}$. For every $h = 1, \ldots, N$, we define the sequences of points $\{x_k(h)\}_{k\in\mathbb{N}}$ such that

$$\left|\mathcal{E}_k(h) \cap B(x_k(h), 1)\right| \geq \min\left\{c(n)\frac{\mathcal{E}_k(h)}{P(\mathcal{E}_k(h))}, \frac{1}{2n}\right\}^n \geq \min\left\{c(n)\frac{\mathbf{m}_{\min}}{p_{\max}}, \frac{1}{2n}\right\}^n \tag{1.40}$$

This property is ensured by remark 1.12. Define $S > 0$ such that

$$\frac{\omega_n S^n}{2} = \sum_{j=1}^{N} \mathbf{m}(j) = \sum_{j=1}^{N} \left|\mathcal{E}_k(j)\right|$$

Without loss of generality, up to rescaling the volumes $\mathbf{m}(h)$, we can assume $S = 1$. By the just given definition, it follows that $\bigcup_{j=1}^{N}(\mathcal{E}_k(j) \cap B(x_k(h), S))$ has a volume at most half of $B(x_k(h), S)$ and so

$$\left|\mathcal{E}_k(0) \cap B(x_k(h), S)\right| \geq \frac{\omega_n S^n}{2}$$

We say that $\{x_k(h)\}_k$ and $\{x_k(h')\}_k$ are asymptotically close whenever

$$\liminf_{k\to\infty} \left|x_k(h) - x_k(h')\right| < S$$

Moreover we say that they don't tear apart if there exist $\{h_1, \dots, h_l\}$ with $\{x_k(h_i)\}_k, \{x_k(h_{i+1})\}_k$ asymptotically close for $i = 1, \dots, l-1$ and $h_1 = h, h_l = h'$. Then we can partitionate $\{1, \dots, N\}$ into $\{\Lambda_j\}_{j=1}^{s}$ in such a way that for every $j = 1, \dots, s$ and $h, h' \in \Lambda_j$, the sequences $\{x_k(h)\}_k, \{x_k(h')\}_k$ don't tear apart. Up to extracting subsequences, there exists, for $j = 1, \dots, s$ and $h, h' \in \Lambda_j$, the limit

$$\lim_{k\to\infty} x_k(h) - x_k(h') \in B_{NS}$$

For every $j = 1, \dots, s$, we choose $h_j \in \Lambda_j$ and we define, for $h \in \Lambda_j$,

$$\omega(h) = \lim_{k\to\infty} x_k(h) - x_k(h_j)$$

Provided k is large enough, we have

$$\bigcup_{h\in\Lambda_j} B(x_k(h), S) \subset\subset B(x_k(h_j), 2NS)$$

In fact, if $x \in B(x_k(h), S)$ and $\left|x_k(h) - x_k(h_j)\right| \leq NS + 1/2$ for k sufficiently large, then

$$\left|x - x_k(h_j)\right| \leq \left|x - x_k(h)\right| + \left|x_k(h) - x_k(h_j)\right| < 2NS$$

Now we are going to construct a new cluster $\mathcal{E}_k^*$. Let's proceed in this way. We notice that we have determined s pieces for each chamber $\mathcal{E}_k(h)$:

$$\mathcal{E}_k(h) \cap \bigcup_{l\in\Lambda_j} B(x_k(l), S), \quad j = 1, \dots, s$$

Let us define the translation vectors $y_{k,j}$ as

$$v_j = 4(N+1)Sje_n$$
$$y_{k,j} = v_j - x_k(h_j)$$

and the new cluster $\mathcal{E}_k^*$ as

$$\mathcal{E}_k^*(h) = \bigcup_{j=1}^{s}\left(y_{k,j} + \left(\mathcal{E}_k(h) \cap \bigcup_{l\in\Lambda_j} B(x_k(l), S)\right)\right)$$

It is easy to see that since

$$\mathcal{E}_k(h) \cap \bigcup_{h \in \Lambda_j} B(x_k(h), S) \subset\subset B(x_k(h_j), 2NS)$$

then

$$y_{k,j} + \left(\mathcal{E}_k(h) \cap \bigcup_{h \in \Lambda_j} B(x_k(h), S) \right) \subset\subset y_{k,j} + B(x_k(h_j), 2NS) = B(v_j, 2NS) \tag{1.41}$$

Notice that, by the previous choice of the vectors $\{v_j\}_j$, the balls $\{B(v_j, 2NS)\}_j$ are disjoint and at mutually distance $4S$. Recall that, by (1.40), for $h \in \Lambda_j$ we have

$$\begin{aligned} \left|\mathcal{E}_k^*(h) \cap B(x_k(h) + y_{k,j}, S)\right| &\geq \left|\mathcal{E}_k(h) \cap B(x_k(h), S)\right| \\ &\geq \min\left\{ c(n) \frac{\mathbf{m}_{\min}}{p_{\max}}, \frac{1}{2n} \right\}^n > 0 \end{aligned}$$

In particular we have

$$\inf_{k \in \mathbb{N}} \min_{h=1,\dots,N} \left|\mathcal{E}_k^*(h)\right| > 0 \tag{1.42}$$

Moreover

$$\begin{aligned} P(\mathcal{E}_k^*(h)) &\leq \sum_{j=1}^{s} P\Big(\mathcal{E}_k(h) \cap \bigcup_{h \in \Lambda_j} B(x_k(h), S) \Big) \\ &\leq \sum_{j=1}^{s} P(\mathcal{E}_k(h)) + \sum_{j=1}^{s} \sum_{h \in \Lambda_j} P\big(B(x_k(h), S)\big) \\ &\leq s p_{\max} + N n \omega_n S^{n-1} \end{aligned}$$

which implies

$$\sup_{k \in \mathbb{N}} P(\mathcal{E}_k^*) < \infty \tag{1.43}$$

Thus, using (1.41), (1.42), (1.43) and the compactness criterion 1.9, we can state that there exists a minimizing subsequence converging to a certain cluster $\mathcal{E}^*$. By convenience, we denote again with $\{\mathcal{E}_k\}_k$ this subsequence. Moreover we notice that

$$\begin{aligned} \left|\mathcal{E}^*(h) \cap B(v_j + \omega(h), S)\right| &\geq \min\left\{ c(n) \frac{\mathbf{m}_{\min}}{p_{\max}}, \frac{1}{2n} \right\}^n \\ \left|\mathcal{E}^*(0) \cap B(v_j + \omega(h), S)\right| &\geq \frac{\omega_n S^n}{2} \end{aligned}$$

whenever $h \in \Lambda_j$. These inequalities follow from the convergence

$$\mathcal{E}_k^*(h) \cap B(x_k(h) + y_{k,j}, S) \to \mathcal{E}^*(h) \cap B(v_j + \omega(h), S)$$

and the continuity of the norm $|\cdot|$ with respect to the measure convergence. The two inequalities imply that $\mathcal{E}^*(h), \mathcal{E}^*(0)$ are linked in $B(v_j + \omega(h), S)$. Thus consider

$$A = \bigcup_{j=1}^{s} \bigcup_{h \in \Lambda_j} B(v_j + \omega(h), S)$$

Thanks to the restoration volume theorem, we can assert that there exist positive constants ε, C depending just on $\mathcal{E}^*$ and maps

$$\Phi_k \colon ((-\varepsilon, \varepsilon)^{N+1} \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\begin{aligned}
&\{x \in \mathbb{R}^N : x \neq \Phi_k(a, x)\} \subset\subset A \\
&\left|\Phi_k(a, \mathcal{E}_k^*(h)) \cap A\right| = \left|\mathcal{E}_k^*(h) \cap A\right| + a(h) \\
&\left|\mathcal{H}^{n-1}\left(\Phi_k(a, \Sigma)\right) - \mathcal{H}^{n-1}(\Sigma)\right| \leq C\mathcal{H}^{n-1}(\Sigma) \sum_{h=0}^{N} \left|a(h)\right|
\end{aligned}$$

where Σ is an arbitrary $\mathcal{H}^{n-1}$-rectifiable set of $\mathbb{R}^n$. We would like to construct maps Ψ_k with the following properties:

$$\begin{aligned}
&\{x \in \mathbb{R}^n : x \neq \Psi_k(a, x)\} \subset\subset \bigcup_{h=1}^{N} B(x_k(h), S) \\
&\left|\Psi_k(a, \mathcal{E}_k(h))\right| = \left|\mathcal{E}_k(h)\right| + a(h) \\
&\left|\mathcal{H}^{n-1}\left(\Psi_k(a, \Sigma)\right) - \mathcal{H}^{n-1}(\Sigma)\right| \leq C\mathcal{H}^{n-1}(\Sigma) \sum_{h=0}^{N} \left|a(h)\right|
\end{aligned}$$

This is the basic idea. Since, for $h \in \Lambda_j$, it holds

$$B(x_k(h) + y_{k,j}, S) = B(x_k(h) + v_j - x_k(h_j), S) \to B(v_j + \omega(h), S)$$

as $k \to \infty$, and the diffeomorphisms Φ_k have variations on a compact subset of $\bigcup_{j=1}^{s} \bigcup_{h \in \Lambda_j} B(v_j + \omega(h), S)$, then we can define Ψ_k as

$$\Psi_k(a, x) = -y_{k,j} + \Phi_k(a, x + y_{k,j})$$

if $x + y_{k,j} \in \bigcup_{h \in \Lambda_j} B(v_j + \omega(h), S)$.
The map is well defined because, if

$$C_{k,j} = \left\{x \in \bigcup_{h \in \Lambda_j} B(v_j + \omega(h), S) : x \neq \Phi_k(a, x)\right\}$$

then the translated sets $\tilde{C}_{k,j} = C_{k,j} - y_{k,j}$, $j = 1, \ldots, s$ are always disjoint. Indeed, looking at the proof of theorem 1.15, we can choose the diffeomorphisms in such a way that the support is small enough. As a consequence, it is easy to see that the sets $C_{k,i} - y_{k,i}, C_{k,j} - y_{k,j}$ for $i \neq j$ are disjoint. Let us prove the validity of the three above properties.

- We have
$$\{x \in \mathbb{R}^n : x \neq \Psi_k(a,x)\} = \bigcup_{j=1}^{s}(-y_{k,j} + C_{k,j})$$
since $x \neq \Psi_k(a,x)$ if and only if $x + y_{k,j} \neq \Phi_k(a, x + y_{k,j})$ for $x + y_{k,j} \in \bigcup_{h\in\Lambda_j} B(v_j + \omega(h), S)$. Moreover it holds
$$C_{k,j} - y_{k,j} \subset\subset \bigcup_{h\in\Lambda_j} B(v_j + \omega(h), S) - y_{k,j} = \bigcup_{h\in\Lambda_j} B(x_k(h_j) + \omega(h), S)$$
Since for k large $x_k(h) \sim \omega(h) + x_k(h_j)$ (that is they are enough close), then
$$C_{k,j} - y_{k,j} \subset\subset \bigcup_{h\in\Lambda_j} B(x_k(h), S)$$
This proves the first of the three properties.

- Set $\tilde{C}_k = \bigcup_{j=1}^{s} \tilde{C}_{k,j}$ as the "support" of Ψ_k. Let's compute $\left|\Psi_k(a, \mathcal{E}_k(h))\right|$. We have
$$\begin{aligned}
\left|\Psi_k(a, \mathcal{E}_k(h)) \cap \tilde{C}_k\right| &= \left|\Psi_k(a, \mathcal{E}_k(h) \cap \tilde{C}_k)\right| \\
&= \sum_{j=1}^{s} \left|\Phi_k\big(a, (\mathcal{E}_k(h) \cap \tilde{C}_{k,j}) + y_{k,j}\big)\right| \\
&= \left|\bigcup_{j=1}^{s} \Phi_k\big(a, (y_{k,j} + \mathcal{E}_k(h)) \cap C_{k,j}\big)\right| \\
&= a(h) + \left|\bigcup_{j=1}^{s} \big((y_{k,j} + \mathcal{E}_k(h)) \cap C_{k,j}\big)\right| \\
&= a(h) + \sum_{j=1}^{s} \left|\mathcal{E}_k(h) \cap \tilde{C}_{k,j}\right| \\
&= a(h) + \left|\mathcal{E}_k(h) \cap \tilde{C}\right|
\end{aligned}$$

- Let Σ be a $\mathcal{H}^{n-1}$-rectifiable set. Without loss of generality, we assume that Σ is contained in $\tilde{C}_{k,j}$. Then
$$\begin{aligned}
\left|\mathcal{H}^{n-1}\left(\Psi_k(a,\Sigma)\right) - \mathcal{H}^{n-1}(\Sigma)\right| &= \left|\mathcal{H}^{n-1}\left(\Phi_k(a, \Sigma + y_{k,j})\right) - \mathcal{H}^{n-1}\left(\Sigma + y_{k,j}\right)\right| \\
&\leq C\mathcal{H}^{n-1}\left(\Sigma + y_{k,j}\right) \sum_{h=0}^{N} \left|a(h)\right| \\
&= C\mathcal{H}^{n-1}(\Sigma) \sum_{h=0}^{N} \left|a(h)\right|
\end{aligned}$$

Part two. Let ε_0 satisfying the hypotheses of the Nucleation Lemma for every $\mathcal{E}_k(h)$, that is

$$\varepsilon_0 \leq \min\left\{\mathbf{m}_{\min}, \frac{p_{\min}}{2nc(n)}\right\}$$

Then there exist points $\{x_k(h,i)\}_{i=1}^{L(h,k)}$ such that

$$\left|\mathcal{E}_k(h) \setminus \bigcup_{i=1}^{L(h,k)} B(x_k(h,i),2)\right| \leq \varepsilon_0$$

$$L(h,k) \leq \mathbf{m}_{\max}\left(\frac{p_{\max}}{c(n)\varepsilon_0}\right)^n$$

Define

$$F_k = \left(\bigcup_{h=1}^{N} \bar{B}(x_k(h),S)\right) \cup \left(\bigcup_{h=1}^{N}\bigcup_{i=1}^{L(h,k)} \bar{B}(x_k(h,i),2)\right)$$

Since

$$\sum_{h=1}^{N} \left|\mathcal{E}_k(h) \setminus F_k\right| \leq N\varepsilon_0$$

we may apply the Truncation Lemma to $\mathcal{E}_k, F_k, \alpha = N\varepsilon_0$. It follows that there exist $r_k \in [0, 7n(N\varepsilon_0)^{1/n}]$ such that the cluster $\mathcal{E}'$ defined as

$$\mathcal{E}_k'(h) = \mathcal{E}_k(h) \cap \{u_k \leq r_k\}, \quad u_k(x) = \operatorname{dist}(x, F_k)$$

satisfies

$$P(\mathcal{E}_k') \leq P(\mathcal{E}_k) - \frac{d(\mathcal{E}_k, \mathcal{E}_k')}{4(N\varepsilon_0)^{1/n}}$$

We notice that

$$\begin{aligned} d(\mathcal{E}_k, \mathcal{E}_k') &= \sum_{h=1}^{N} \left|\mathcal{E}_k(h) \setminus \mathcal{E}_k'(h)\right| = \sum_{h=1}^{N} \left|\mathcal{E}_k(h) \cap \{u_k > r_k\}\right| \\ &\leq \sum_{h=1}^{N} \left|\mathcal{E}_k(h) \setminus \bigcup_{i=1}^{L(h,k)} \bar{B}(x_k(h,i),2)\right| < N\varepsilon_0 \end{aligned}$$

because $\{u_k > r_k\} \subseteq \mathbb{R}^n \setminus F_k \subseteq \mathbb{R}^n \setminus \bigcup_{i=1}^{L(h,k)} \bar{B}(x_k(h,i),2)$

Now we would like to define new clusters $\mathcal{E}_k''$ in order to restore the chambers measures of $\mathcal{E}_k$, through the diffeomorphisms previously constructed. We set

$$\begin{aligned} a_k(h) &= \left|\mathcal{E}_k(h)\right| - \left|\mathcal{E}_k'(h)\right| = \left|\mathcal{E}_k(h) \cap \{u_k > r_k\}\right|, \quad h = 1, \dots, N \\ a_k(0) &= -\sum_{h=1}^{N} a_k(h) \end{aligned}$$

Provided that ε_0 is small enough, $a_k \in (-\varepsilon_1, \varepsilon_1)^{N+1} \cap V$. Define the new cluster $\mathcal{E}''$ as

$$\mathcal{E}''_k(h) = \Psi_k(a_k, \mathcal{E}'_k(h)), \quad h = 1, \dots, N, \, k \in \mathbb{N}$$

Since $\{x \neq \Psi_k(a_k, x)\} \subset\subset \bigcup_{h=1}^N B(x_k(h), S) \subseteq F_k \subseteq \{u_k \leq r_k\}$, then

$$\Psi_k(a_k, \mathcal{E}_k(h)) \cap \{u_k \leq r_k\} = \Psi_k(a_k, \mathcal{E}_k(h) \cap \{u_k \leq r_k\}) = \Psi_k(a_k, \mathcal{E}'_k(h))$$

In particular, recalling one of the three properties characterizing Ψ_k, we get

$$\begin{aligned} a_k(h) &= \big|\Psi_k(a_k, \mathcal{E}_k(h))\big| - \big|\mathcal{E}_k(h)\big| \\ &= \big|\Psi_k(a_k, \mathcal{E}_k(h)) \cap \{u_k \leq r_k\}\big| - \big|\mathcal{E}_k(h) \cap \{u_k \leq r_k\}\big| \\ &= \big|\Psi_k(a_k, \mathcal{E}'_k(h))\big| - \big|\mathcal{E}'_k(h)\big| \end{aligned}$$

Then we deduce

$$\begin{aligned} \big|\mathcal{E}''_k(h)\big| &= \big|\Psi_k(a_k, \mathcal{E}'_k(h))\big| \\ &= a_k(h) + \big|\mathcal{E}'_k(h)\big| = \big|\mathcal{E}_k(h)\big| \end{aligned}$$

that is

$$\mathbf{m}(\mathcal{E}'') = \mathbf{m}(\mathcal{E})$$

Moreover, we can provide the following perimeter estimation.

$$\begin{aligned} P(\mathcal{E}''_k) &= \sum_{0 \leq h < l \leq N} \mathcal{H}^{n-1}(\Psi_k(a_k, \mathcal{E}'_k(h,l))) \\ &\leq \sum_{0 \leq h < l \leq N} \Big(\mathcal{H}^{n-1}(\mathcal{E}'_k(h,l)) + C\mathcal{H}^{n-1}\left(\mathcal{E}'_k(h,l)\right) \sum_{j=0}^N \big|a(j)\big|\Big) \\ &= P(\mathcal{E}'_k) + C_1 P(\mathcal{E}'_k) \sum_{j=0}^N \big|a(j)\big| \\ &\leq P(\mathcal{E}_k) - \frac{d(\mathcal{E}_k, \mathcal{E}'_k)}{4(N\varepsilon_0)^{1/n}} + C_1 P(\mathcal{E}'_k) \sum_{j=0}^N \big|a(j)\big| \end{aligned}$$

If $P(\mathcal{E}_k) \to \gamma$ then $P(\mathcal{E}_k) \leq 2\gamma$ for k sufficiently large. Moreover

$$\sum_{h=0}^N \big|a_k(h)\big| \leq 2 \sum_{h=1}^N \big|a_k(h)\big| = 2d(\mathcal{E}_k, \mathcal{E}'_k)$$

Thus we can conclude that

$$\begin{aligned} P(\mathcal{E}''_k) &\leq P(\mathcal{E}_k) - \frac{d(\mathcal{E}_k, \mathcal{E}'_k)}{4(N\varepsilon_0)^{1/n}} + 4C_1\gamma d(\mathcal{E}_k, \mathcal{E}'_k) \\ &\leq P(\mathcal{E}_k) + \left(4C_1\gamma - \frac{1}{4(N\varepsilon_0)^{1/n}}\right) d(\mathcal{E}_k, \mathcal{E}'_k) \end{aligned}$$

The last term is smaller than $P(\mathcal{E}_k)$ provided that ε_0 is sufficiently small. Hence $\{\mathcal{E}''_k\}_k$ is a minimizing sequence satisfying the hypotheses of proposition 1.8. Notice that we have just to prove (1.12). This follows from the fact that

$$\mathcal{E}''_k(h) = \Psi_k(a_k, \mathcal{E}'_k(h)) = \Psi_k(a_k, \mathcal{E}_k(h)) \cap \{u_k \leq r_k\} \subseteq \{u_k \leq r_k\}$$

and that

$$\{u_k \leq r_k\} \subseteq \bigcup_{h=1}^{N} \left(\bar{B}(x_k(h), S + r_k) \cup \bigcup_{i=1}^{L(h,k)} \bar{B}(x_k(h,i), 2 + r_k) \right)$$

In fact for every $k \in \mathbb{N}$ we have $r_k \leq 7n(N\varepsilon_0)^{1/n}$ and the number of points $\{x_k(h)\}_h$, $\{x_k(h,i)\}_{h,i}$ can be uniformly bounded in k. Then

$$\mathcal{E}''_k(h) \subseteq \bigcup_{x \in \Omega_k} B(x, R)$$

for some $R > 0$ and sets $\{\Omega_k\}_{k \in \mathbb{N}}$ with uniformly bounded cardinality.

Thus, thank to proposition 1.8, we conclude that $\{\mathcal{E}''_k\}_k$ admits a converging subsequence $\mathcal{E}''_{k(l)} \to \mathcal{E}$. The N-cluster $\mathcal{E}$ is a minimum of the original partitioning problem.

Finally we prove that such a minimizer is bounded. Indeed if $\mathcal{E}$ is not bounded then, applying the previous construction to the sequence $\{\mathcal{E}_k\}_k = \mathcal{E}$, we have another sequence $\{\mathcal{E}''_k\}_k$ of cluster with

$$P(\mathcal{E}''_k) < P(\mathcal{E}_k) = P(\mathcal{E}), \quad \mathbf{m}(\mathcal{E}''_k) = \mathbf{m}(\mathcal{E}_k) = \mathbf{m}$$

which is a contradiction.

□

CHAPTER 2

Regularity and planar cases

In the first chapter we showed the existence of the minimum for a partitioning problem. Now we would like to deepen the topic and see if we are able to get informations about the minimizers.

In the first part of this second chapter, we are going to detail the study of the minimal clusters in $\mathbb{R}^n$, in particular about their regularity. We'll see that the interfaces of a minimal cluster are analytic hypersurfaces of $\mathbb{R}^n$ with constant mean curvature. Later, we are going to focus on some specific cases in $\mathbb{R}^2$. First of all we will prove the regularity properties of the cluster in the plane: the most relevant one is the 120° rule. Then we are going to characterize entirely the 2-minimizing clusters and to provide a symmetry property for the 4-cluster.

2.1 Regularity of the minimizers in $\mathbb{R}^n$

The aim of this section is to prove the following theorem.

Theorem 2.1. *Let $\mathcal{E}$ be a minimizing cluster in $\mathbb{R}^n$. Then for every $0 \leq h < k \leq N$, the interface $\mathcal{E}(h,k)$ is an analytic constant mean curvature surface in $\mathbb{R}^n$ which satisfies*

$$\sum_{h=0}^{N} \mathcal{H}^{n-1}\left(\partial\mathcal{E}(h) \setminus \partial^*\mathcal{E}(h)\right) = 0$$

In order to prove the theorem, we are going to use the following lemma and its corollary. The lemma states that, if Λ is a subset of indices and the measure of the parts of $\mathcal{E}(h), h \in \Lambda$ inside a certain ball $B(x,r)$ is sufficiently small, then each $\mathcal{E}(h), h \in \Lambda$ is disjoint in measure with respect to $B(x, r/2)$.

Lemma 2.2 (Infiltration). *If $\mathcal{E}$ is a minimizing cluster in $\mathbb{R}^n$, there exist positive constants $\varepsilon_0 < \omega_n, r_0 > 0$ such that for every $x \in \mathbb{R}^n$, $r < r_0$ and $\Lambda \subseteq \{0, \dots, N\}$ with*

$$\sum_{h\in\Lambda} \left|\mathcal{E}(h) \cap B(x,r)\right| \leq \varepsilon_0 r^n \tag{2.1}$$

it holds

$$\left|\mathcal{E}(h) \cap B(x, r/2)\right| = 0, \quad h \in \Lambda$$

In the next corollary we are going to prove that the sets $\mathcal{E}(h,k) \cap B(x, r_x)$, $0 \le h < k \le N$, for a certain $r_x > 0$, are analytic hypersurfaces with constant mean curvature. In order to prove that, we are going to apply some powerful theorems concerning the regularity theory of minimal sets of finite perimeter. (see [6], chapters $21, 27, 28$). Later we are going to state and prove a simplified version of these theorems.

COROLLARY 2.3. *Let $\mathcal{E}$ be a minimizing cluster in $\mathbb{R}^n$ and $x \in \partial\mathcal{E}(h) \cap \partial\mathcal{E}(k)$ such that*

$$\lim_{r \to 0^+} \frac{\left|\mathcal{E}(h) \cap B(x,r)\right|}{\omega_n r^n} + \frac{\left|\mathcal{E}(k) \cap B(x,r)\right|}{\omega_n r^n} = 1 \tag{2.2}$$

Then there exists $r_x > 0$ such that

$$\left|\mathcal{E}(i) \cap B(x, r_x)\right| = 0$$

whenever $i \neq h, k$. Moreover $\mathcal{E}(h), \mathcal{E}(k)$ are volume-constrained perimiter minimizers inside $B(x, r_x)$. In particular, if either $2 \le n \le 7$ or $n \ge 8$ but $x \in \mathcal{E}(h,k)$ then

$$\partial\mathcal{E}(h) \cap \partial\mathcal{E}(k) \cap B(x, r_x) = \mathcal{E}(h,k) \cap B(x, r_x)$$

is a constant mean curvature analytic hypersurface in $\mathbb{R}^n$.

Proof. Let ε_0, r_0 be the constants of the previous lemma. By (2.2) and setting $\Lambda = \{0, \dots, N\} \setminus \{h, k\}$, it follows that

$$\lim_{r \to 0^+} \frac{\left|\bigcup_{i \in \Lambda} \mathcal{E}(i) \cap B(x,r)\right|}{\omega_n r^n} = 0$$

and so, for a certain fixed $\tilde{r} < r_0$, it holds

$$\frac{\left|\bigcup_{i \in \Lambda} \mathcal{E}(i) \cap B(x,\tilde{r})\right|}{\omega_n \tilde{r}^n} \le \frac{\varepsilon_0}{\omega_n}$$

Then, using the lemma, we have

$$\left|\mathcal{E}(i) \cap B(x, \tilde{r}/2)\right| = 0, \quad i \neq h, k$$

Thus we can set $r_x = \tilde{r}/2$ and the first part of corollary is proved.

Now we are going to show that $\mathcal{E}(h)$ is a volume-constrained perimeter minimizer inside $B(x, r_x)$. Let $F \subseteq \mathbb{R}^n$ be a set with

$$|F| = \left|\mathcal{E}(h)\right|, \quad \mathcal{E}(h) \Delta F \subset\subset B(x, r_x)$$

We define the cluster $\mathcal{E}'$ as

$$\begin{aligned}
&\mathcal{E}'(h) = (\mathcal{E}(h) \setminus B(x,r_x)) \cup (B(x,r_x) \cap F) \equiv F \\
&\mathcal{E}'(k) = (\mathcal{E}(k) \setminus B(x,r_x)) \cup (B(x,r_x) \setminus F) \\
&\mathcal{E}'(i) = \mathcal{E}(i), \quad i \neq h,k
\end{aligned}$$

Notice that $\mathcal{E}'(h)$ coincides with F, because $\mathcal{E}(h)$ and F do not differ outside $B(x,r_x)$ and $\mathcal{E}'(h)$ is defined as F inside $B(x,r_x)$. The new cluster is a competitor because $\big|\mathcal{E}'(h)\big| = |F| = \big|\mathcal{E}(h)\big|$ and

$$\begin{aligned}
\big|\mathcal{E}'(k)\big| &= \big|\mathcal{E}(k) \setminus B(x,r_x)\big| + \big|B(x,r_x) \setminus F\big| \\
&= \big|\mathcal{E}(k) \setminus B(x,r_x)\big| + \big|B(x,r_x)\big| - \big|B(x,r_x) \cap F\big| \\
&= \big|\mathcal{E}(k) \setminus B(x,r_x)\big| + \big|B(x,r_x)\big| - \big|B(x,r_x) \cap \mathcal{E}(h)\big| \\
&= \big|\mathcal{E}(k) \setminus B(x,r_x)\big| + \big|B(x,r_x) \cap \mathcal{E}(k)\big| \\
&= \big|\mathcal{E}(k)\big|
\end{aligned}$$

Thus $\mathbf{m}(\mathcal{E}) = \mathbf{m}(\mathcal{E}')$. By minimality, we have

$$P(\mathcal{E}) \leq P(\mathcal{E}')$$

and so

$$P(\mathcal{E}(h)) + P(\mathcal{E}(k)) \leq P(\mathcal{E}'(h)) + P(\mathcal{E}'(k)) \tag{2.3}$$

Since, up to null $\mathcal{L}^n$ measure sets, $\mathcal{E}(k)\Delta\mathcal{E}'(k) = \mathcal{E}(h)\Delta F \subset\subset B(x,r_x)$, then from (2.3) it follows that

$$P(\mathcal{E}(h); B(x,r_x)) + P(\mathcal{E}(k); B(x,r_x)) \leq P(\mathcal{E}'(h); B(x,r_x)) + P(\mathcal{E}'(k); B(x,r_x))$$

Thus either

$$P(\mathcal{E}(h); B(x,r_x)) \leq P(\mathcal{E}'(h); B(x,r_x))$$

or

$$P(\mathcal{E}(k); B(x,r_x)) \leq P(\mathcal{E}'(k); B(x,r_x))$$

In the first case we would have $P(\mathcal{E}(h); B(x,r_x)) \leq P(F; B(x,r_x))$. Now let's consider the second case. Since $P(\mathcal{E}(i); B(x,r_x)) = 0$ as a consequence of $\big|\mathcal{E}(i) \cap B(x,r_x)\big| = 0$, we have

$$\begin{aligned}
P(\mathcal{E}(h); B(x,r_x)) &= \sum_{\substack{j=0 \\ j\neq h}}^{N} \mathcal{H}^{n-1}\big(\mathcal{E}(h,j) \cap B(x,r_x)\big) = \mathcal{H}^{n-1}\big(\mathcal{E}(h,k) \cap B(x,r_x)\big) \\
&= P(\mathcal{E}(k); B(x,r_x))
\end{aligned}$$

Moreover, as $\big|(\mathcal{E}'(k)\Delta F^c) \cap B(x,r_x)\big| = 0$,

$$P(\mathcal{E}'(k); B(x,r_x)) = P(F^c; B(x,r_x)) = P(F; B(x,r_x))$$

Then, we can finally conclude that

$$P(\mathcal{E}(h); B(x, r_x)) \leq P(F; B(x, r_x))$$

This proves that $\mathcal{E}(h)$ is a volume-constrained perimeter minimizer.

Thanks to the just shown minimality of $\mathcal{E}(h), \mathcal{E}(k)$ and some regularity theorems that we are not going to deepen, if either $2 \leq n \leq 7$ or $n \geq 8$ but $x \in \mathcal{E}(h, k)$, then $\mathcal{E}(h, k) \cap B(x, r_x) = \partial\mathcal{E}(h) \cap \partial\mathcal{E}(k) \cap B(x, r_x)$ is a constant mean curvature analytic hypersurface in $\mathbb{R}^n$.

□

REMARK 2.4. By the previous corollary, we know that, if $x \in \mathcal{E}(h, k)$ there exists r_x such that

$$\mathcal{E}(h, k) \cap B(x, r_x)$$

is an analytic constant mean curvature hypersurface in $\mathbb{R}^n$. Then, covering $\mathcal{E}(h, k)$ with $\cup_{x \in \mathcal{E}(h,k)} \mathcal{E}(h, k) \cap B(x, r_x)$ and using the compactness of $\mathcal{E}(h, k)$, we deduce that $\mathcal{E}(h, k)$ is an analytic hypersurface too. Moreover, each connected component of $\mathcal{E}(h, k)$, has constant mean curvature.

Proof of the lemma. We are going to consider only the case $\Lambda = \{2, \dots, N\}$ and $x = 0$. By corollary 1.18, we know that there exist constant $r_0, C > 0$ such that, given an N-cluster $\mathcal{F}$ with $\mathcal{F}(h)\Delta\mathcal{E}(h) \subset\subset B_{r_0}$ for every $h = 1, \dots, N$, then

$$P(\mathcal{E}) \leq P(\mathcal{F}) + C\left|\mathbf{m}(\mathcal{E}) - \mathbf{m}(\mathcal{F})\right|$$

Define, for $s > 0$,

$$E_s = B_s \cap \bigcup_{h=2}^{N} \mathcal{E}(h)$$

and $m\colon (0, +\infty) \to (0, +\infty)$ as

$$m(s) = |E_s|$$

It is known that, for almost every $s > 0$,

$$m'(s) = \sum_{h=2}^{N} \mathcal{H}^{n-1}\left(\mathcal{E}(h) \cap \partial B_s\right)$$

Moreover, for almost every $s > 0$,

$$\sum_{h=0}^{N} \mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(h) \cap \partial B_s\right) = 0$$

In fact, since $\{\partial B_s\}_s$ is a family of disjoint Borel sets and $\mathcal{H}^{n-1} \llcorner \partial^*\mathcal{E}(h)$ is a Radon measure, then $\mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(h) \cap \partial B_s\right) > 0$ at most in a countable set of indices s.

Now there could be two possibilities:

$$\sum_{h\geq 2} \mathcal{H}^{n-1}\left(B_s \cap \mathcal{E}(h,1)\right) \geq \sum_{h\geq 2} \mathcal{H}^{n-1}\left(B_s \cap \mathcal{E}(h,0)\right)$$

or

$$\sum_{h\geq 2} \mathcal{H}^{n-1}\left(B_s \cap \mathcal{E}(h,0)\right) \geq \sum_{h\geq 2} \mathcal{H}^{n-1}\left(B_s \cap \mathcal{E}(h,1)\right)$$

Without loss of generality, we can assume that we are in the first case. We define the new cluster $\mathcal{F}$ as

$$\begin{aligned}
&\mathcal{F}(0) = \mathcal{E}(0)\\
&\mathcal{F}(1) = \mathcal{E}(1) \cup E_s\\
&\mathcal{F}(h) = \mathcal{E}(h) \setminus B_s, \quad h = 2, \ldots, N
\end{aligned}$$

Notice that, since $\mathcal{E}$ and $\mathcal{F}$ differ just on B_s, if $s < r_0$ then $\mathcal{E}(h)\Delta\mathcal{F}(h) \subseteq B_s \subset\subset B_{r_0}$. Thus, in this case,

$$P(\mathcal{E}) \leq P(\mathcal{F}) + C\left|\mathbf{m}(\mathcal{E}) - \mathbf{m}(\mathcal{F})\right| \tag{2.4}$$

The following estimations hold:

$$P(\mathcal{E}; \mathbb{R}^n \setminus \bar{B}_s) = P(\mathcal{F}; \mathbb{R}^n \setminus \bar{B}_s) \tag{2.5}$$

$$P(\mathcal{E}; \partial B_s) = \frac{1}{2}\sum_{h=0}^{N} \mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(h) \cap \partial B_s\right) = 0, \quad \text{for a.e. } s > 0 \tag{2.6}$$

$$P(\mathcal{F}; \partial B_s) = \frac{1}{2}\sum_{h=0}^{N} P(\mathcal{F}(h); \partial B_s) = m'(s), \quad \text{for a.e. } s > 0 \tag{2.7}$$

Let us briefly show the last equality. If $h \geq 2$ then, for almost every $s > 0$,

$$\partial^*\mathcal{F}(h) \cap \partial B_s = (\partial^*(\mathcal{E}(h) \setminus B_s)) \cap \partial B_s \approx \partial B_s \cap \mathcal{E}(h)^{(1)}$$

and so

$$P(\mathcal{F}(h); \partial B_s) = \mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(h) \cap \partial B_s\right) = \mathcal{H}^{n-1}\left(\mathcal{E}(h)^{(1)} \cap B_s\right)$$

If $h = 1$ then, for almost every $s > 0$,

$$\partial^*\mathcal{F}(1) \cap \partial B_s \approx \left(\bigcup_{h\geq 2} \mathcal{E}(h)\right)^{(1)} \cap \partial B_s$$

and so

$$P(\mathcal{F}(1); \partial B_s) = \mathcal{H}^{n-1}\left(\partial^*\mathcal{F}(1) \cap \partial B_s\right) = \sum_{h\geq 2} \mathcal{H}^{n-1}\left(\mathcal{E}(h)^{(1)} \cap \partial B_s\right)$$

If $h = 0$ then, for almost every $s > 0$,

$$P(\mathcal{F}(0); \partial B_s) = \mathcal{H}^{n-1}\left(\partial^* \mathcal{E}(0) \cap \partial B_s\right) = 0$$

Hence, putting all these equalities together, we get

$$\begin{aligned}
P(\mathcal{F}; \partial B_s) &= \frac{1}{2} \sum_{h=0}^{N} P(\mathcal{F}(h); \partial B_s) \\
&= \frac{1}{2}\left(P(\mathcal{F}(0); \partial B_s) + P(\mathcal{F}(1); \partial B_s) + \sum_{h \geq 2} P(\mathcal{F}(h); \partial B_s) \right) \\
&= \frac{1}{2}\left(\sum_{h \geq 2} \mathcal{H}^{n-1}\left(\mathcal{E}(h)^{(1)} \cap \partial B_s\right) + \sum_{h \geq 2} \mathcal{H}^{n-1}\left(\mathcal{E}(h)^{(1)} \cap \partial B_s\right) \right) \\
&= \sum_{h \geq 2} \mathcal{H}^{n-1}\left(\mathcal{E}(h)^{(1)} \cap \partial B_s\right) = m'(s)
\end{aligned}$$

Finally we have also

$$\begin{aligned}
\left|\mathbf{m}(\mathcal{E}) - \mathbf{m}(\mathcal{F})\right| &\leq \sum_{h=1}^{N} \left| \left|\mathcal{E}(h)\right| - \left|\mathcal{F}(h)\right| \right| \\
&= \left| \left|\mathcal{E}(1)\right| - \left|\mathcal{F}(1)\right| \right| + \sum_{h \geq 2} \left| \left|\mathcal{E}(h)\right| - \left|\mathcal{F}(h)\right| \right| \\
&= |E_s| + \sum_{h \geq 2} \left|\mathcal{E}(h) \cap B_s\right| = 2\,|E_s|
\end{aligned} \tag{2.8}$$

Thus, by (2.4) and (2.8), we get

$$P(\mathcal{E}; B_s) + P(\mathcal{E}; \partial B_s) + P(\mathcal{E}; B_s^c) \leq P(\mathcal{F}; B_s) + P(\mathcal{F}; \partial B_s) + P(\mathcal{F}; B_s^c) + 2C\,|E_s|$$

and, by (2.5)-(2.7), it holds for almost every $s > 0$

$$P(\mathcal{E}; B_s) \leq P(\mathcal{F}; B_s) + m'(s) + 2C\,|E_s| \tag{2.9}$$

Now we claim that

$$P(\mathcal{E}; B_s) - P(\mathcal{F}; B_s) \geq \frac{m(s)^{(n-1)/n} - m'(s)}{2}$$

Let's prove the claim. It holds

$$\begin{aligned}
\sum_{h=2}^{N} P(\mathcal{E}(h); B_s) \geq P(E_s; B_s) &= P(E_s \cap B_s) - \mathcal{H}^{n-1}\left(\partial B_s \cap E_s\right) \\
&\geq P(E_s \cap B_s) - \mathcal{H}^{n-1}\left(\partial B_s \cap \bigcup_{h=2}^{N} \mathcal{E}(h)\right) \\
&= P(E_s) - m'(s) \geq |E_s|^{(n-1)/n} - m'(s) \\
&= m(s)^{(n-1)/n} - m'(s)
\end{aligned}$$

As $\left|\mathcal{F}(h)\cap B_s\right| = 0$ for $h\geq 2$, then $P(\mathcal{F}(h);B_s)=0$. Thus we have

$$\begin{aligned}
&2\left(P(\mathcal{E};B_s)-P(\mathcal{F};B_s)\right) = P(\mathcal{E}(1);B_s)+\sum_{h\geq 2}P(\mathcal{E}(h);B_s)-P(\mathcal{F}(1);B_s)\\
&\geq P(\mathcal{E}(1);B_s)-P(\mathcal{E}(1)\cup E_s;B_s)+m(s)^{(n-1)/n}-m'(s)\\
&\geq m(s)^{(n-1)/n}-m'(s)
\end{aligned}$$

because

$$\begin{aligned}
&P(\mathcal{E}(1);B_s)-P(\mathcal{E}(1)\cup E_s;B_s)=\\
&=\sum_{\substack{h=0\\h\neq 1}}^{N}\mathcal{H}^{n-1}\left(\mathcal{E}(h,1)\cap B_s\right)-\sum_{\substack{k=0\\k\neq 0}}^{N}\mathcal{H}^{n-1}\left(\mathcal{E}(k,0)\cap B_s\right)\\
&=\sum_{h\geq 2}\mathcal{H}^{n-1}\left(\mathcal{E}(h,1)\cap B_s\right)-\sum_{k\geq 2}\mathcal{H}^{n-1}\left(\mathcal{E}(k,1)\cap B_s\right)\geq 0
\end{aligned}$$

by the initial hypothesis. Thus the claim is proved.

Recalling also (2.9), we get

$$P(\mathcal{F};B_s)+\frac{m(s)^{(n-1)/n}-m'(s)}{2}\leq P(\mathcal{E};B_s)\leq P(\mathcal{F};B_s)+m'(s)+2Cm(s)$$

and so

$$m(s)^{(n-1)/n}\leq 3m'(s)+4Cm(s)$$

Let $\varepsilon_0\leq\left(\frac{1}{12n}\right)^n<1$ and (2.1) be valid. Up to decreasing the value of $r_0<1/(8C)$, a simple computation leads to

$$4Cm(s)\leq\frac{m(s)^{(n-1)/n}}{2}$$

for a.e. $s\in(0,r)$. Taking into account the last two inequalities, we get

$$m(s)^{(n-1)/n}\leq 6m'(s),\qquad\text{for a.e. } s\in(0,r)$$

which can be written also as

$$1\leq 6\frac{m'(s)}{m(s)^{(n-1)/n}}=6n(m(s)^{1/n})'$$

Let $[r_*,+\infty)$ be the support of m, $r_*\geq 0$. Then, considering without loss of generality $r>r_*$ and integrating on (r_*,r) the last inequality, we get

$$r-r_*=6n(m(r)^{1/n}-m(r_*)^{1/n})=6nm(r)^{1/n}\leq 6n(\varepsilon_0 r^n)^{1/n}\leq 6n\frac{1}{12n}r$$

that is

$$r_*\geq\frac{r}{2}$$

Since $m(r_*)=0$ and m is increasing, we have $m(\frac{r}{2})=0$.

$\square$

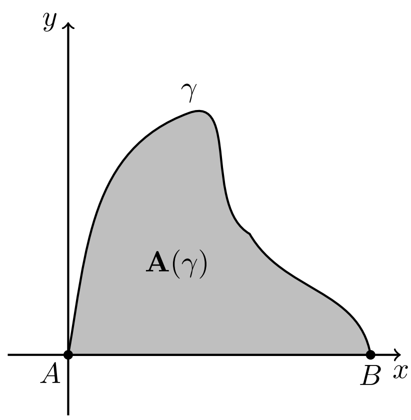


Figure 2.1: In the figure is represented a Lipschitz curve γ from A to B with prescribed area $\mathbf{A}(\gamma)$.

2.2 A PERIMETER MINIMIZING VARIATIONAL PROBLEM

Let's start explaining the general idea of this problem. Given two fixed distinct points A and B, we would like to find the Lipschitz curve of minimal length with A, B as endpoints and with fixed enclosed area. Namely, if $\gamma\colon [0,1] \to \mathbb{R}^2$ is a Lipschitz curve, we define the functional $\mathbf{L}$ and $\mathbf{A}$ as

$$\mathbf{L}(\gamma) = \int_0^1 |\dot{\gamma}| \, \mathrm{d}t$$
$$\mathbf{A}(\gamma) = \frac{1}{2} \int_0^1 \gamma \cdot \dot{\gamma}^{\perp} \mathrm{d}t$$

Here $x^{\perp} = (x_2, -x_1)$ if $x = (x_1, x_2)$. The first functional represents the length of the curve γ. The second one, up to the sign, is the area of the region enclosed by the curve and the line segment AB.

Define also the set of the admissible curves as

$$\mathcal{A} = \{\gamma\colon [0,1] \to \mathbb{R}^2 : \gamma \in \ \mathrm{Lip}([0,1], \mathbb{R}^2), \gamma(0) = A, \gamma(1) = B, \mathbf{A}(\gamma) = v\}$$

with $A = (0,0), B = (1,0)$, $v \in \mathbb{R}$. We are going to study the problem

$$\min_{\gamma \in \mathcal{A}} \mathbf{L}(\gamma) \tag{2.10}$$

Existence. Let's start proving the existence of the minimizer. Consider a minimizing sequence $\{\gamma^{(n)}\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, that is

$$\mathbf{L}(\gamma^{(n)}) \to l \coloneqq \inf_{\gamma \in \mathcal{A}} \mathbf{L}(\gamma)$$

Since $\mathbf{L} < \infty$, we can assume that, for a certain $C > 0$ and for every $n \in \mathbb{N}$,

$$\mathbf{L}(\gamma^{(n)}) \leq C < \infty$$

We recall that by the Poincaré inequality, if $f \in W_0^{1,1}(0,1)$ then there exists a constant $C_1 > 0$ such that

$$\int_0^1 \big|f(x)\big| \,\mathrm{d}x \leq C_1 \int_0^1 \big|f'(x)\big| \,\mathrm{d}x$$

Then, for a fixed $\tilde{\gamma} \in \mathcal{A}$, we have $\gamma^{(n)} - \tilde{\gamma} \in W_0^{1,1}((0,1), \mathbb{R}^2)$ and so

$$\begin{aligned}
\int_0^1 \Big|\gamma^{(n)}\Big| \,\mathrm{d}t &\leq \int_0^1 \Big|\gamma^{(n)} - \tilde{\gamma}\Big| \,\mathrm{d}t + \int_0^1 |\tilde{\gamma}| \,\mathrm{d}t \\
&\leq C_1 \bigg(\int_0^1 \Big|\dot{\gamma}^{(n)} - \dot{\tilde{\gamma}}\Big| \,\mathrm{d}t \bigg) + \int_0^1 |\tilde{\gamma}| \,\mathrm{d}t \\
&\leq C_1 \bigg(\int_0^1 \Big|\dot{\gamma}^{(n)}\Big| \,\mathrm{d}t + \int_0^1 \Big|\dot{\tilde{\gamma}}\Big| \,\mathrm{d}t \bigg) + \int_0^1 |\tilde{\gamma}| \,\mathrm{d}t \leq D
\end{aligned}$$

for a certain $D \in \mathbb{R}$, independent on n. Thus $\{\gamma^{(n)}\}_n$ is bounded in $W^{1,1}((0,1), \mathbb{R}^2)$. In particular, it admits a subsequence converging to $\bar{\gamma}$ weakly in $W^{1,1}((0,1), \mathbb{R}^2)$. We recall this subsequence $\gamma^{(n)}$. Moreover we can assume that this subsequence converges to $\bar{\gamma}$ also in L^1.

By Tonelli's theorem, we deduce that $\mathbf{L}$ is lower semicontinuous with respect to the weak topology of $W^{1,1}((0,1), \mathbb{R}^2)$. Then

$$\mathbf{L}(\bar{\gamma}) \leq \liminf_{n \to \infty} \mathbf{L}(\gamma^{(n)}) = \inf_{\gamma \in \mathcal{A}} \mathbf{L}(\gamma)$$

Thus the problem admits minimum in $W^{1,1}((0,1), \mathbb{R}^2)$. Moreover, by the strictly convexity of $\mathbf{L}$, the minimum is unique.

Let's prove that $\mathbf{A}(\bar{\gamma}) = v$. It holds

$$\Big|\mathbf{A}(\bar{\gamma}) - \mathbf{A}(\gamma^{(n)})\Big| =$$

$$\left|\int_0^1 \bar{\gamma} \cdot \dot{\bar{\gamma}}^{\perp} - \gamma^{(n)} \cdot \dot{\gamma}^{(n)\perp} \mathrm{d}t\right| \leq$$

$$\left|\int_0^1 \bar{\gamma} \cdot \dot{\bar{\gamma}}^{\perp} - \bar{\gamma} \cdot \dot{\gamma}^{(m)\perp} + \bar{\gamma} \cdot \dot{\gamma}^{(m)\perp} - \gamma^{(n)} \cdot \dot{\gamma}^{(m)\perp} + \gamma^{(n)} \cdot \dot{\gamma}^{(m)\perp} - \gamma^{(n)} \cdot \dot{\gamma}^{(n)\perp} \mathrm{d}t\right| \leq$$

$$\left|\int_0^1 \bar{\gamma} \cdot (\dot{\bar{\gamma}} - \dot{\gamma}^{(m)})^{\perp} \mathrm{d}t\right| + \left|\int_0^1 \dot{\gamma}^{(m)\perp} \cdot (\bar{\gamma} - \gamma^{(n)}) \mathrm{d}t\right| + \left|\int_0^1 (\dot{\gamma}^{(m)} - \dot{\gamma}^{(n)})^{\perp} \cdot (\gamma^{(n)} - \bar{\gamma} + \bar{\gamma}) \mathrm{d}t\right|$$

Choose $\varepsilon > 0$. Using the weak convegence of $\dot{\gamma}^{(n)} \overset{L^1}{\rightharpoonup} \dot{\bar{\gamma}}$ and the boundness of $\bar{\gamma} \in AC((0,1), \mathbb{R}^2)$, we find that the first term goes to zero as $m \to \infty$. Then determine $\overline{m}$ such that

$$\left|\int_0^1 \bar{\gamma} \cdot (\dot{\bar{\gamma}} - \dot{\gamma}^{(m)})^{\perp} \mathrm{d}t\right| \leq \varepsilon$$

for $m \geq \overline{m}$. By the strong convergence $\gamma^{(n)} \xrightarrow{L^1} \bar{\gamma}$ and the estimation $\left|\dot{\gamma}^{(m)\perp}\right| \leq L_m$ for a certain $L_m \in \mathbb{R}_+$ (since $\gamma^{(m)}$ is Lipschitz continuous), we get that the second term goes to zero too as $n \to \infty$. In particular determine $\overline{n}$ such that, for every $n \geq \overline{n}$,

$$\left|\int_0^1 \dot{\gamma}^{(\overline{m})\perp} \cdot (\bar{\gamma} - \gamma^{(n)}) \mathrm{d}t\right| \leq \varepsilon$$

Finally, with similar arguments, it is easy to show that also the third term is smaller that ε if $m = \overline{m}$ and n is sufficiently large. This proves that

$$\left|\mathbf{A}(\bar{\gamma}) - \mathbf{A}(\gamma^{(n)})\right| \leq 3\varepsilon$$

for n large. Then $v = \mathbf{A}(\gamma^{(n)}) \overset{n\to\infty}{\longrightarrow} \mathbf{A}(\bar{\gamma})$ and so $\mathbf{A}(\bar{\gamma}) = v$.

Characterization of the solution. Now we would like to determine the expression of $\bar{\gamma}$. By convenience, let

$$\begin{aligned} f(\xi) &= |\xi| \\ g(u, \xi) &= \frac{1}{2} u \cdot \xi^{\perp} - v \end{aligned}$$

With this definition we have

$$\begin{aligned} \mathbf{L}(\bar{\gamma}) &= \int_0^1 f(\dot{\bar{\gamma}}) \mathrm{d}t \\ \int_0^1 g(\bar{\gamma}, \dot{\bar{\gamma}}) \mathrm{d}t &= \mathbf{A}(\bar{\gamma}) - v = 0 \end{aligned}$$

Define, for fixed $\varphi, \psi \in C_c^{\infty}((0,1), \mathbb{R}^2)$, the functions

$$\begin{aligned} F(\varepsilon, h) &= \mathbf{L}(\bar{\gamma} + \varepsilon\varphi + h\psi) = \int_0^1 f(\dot{\bar{\gamma}} + \varepsilon\dot{\varphi} + h\dot{\psi})\,\mathrm{d}t \\ G(\varepsilon, h) &= \mathbf{A}(\bar{\gamma} + \varepsilon\varphi + h\psi) - v = \int_0^1 g(\bar{\gamma} + \varepsilon\varphi + h\psi, \dot{\bar{\gamma}} + \varepsilon\dot{\varphi} + h\dot{\psi})\,\mathrm{d}t \end{aligned}$$

Notice that

$$G(0,0) = 0, \qquad G_h(0,0) = \int_0^1 (g_u(\bar{\gamma}, \dot{\bar{\gamma}})\psi + g_\xi(\bar{\gamma}, \dot{\bar{\gamma}})\dot{\psi})\,\mathrm{d}t$$

Since

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g_\xi(u, \xi) \neq g_u(u, \xi)$$

then there exists $\psi \in C_c^{\infty}((0,1), \mathbb{R}^2)$ such that

$$\int_0^1 \big(g_\xi(\bar{\gamma}, \dot{\bar{\gamma}})\dot{\psi} + g_u(\bar{\gamma}, \dot{\bar{\gamma}})\psi\big) \mathrm{d}t = 1$$

that is

$$G_h(0,0) = 1 \neq 0$$

Thus, by Dini's theorem, we deduce that there exists $h(\varepsilon)$ such that

$$G(\varepsilon, h(\varepsilon)) = 0, \tag{2.11}$$

for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. This means that the curves $\bar{\gamma} + \varepsilon\varphi + h(\varepsilon)\psi$ enclose the corrected area. By (2.11), we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G(\varepsilon, h(\varepsilon))_{|\varepsilon=0} = G_\varepsilon(0,0) + G_h(0,0)h'(0)$$

and so

$$h'(0) = -G_\varepsilon(0,0)$$

Finally, by the minimality of $\bar{\gamma}$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\varepsilon, h(\varepsilon))_{|\varepsilon=0} = F_\varepsilon(0,0) + F_h(0,0)h'(0) = F_\varepsilon(0,0) + \lambda G_\varepsilon(0,0)$$

with $\lambda = -F_h(0,0)$. Since

$$f_\xi(\xi) = \frac{\xi}{|\xi|}, \quad g_u(u,\xi) = \frac{1}{2}\xi^\perp, \quad g_\xi(u,\xi) = -\frac{1}{2}u^\perp$$

exploiting the last equality we get

$$\begin{aligned}
0 &= \frac{\partial}{\partial \varepsilon}\left(\int_0^1 f(\dot{\bar{\gamma}} + \varepsilon\dot{\varphi} + h\dot{\psi}) + \lambda g(\bar{\gamma} + \varepsilon\varphi + h\psi, \dot{\bar{\gamma}} + \varepsilon\dot{\varphi} + h\dot{\psi})\right)_{|\varepsilon=0,h=0} \\
&= \int_0^1 f_\xi(\bar{\gamma}) \cdot \dot{\varphi} + \lambda\big(g_u(\bar{\gamma}, \dot{\bar{\gamma}}) \cdot \varphi + g_\xi(\bar{\gamma}, \dot{\bar{\gamma}})\dot{\varphi}\big)\mathrm{d}t \qquad (2.12) \\
&= \int_0^1 \frac{\dot{\bar{\gamma}}}{|\dot{\bar{\gamma}}|} \cdot \dot{\varphi} + \lambda\left(\frac{1}{2}\dot{\bar{\gamma}}^\perp \cdot \varphi - \frac{1}{2}\bar{\gamma}^\perp \cdot \dot{\varphi}\right)\mathrm{d}t
\end{aligned}$$

We notice that

$$\begin{aligned}
\int_0^1 \varphi(t) \cdot \dot{\bar{\gamma}}^\perp(t)\mathrm{d}t &= \int_0^1 \left(\int_0^1 \dot{\varphi}(\tau)\chi_{[0,t]}(\tau)\mathrm{d}\tau\right) \cdot \dot{\bar{\gamma}}^\perp(t)\mathrm{d}t \\
&= \int_0^1 \left(\int_\tau^1 \dot{\bar{\gamma}}^\perp(t)\mathrm{d}t\right) \cdot \dot{\varphi}(\tau)\mathrm{d}\tau \\
&= \int_0^1 \left(\bar{\gamma}^\perp(1) - \bar{\gamma}^\perp(\tau)\right) \cdot \dot{\varphi}(\tau)\mathrm{d}\tau \\
&= -\int_0^1 \bar{\gamma}^\perp(\tau) \cdot \dot{\varphi}(\tau)\mathrm{d}\tau
\end{aligned}$$

Thus the condition (2.12) becomes

$$\int_0^1 \left(\frac{\dot{\bar{\gamma}}}{|\dot{\bar{\gamma}}|} - \lambda \bar{\gamma}^{\perp} \right) \cdot \dot{\varphi}\, \mathrm{d}t = 0$$

for every $\varphi \in C_c^\infty((0,1),\mathbb{R}^2)$. This implies that there exists a constant vector $c = (c_1, c_2) \in \mathbb{R}^2$ such that

$$\frac{\dot{\bar{\gamma}}}{|\dot{\bar{\gamma}}|} - \lambda \bar{\gamma}^{\perp} = c, \qquad \text{for a.e. } t \in (0,1) \tag{2.13}$$

Let's consider the arc parametrization of $\bar{\gamma}$:

$$\tilde{\gamma}(s) = \bar{\gamma}(t(s)), \quad s \in [0, L]$$

with $t(s) = \mathrm{len}^{-1}(s), \mathrm{len}(t) = \int_0^t \left|\dot{\bar{\gamma}}(\tau)\right| \mathrm{d}\tau$. Then (2.13) becomes

$$\dot{\tilde{\gamma}}(s) = \lambda \tilde{\gamma}^{\perp}(s) + c \quad \text{for a.e. } s \in [0, L] \tag{2.14}$$

that is, if $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$,

$$\begin{cases} \dot{\tilde{\gamma}}_1 = \lambda \tilde{\gamma}_2 + c_1 \\ \dot{\tilde{\gamma}}_2 = -\lambda \tilde{\gamma}_1 + c_2 \end{cases}$$

almost everywhere on $(0, L)$. By (2.14), we deduce that $\dot{\tilde{\gamma}}$ is continuous and, iterating a similar argument, that it belongs to $C^\infty((0,1),\mathbb{R}^2)$. In particular $\tilde{\gamma}$ is Lipschitz continuous. In fact $\tilde{\gamma}(s) = \int_0^s \dot{\tilde{\gamma}}(\tau)\mathrm{d}\tau$ can be represented as the integral of a continuous function. Then its derivative can be computed for every $s \in (0, L)$. Thus it is constantly equal to $\lambda \tilde{\gamma}^{\perp}(s) + c$, which is a continuous function. In this way we see that $\tilde{\gamma} \in C^1((0, L), \mathbb{R}^2)$.

Now let's solve the differential equation (2.14) with initial condition $\tilde{\gamma}(0) = (0, 0)$. It is easy to see that the solution is

$$\begin{cases} \tilde{\gamma}_1(s) = \frac{1-\cos(\lambda s)}{\lambda} c_2 + \frac{\sin(\lambda s)}{\lambda} c_1 \\ \tilde{\gamma}_2(s) = \frac{\sin(\lambda s)}{\lambda} c_2 - \frac{1-\cos(\lambda s)}{\lambda} c_1 \end{cases}$$

and that it represents a circular arc with center in $(\frac{1}{2}, \sqrt{(\frac{1}{\lambda})^2 - \frac{1}{4}})$ and radius $r = \frac{1}{|\lambda|}$. We notice that the solution has constant mean curvature. The constant c_1, c_2 satisfy the relationship

$$c_1^2 + c_2^2 = 1$$

and they can be determined using the constrain $\mathbf{A}(\bar{\gamma}) = v$. This ends the proof of the existence and characterization of the solution of (2.10). Thus we can affirm that the perimeter minimizer with fixed endpoints and enclosed area is a circular arc.

2.3 Regularity of minimal planar clusters

Now we are going to focus on the minimal clusters in $\mathbb{R}^2$. We already know that, by the general regularity theorem, the minimizers have interfaces which are constant mean curvature hypersurfaces. In $\mathbb{R}^2$ this means that they are necessarily circular arcs or line segments. Moreover, since we are in the particular case of the plane, we will be able to get other details about the minimizers (Theorem 2.5). In the following sections, we are going to analyse and characterize the minimal 2-clusters and 4-clusters.

Theorem 2.5. *Let $\mathcal{E} = \{\mathcal{E}(1), \dots, \mathcal{E}(N)\}$ be the perimeter minimizer for a partitioning problem in $\mathbb{R}^2$. Then $\bigcup_{h=0}^{N} \partial^*\mathcal{E}(h)$ is a finite union of circular arcs or line segment meeting in threes at $120°$ angles at finitely many points. Moreover, for every $h, k = 0, \dots, N$, each arc belonging to $\mathcal{E}(h,k)$ has the same mean curvature and the set $\partial\mathcal{E}(h) \setminus \partial^*\mathcal{E}(h)$ is discrete.*

Remark 2.6. By theorem 1.15, we know that, given clusters $\mathcal{E}, \mathcal{E}'$ sufficiently close in a measure sense, we can find another cluster $\mathcal{E}''$ with $\mathbf{m}(\mathcal{E}'') = \mathbf{m}(\mathcal{E})$ which, setting $C = 2C_1 \sup\{P(\mathcal{E}), P(\mathcal{E}')\} > 0$, satisfies the estimation

$$P(\mathcal{E}'') \leq P(\mathcal{E}') + CN \max_{h=1,\dots,N} \big|a(h)\big|$$

where $a(h) = \big|\mathcal{E}(h)\big| - \big|\mathcal{E}'(h)\big|$. This can be done applying the diffeomorphism to the chambers of $\mathcal{E}'$ and restoring the areas of $\mathcal{E}$.

Remark 2.7. Since a segment is a circular arc with zero curvature, from now we are going to call arc both a circular arc and a line segment.

Proof of the theorem. We are going to proceed in this way. We claim that:

1. $\partial\mathcal{E}(h)$ is a finite collection of rectifiable cycles;
2. $\partial^*\mathcal{E}(h)$ is the union of finitely many arcs;
3. the meeting arcs form $120°$ angles;
4. every arc contained in $\mathcal{E}(h,k)$ has the same mean curvature.

Let's prove these statements.

1. Since $\partial\mathcal{E}(h) = M \cup \partial^*\mathcal{E}(h)$, for some $\mathcal{H}^{n-1}$-null measure set $M \in \mathbb{R}^2$ and $\partial^*\mathcal{E}(h)$ is a rectifiable set, then $\partial\mathcal{E}(h)$ is formed by at most countably many closed curves. Assume by contradiction that $\partial\mathcal{E}(h)$ has not a finite number of cycles. Since the perimeter is finite, necessarily there exists a cycle $\mathcal{C}$ of length $0 < \varepsilon < 4\pi/N^2\tilde{C}$, with $\tilde{C} = 2C_1P(\mathcal{E})$. For

sure, for some $k \neq h$, it holds $\mathcal{H}^1(\mathcal{C} \cap \partial^* \mathcal{E}(k)) \geq \varepsilon/N$. If R is the region inside $\mathcal{C}$, we define $\mathcal{E}'$ setting

$$\begin{aligned} \mathcal{E}'(k) &= \mathcal{E}(k) \cup R \\ \mathcal{E}'(h) &= \mathcal{E}(h) \setminus R \\ \mathcal{E}'(i) &= \mathcal{E}(i), \quad i \neq h, k \end{aligned}$$

Then

$$P(\mathcal{E}') \leq P(\mathcal{E}) - \varepsilon/N$$

Notice that, since $P(\mathcal{E}') \leq P(\mathcal{E})$, we get $\tilde{C} = C = 2C_1 P(\mathcal{E})$. As a consequence of the isoperimetric formula, the area of R is at most $\varepsilon^2/4\pi$ and so it can be made as small as we want. In particular, for ε sufficiently small, it is in $(-\eta, \eta)$. Applying the diffeomorphism, we get a new cluster $\mathcal{E}''$ with $\mathbf{m}(\mathcal{E}'') = \mathbf{m}(\mathcal{E})$ and

$$P(\mathcal{E}) \leq P(\mathcal{E}'') \leq P(\mathcal{E}') + CN\frac{\varepsilon^2}{4\pi} \leq P(\mathcal{E}) - \frac{\varepsilon}{N} + CN\frac{\varepsilon^2}{4\pi}$$

However this would imply that

$$\frac{4\pi}{N^2 C} \leq \varepsilon$$

which is a contradiction with the initial choice of ε. Thus each chamber has a finite number of cycles.

2. Now we prove that $\partial\mathcal{E}(h)$ is the union of finitely many arcs. First of all, we show that there could not be infinite arcs meeting in threes, fours, etc. We give a sketch of the proof of this fact. We say that a point A is a *k-point* (for the cluster $\mathcal{E}$) if there are k distinct arcs meeting in A. In particular we say that A is a k-point for $\mathcal{E}(h_1), \mathcal{E}(h_2), \ldots, \mathcal{E}(h_k)$ if the arcs meeting in A belongs to $\mathcal{E}(h_1), \ldots, \mathcal{E}(h_k)$. We want to prove that there not exist infinite 3-points, 4-points, etc. By simplicity we prove only the case of of 3-points. Assume by contradiction that A is a 3-point for $\mathcal{E}(h), \mathcal{E}(k), \mathcal{E}(j)$. Let us denote with $\mathcal{E}(h)(1), \mathcal{E}(k)(1), \mathcal{E}(j)(1)$ the connected components of $\mathcal{E}(h), \mathcal{E}(k), \mathcal{E}(j)$ which contain A in their boundaries. Let B, C other k-points, $k \geq 3$, satisfying

$$\begin{aligned} B &\in \partial\mathcal{E}(h)(1) \cap \partial\mathcal{E}(j)(1) \\ C &\in \partial\mathcal{E}(h)(1) \cap \partial\mathcal{E}(k)(1) \end{aligned}$$

Let D be another 3-point for $\mathcal{E}(h), \mathcal{E}(k), \mathcal{E}(j)$. If $D \notin \partial\mathcal{E}(h)(1)$, then $D \in \partial\mathcal{E}(h)(2)$, for another connected component $\mathcal{E}(h)(2)$ of $\mathcal{E}(h)$. Otherwise $D \in \partial\mathcal{E}(h)(1)$. Let us write $\widehat{AB}$ to denote the path from A to B along the boundary of $\mathcal{E}(h)(1)$. In this case there could be only these possibilities:

- if $D \in \widehat{AC}$ then $D \in \partial\mathcal{E}(j)(2)$, for some connected components $\mathcal{E}(j)(2)$ of $\mathcal{E}(j)$;
- if $D \in \widehat{AB}$ then $D \in \partial\mathcal{E}(k)(2)$, for some connected components $\mathcal{E}(k)(2)$ of $\mathcal{E}(k)$;
- if $D \in \widehat{BC}$ then $D \in \partial\mathcal{E}(k)(2) \cap \partial\mathcal{E}(j)(2)$, for some connected components $\mathcal{E}(k)(2), \mathcal{E}(j)(2)$ of $\mathcal{E}(k), \mathcal{E}(j)$.

For each case, we have determined another connected component of $\mathcal{E}(h), \mathcal{E}(k)$ or $\mathcal{E}(j)$. Then, if there exist infinite 3-points, there are infinite cycles. This is a contradiction.

Now we are going to use the variational problem of the previous section in order to prove that the number of arcs in a minimal cluster is finite. It is easy to see that we can always assume to have at least one k-point, $k \geq 3$. Let A be a k-point, $k \geq 3$, and B another k-point, $k \geq 3$, found along one of the k arcs starting in A. We know that we can strictly reduce the length of this curve with a single circular arc and without changing the enclosed area. This clearly works also if A, B coincides (by the isoperimetric inequality). Notice that A, B are k-points, $k \geq 3$, again. We can repeat this proceeding for every k-points, $k \geq 3$, of the cluster. Notice also that if there is a path with infinite 2-points and with two k-points, $k \geq 3$, as endpoints then, in the new cluster, these 2-points are not present. Instead, if there is a path with only 2-points, then we can replace it with a single circumference.

Thus we have proved that, in the minimal cluster, there are not 2-points and the k-points, $k \geq 3$, are finite. In particular the number of arcs is finite.

3. Now we are going to prove that there are only 3-points and that, in each of them, the arcs meet at $120°$ angles. Without loss of generality, let 0 be the k-point, $k \geq 3$, and assume that there are arcs of $\partial\mathcal{E}(1), \partial\mathcal{E}(2), \partial\mathcal{E}(3)$ meeting in 0. Define

$$T = \bigcup_{h=1}^{3} \partial\mathcal{E}(h)$$
$$T_r = T \cap B_r = \Gamma_1(r) \cup \Gamma_2(r) \cup \Gamma_3(r)$$
$$T'_r = \frac{1}{r} T_r = \Gamma'_1(r) \cup \Gamma'_2(r) \cup \Gamma'_3(r), \quad \Gamma'_i(r) = \frac{1}{r}\Gamma_i(r), \quad i = 1, 2, 3$$
$$p_i(r) = \Gamma_i(r) \cap \partial B_r, \quad p'_i(r) = \Gamma'_i(r) \cap \partial B_1 = \frac{1}{r} p_i(r), \quad i = 1, 2, 3$$

Finally we set T'_0 as the limit set. The limit is in the blow-up sense, that is the blow-ups of the sets $\mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3)$ converge to some sets and T'_0 is the union of the boundaries of those sets. The limit T'_0 is the

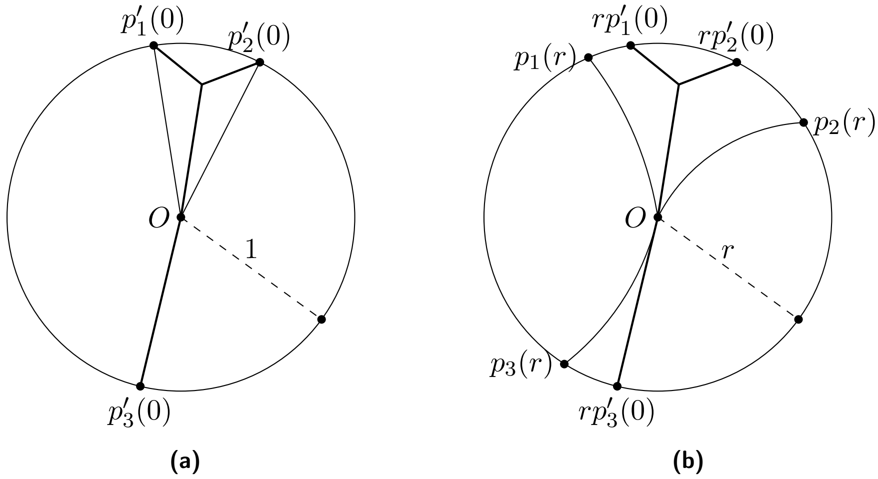


Figure 2.2: In figure **(a)** we can see the set T'_0 which is formed by the three lines connecting O to $p'_1(0), p'_2(0), p'_3(0)$ and the set $\tilde{T}'_0$ which is defined as the one with the bold lines. It is constructed in such a way that the three segments, which start from $O, p'_1(0), p'_2(0)$ and have a common endpoint, meet forming $120°$ angles. In figure **(b)** we see the set T_r formed by the three arcs starting from O and reaching $p_1(r), p_2(r), p_3(r)$ and the set $\tilde{T}_r$.

union of three line segments; we prove that they meet at $120°$ angles at 0. Assume by contradiction that there is an angle $\alpha < 120°$. By a simple geometric computation, we see that the set $\tilde{T}'_0$ (look at Figure 2.2) provides a network with strictly smaller length than T'_0.

This means that

$$P(\tilde{T}'_0) < P(T'_0) - \beta$$

for some $\beta > 0$. Define the sets

$$\tilde{T}_r = r\tilde{T}'_0, \quad r > 0$$

Then we have

$$P(\tilde{T}_r) = rP(\tilde{T}'_0) < rP(T'_0) - \beta r \tag{2.15}$$

By the convergence $P(T'_r) \to P(T'_0)$ for $r \to 0^+$ and the equality $P(T'_r) = \frac{P(T_r)}{r}$ we deduce that

$$P(T_r) = rP(T'_0) + o(r)$$

and so (2.15) becomes

$$P(\tilde{T}_r) < P(T_r) - \beta r + o(r) \tag{2.16}$$

We modify the sets $\tilde{T}_r$, calling it $\tilde{T}_r$ again, adding the arcs contained in ∂B_r connecting $p_1(r)$ and $rp_1'(0)$, $p_2(r)$ and $rp_2'(0)$, $p_3(r)$ and $rp_3'(0)$. With a simple computation, it is seen that $\left|rp_i'(0) - p_i(r)\right| \sim r^2 = o(r)$. Then, also after the modification of $\tilde{T}_r$, (2.16) still holds true.

Let $\tilde{\mathcal{E}}_r$ be the cluster which coincides with $\mathcal{E}$ outside the ball B_r and that has $\tilde{T}_r$ as "boundary" inside B_r. Now we modify $\tilde{\mathcal{E}}_r$ in such a way that it has the same areas of the cluster $\mathcal{E}$. Provided r is sufficiently small, we can apply the restoration volume theorem 1.15. Then, since $\left|a(h)\right| \leq \pi r^2$, we can restore the original areas with a quadratic cost. If we keep calling $\tilde{T}_r$ the boundary of the cluster $\tilde{\mathcal{E}}_r$ inside B_r, we have that $\tilde{\mathcal{E}}_r$ is a N-cluster with $\mathbf{m}(\mathcal{E}) = \mathbf{m}(\tilde{\mathcal{E}}_r)$ and

$$P(\tilde{T}_r) < P(T_r) - \beta r + o(r) \leq P(T_r) - \tfrac{\beta}{2} r$$

This clearly contradicts the minimality of $\mathcal{E}$ because we would have

$$P(\tilde{\mathcal{E}}_r) \leq P(\mathcal{E}) - \tfrac{\beta}{2} r$$

for a certain r small enough.

Thus the initial hypothesis of the existence of an angle $\alpha < 120°$ leads to a contradiction. Then the arcs necessarily meet in a 3-point with three angles of exactly $120°$.

4. Finally we show that if $\mathcal{C}_1, \mathcal{C}_2$ are two arcs of $\mathcal{E}(h,k)$ then $\mathcal{C}_1, \mathcal{C}_2$ have the same curvature.

We are going to use this notation. Let $\mathcal{C}$ be an arc with A, B as endpoints. We denote with R its radius, with $\mathbf{A}$ the area enclosed by $\mathcal{C}$ and the segment AB, with d the length of the segment connecting A, B, with l the length of the arc and with θ the angle between AB and the tangent to $\mathcal{C}$ in A (or equivalently in B). The following formulas hold

$$\begin{aligned} \mathbf{A}(R,\theta) &= R^2(\theta - \sin(\theta)\cos(\theta)) \\ l &= 2R\theta \\ R &= \frac{d}{2\sin(\theta)} \end{aligned}$$

We say that an arc $\mathcal{C} \subseteq \mathcal{E}(h,k)$ is *convex* in $\mathcal{E}(h)$ if, fixing the endpoints and decreasing θ a little, the new arc is contained in $\mathcal{E}(h)$. Instead $\mathcal{C}$ is said *concave* in $\mathcal{E}(k)$ if, fixing the endpoints and increasing θ a little, we get an arc contained in $\mathcal{E}(k)$. Namely $\mathcal{C}$ is *convex* in $\mathcal{E}(h)$ (respectively *concave* in $\mathcal{E}(k)$) if there exists $\delta\theta_{\max} > 0$ such that each arc with the same endpoints of $\mathcal{C}$ and with angle $\theta - \delta\theta, \delta\theta \in (0, \delta\theta_{\max})$ (respectively $\theta + \delta\theta, \delta\theta \in (0, \delta\theta_{\max})$) is contained in $\mathcal{E}(h)$ (in $\mathcal{E}(k)$).

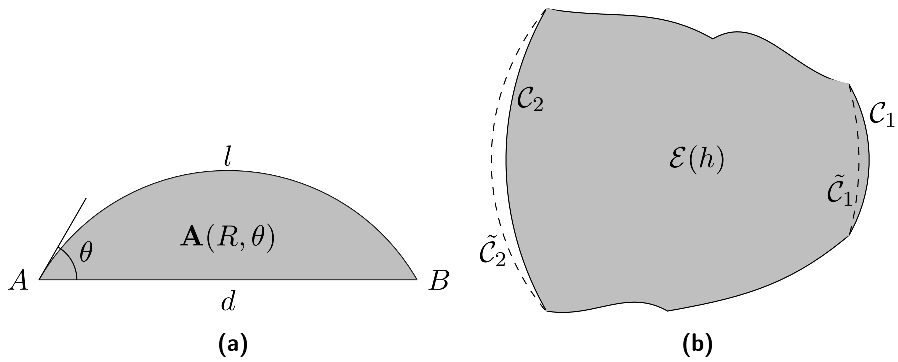


Figure 2.3: In figure **(b)** we can see the original chamber $\mathcal{E}(h)$ and the modified one. The new arcs $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ are represented with dashed lines.

Assume that $\mathcal{C}_1, \mathcal{C}_2$ are both convex in $\mathcal{E}(h)$ (see Figure 2.3 (b)). Moreover assume by contradiction that $\mathcal{C}_1, \mathcal{C}_2$ have different curvature, that is $R_1 \neq R_2$. Without loss of generality, let $R_1 < R_2$, that $\frac{d_1}{\sin(\theta_1)} < \frac{d_2}{\sin(\theta_2)}$. Notice that we can assume also that $R_1 < \infty$. In fact, otherwise, we would have $R_2 = \infty$ and so $\mathcal{C}_1, \mathcal{C}_2$ would have the same curvature. The two arcs enclose the areas

$$A_1 = \frac{d_1^2}{4} \frac{\theta_1 - \sin(\theta_1)\cos(\theta_1)}{\sin^2(\theta_1)} = \mathbf{A}(R_1, \theta_1) := \mathbf{A}_1(\theta_1)$$
$$A_2 = \frac{d_2^2}{4} \frac{\theta_2 - \sin(\theta_2)\cos(\theta_2)}{\sin^2(\theta_2)} = \mathbf{A}(R_2, \theta_2) := \mathbf{A}_2(\theta_2)$$

for some angles θ_1, θ_2. The idea is the following. Keeping the endpoints of $\mathcal{C}_1$ fixed, we reduce the angle θ_1 in $\tilde{\theta}_1 < \theta_1$ and thus we have a new arc $\tilde{\mathcal{C}}_1$. Since the area enclosed by $\tilde{\mathcal{C}}_1$ is smaller than the one inside $\mathcal{C}_1$, we have to increase the angle θ_2 in $\tilde{\theta}_2 > \theta_2$, again keeping fixed the endpoints of $\mathcal{C}_2$, in order to restore the correct area. The new arc $\tilde{\mathcal{C}}_2$ has to satisfy the condition

$$A_1 + A_2 = \mathbf{A}_1(\tilde{\theta}_1) + \mathbf{A}_2(\tilde{\theta}_2)$$

As a consequence, the sum of length of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ changes. In this way, there should be a reduction of the length.

Let us detail this proceeding. The value of $\tilde{\theta}_2$ is determined by

$$\mathbf{A}_2(\tilde{\theta}_2) = A_1 + A_2 - \mathbf{A}_1(\tilde{\theta}_1)$$

Let $\tilde{l}_1, \tilde{l}_2$ be the length of $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$:

$$\tilde{l}_1 = \frac{d_1 \tilde{\theta}_1}{\sin(\tilde{\theta}_1)}, \qquad \tilde{l}_2 = \frac{d_2 \tilde{\theta}_2}{\sin(\tilde{\theta}_2)}$$

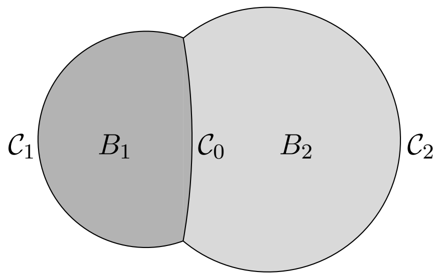


Figure 2.4: Example of a standard double bubble.

and $\tilde{l} = \tilde{l}_1 + \tilde{l}_2$. Then we have

$$\begin{aligned}\frac{\mathrm{d}\tilde{l}}{\mathrm{d}\tilde{\theta}_1} &= d_1 \frac{\sin(\tilde{\theta}_1) - \tilde{\theta}_1 \cos(\tilde{\theta}_1)}{\sin^2(\tilde{\theta}_1)} + d_2 \frac{\sin(\tilde{\theta}_2) - \tilde{\theta}_2 \cos(\tilde{\theta}_2)}{\sin^2(\tilde{\theta}_2)} \tilde{\theta}_2' \\ &= \frac{d_1}{\sin(\tilde{\theta}_1)} \left(1 - \frac{\tilde{\theta}_1 \cos(\tilde{\theta}_1)}{\sin(\tilde{\theta}_1)}\right) + \frac{d_2}{\sin(\tilde{\theta}_2)} \left(1 - \frac{\tilde{\theta}_2 \cos(\tilde{\theta}_2)}{\sin(\tilde{\theta}_2)}\right) \tilde{\theta}_2'\end{aligned}$$

and

$$\tilde{\theta}_2'(\tilde{\theta}_1) = -\frac{d_1^2}{\sin^2(\tilde{\theta}_1)} \frac{\sin^2(\tilde{\theta}_2)}{d_2^2} \left(\frac{1}{1 - \tilde{\theta}_2 \frac{\cos(\tilde{\theta}_2)}{\sin(\tilde{\theta}_2)}}\right) \left(1 - \tilde{\theta}_1 \frac{\cos(\tilde{\theta}_1)}{\sin(\tilde{\theta}_1)}\right)$$

Hence we get

$$\frac{\mathrm{d}\tilde{l}}{\mathrm{d}\tilde{\theta}_1} = \left(1 - \tilde{\theta}_1 \frac{\cos(\tilde{\theta}_1)}{\sin(\tilde{\theta}_1)}\right) \frac{d_1}{\sin(\tilde{\theta}_1)} \frac{\sin(\tilde{\theta}_2)}{\tilde{\theta}_2} \left(\frac{d_2}{\sin(\tilde{\theta}_2)} - \frac{d_1}{\sin(\tilde{\theta}_1)}\right)$$

We notice that, if $\tilde{\theta}_1 = \theta_1$, then $\tilde{l}'(\theta_1) > 0$ by the initial hypothesis on R_1, R_2. Then, if there is a little decrease of θ_1, we have a reduction of the lenght $\tilde{l}$, keeping the enclosed area constant. This fact contradicts the minimality of the cluster and so it proves the statement if $\mathcal{C}_1, \mathcal{C}_2$ are both convex (or concave) in $\mathcal{E}(h)$.

If $\mathcal{C}_1, \mathcal{C}_2$ are not both convex (or concave), there is a similar computation which leads to the same conclusion.

This end the proof of the theorem.

□

2.4 Standard double bubble

In this section we are going to focus on the simplest partitioning problem in the plane: the case with only two chambers. In particular, thanks to the regularity theorems of the previous sections and the fact that we are dealing only with arcs, the arguments will be quite simplified.

From now, a cluster will be called also *soap bubble cluster* and its chambers *bubbles*. With *standard double bubble* (see Figure 2.4) we mean a cluster with two vertices and three arcs meeting in threes with 120° angles. The aim of this section is to prove the following theorem.

Theorem 2.8. *For each $A_1, A_2 > 0$, up to rotations and translations, there exists an unique standard double bubble enclosing the areas A_1, A_2. Moreover this is the unique minimizer for the partitioning problem associated to A_1, A_2.*

Now, in order to prove the theorem, we are going to state and demonstrate a series of lemmas and propositions.

Proposition 2.9. *A 2-minimizer cluster $\mathcal{E}$ in $\mathbb{R}^2$, with both connected bubbles and connected exterior, is a standard double bubble.*

Proof. We notice that we can always assume the set $\mathcal{E}(1) \cup \mathcal{E}(2)$ to be connected. Indeed if it wasn't so, sliding the different connected components up to their boundaries are tangent, we would have constructed a minimizing cluster which contradicts the regularity theorems.

Consider the graph with the endpoints of the arcs of $\mathcal{E}$ as vertices and the same arcs as edges. Thus we can apply Euler's formula for connected planar graphs. If V is the number of vertices, F the number of faces (included the exterior) and E the number of edges of the graph, it holds

$$V - E + F = 2$$

Since in our case $F = 3$, we have

$$2E = \sum_{i=1}^{V} \deg(v_i) = 3V$$

where $\{v_i\}_i$ are the vertices. Then we get

$$V = 2, \qquad E = 3$$

and so $\mathcal{E}$ is a standard double bubble. $\square$

Proposition 2.10. *If a 2-minimizing cluster $\mathcal{E}$ has exterior connected then it is a standard double bubble.*

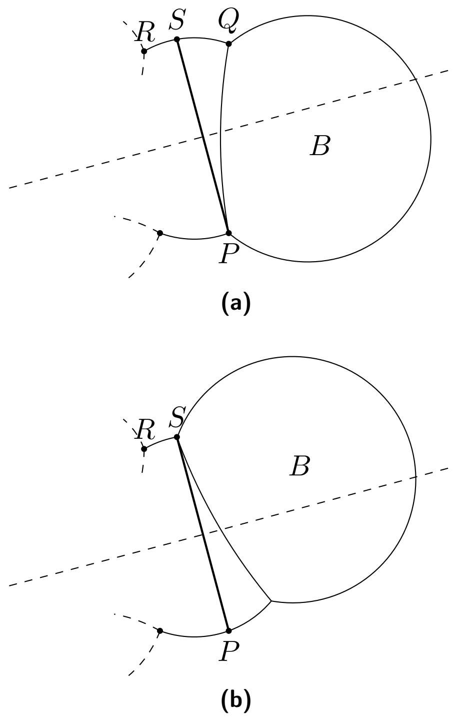


Figure 2.5: In figure **(b)** we see the modified cluster.

Proof. We already know that if a 2-minimizer cluster has connected exterior and connected bubbles then it is standard. Then it is enough to prove that each bubble is connected. Let us assume by contradiction that there exists a disconnected chamber. Let's construct a graph G associated to $\mathcal{E}$ with vertices in the bubbles and edges between adjacent chambers (in this case the exterior is not considered as a bubble). By the regularity of a perimeter minimizer and since the exterior is connected, G has no cycles. Then there exists a vertex of the graph with degree equal to 1. Let's denote with B the connected component of $\mathcal{E}$ associated to this vertex. By definition, B is adjacent just to one connected component of a bubble and to the exterior: then it is composed by two arcs and two vertices P, Q. Now we are going to get a contradiction using the regularity theorems.

In fact we construct the following clusters. Set $S = Q$; we modify the original cluster by a reflection across the axis of PQ as in Figure 2.5. Move S along the arc RQ until the modified bubble is tangent to another bubble. If it is tangent we have a contradiction with the regularity theorem because there is a 4-point. Otherwise $S = R$ and also in this case we have a 4-point. □

The following proposition states the existence of a standard double bubble enclosing the right areas.

PROPOSITION 2.11. *Given $A_1, A_2 > 0$ there exists a standard double bubble which encloses the areas A_1, A_2. Moreover, up to rotations and translations, it is unique.*

Proof. Let $A_1 \leq A_2$ and $\lambda = \frac{A_1}{A_2} \in (0, 1]$. We are going to construct a standard double bubble $\{B_1, B_2\}$ with

$$\frac{\text{area}(B_1)}{\text{area}(B_2)} = \lambda$$

Consider two points A, B (the vertices) at distance 1. For a generic angle $\theta \in [0, \pi/3)$, let's define the standard double bubble through its arcs $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$. Each of them has A, B as endpoints and $\mathcal{C}_0$ forms an angle θ with the segment AB, $\mathcal{C}_2$ is contained in the same half-plane of $\mathcal{C}_0$ and forms an angle $\frac{2\pi}{3} + \theta$ with AB, $\mathcal{C}_1$ belongs to the other half-plane and has an angle $\frac{2\pi}{3} - \theta$ with AB. The bubble B_1 is the one enclosed by $\mathcal{C}_0$ and $\mathcal{C}_1$, the bubble B_2 by $\mathcal{C}_0$ and $\mathcal{C}_2$ (see Figure 2.4).

If $\mathbf{A}(\theta)$ represents the area "inside" an arc with distance 1 between its endpoints and angle θ, then

$$\text{area}(B_1) = \mathbf{A}\left(\frac{2\pi}{3} - \theta\right) + \mathbf{A}(\theta)$$
$$\text{area}(B_2) = \mathbf{A}\left(\frac{2\pi}{3} + \theta\right) - \mathbf{A}(\theta)$$

Define the function

$$R(\theta) = \frac{\text{area}(B_1)(\theta)}{\text{area}(B_2)(\theta)} = \frac{\mathbf{A}\left(\frac{2\pi}{3} - \theta\right) + \mathbf{A}(\theta)}{\mathbf{A}\left(\frac{2\pi}{3} + \theta\right) - \mathbf{A}(\theta)}, \quad \theta \in [0, \pi/3)$$

Notice that, since $\mathbf{A}'' > 0$, area$(B_1)(\theta)$ is strictly decreasing in θ. In fact

$$\text{area}(B_1)'(\theta) = -\mathbf{A}'\left(\frac{2\pi}{3} - \theta\right) + \mathbf{A}'(\theta)$$

and

$$\mathbf{A}'(\theta) < \mathbf{A}'\left(\frac{2\pi}{3} - \theta\right)$$

for $\theta \in [0, \pi/3)$. Similarly, we can see that area(B_2) is increasing in θ. Hence we deduce that $R(\theta)$ is strictly decreasing. Moreover it holds

$$R(0) = 1, \qquad R\left(\frac{\pi}{3}\right) = 0$$

Then $R\colon [0, \pi/3) \to (0, 1]$ is a bijection and thus there exists $\theta \in [0, \pi/3)$ such that

$$R(\theta) = \lambda$$

Finally, if $t \in \mathbb{R}_{>0}$ is such that $\text{area}(tB_1) = A_1$, then

$$\frac{A_1}{A_2} = \lambda = \frac{\text{area}(tB_1)}{\text{area}(tB_2)} = \frac{A_1}{\text{area}(tB_2)}$$

and so $\text{area}(tB_2) = A_2$. Thus the cluster $\{tB_1, tB_2\}$ is a standard double bubble which encloses the correct areas.

□

In the two following lemmas we prove the uniqueness up to rotations and translations of the standard double bubble.

LEMMA 2.12. *Given two points V_1, V_2 and $\theta \in [0, \pi/3)$, there exist exactly two standard double bubbles $\mathcal{E}, \mathcal{E}'$ with V_1, V_2 as vertices which form angles of $\theta, \frac{2\pi}{3} - \theta, \frac{2\pi}{3} + \theta$ with the segment V_1V_2. Moreover each cluster is symmetric w.r.t. the axis of V_1V_2 and $\mathcal{E}$ is the symmetric of $\mathcal{E}'$ w.r.t. the segment V_1V_2.*

Proof. Let $\mathcal{E}$ be the cluster previously constructed and α, β the two half-spaces determined by the segment V_1V_2. We can assume that the arc $\mathcal{C}_0$, which forms with V_1V_2 the angle $\theta \in [0, \pi/3)$, is contained in α. Let $\mathcal{E}'$ be another standard double bubble with the above properties and let $\mathcal{C}_0'$ be the arc of $\mathcal{E}'$ which has the angle $\theta' \in [0, \pi/3)$ with V_1V_2. If $\mathcal{C}_0'$ is contained in α then $\mathcal{C}_0' = \mathcal{C}_0$ and so $\mathcal{E} = \mathcal{E}'$ because of the $120°$ rule. Otherwise, if $\mathcal{C}_0'$ is contained in β, then $\mathcal{C}_0'$ is the symmetric of $\mathcal{C}_0$ w.r.t. V_1V_2. Then the cluster $\mathcal{E}'$ can be uniquely constructed and it is the symmetric of $\mathcal{E}$ w.r.t. V_1V_2. Moreover, by the symmetry properties of the circle, each arc $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_0', \mathcal{C}_1', \mathcal{C}_2'$ is symmetric w.r.t. the axis of V_1V_2, and so also the two clusters $\mathcal{E}, \mathcal{E}'$.

□

LEMMA 2.13. *Let $\mathcal{E}, \mathcal{E}'$ be two standard double bubbles enclosing the areas $A_1, A_2 > 0$. Then, up to rotations and translations, they coincide.*

Proof. Let V_1, V_2 the vertices of $\mathcal{E}$ and V_1', V_2' those of $\mathcal{E}'$; set d, d' as their length. We prove that $d = d'$. Assume by contradiction that $d < d'$. Let θ, θ' the unique angles in $[0, \pi/3)$ formed, respectively, by the arcs of $\mathcal{E}, \mathcal{E}'$ with $V_1V_2, V_1'V_2'$. Let's denote with $\mathbf{A}(d, \theta)$ the area enclosed by an arc with angle θ and with d as distance between its endpoints. Then, with the same notation used in this section, we have

$$\text{area}B_1(d, \theta) = \mathbf{A}\left(d, \frac{2\pi}{3} - \theta\right) + \mathbf{A}(d, \theta)$$
$$\text{area}B_2(d, \theta) = \mathbf{A}\left(d, \frac{2\pi}{3} + \theta\right) - \mathbf{A}(d, \theta)$$

Remember that $\text{area}B_1$ is decreasing in θ, while $\text{area}B_2$ is increasing. Moreover they are clearly increasing in d. Then we have

$$\text{area}B_1(d', \theta') = A_1 = \text{area}B_1(d, \theta) < \text{area}B_1(d', \theta)$$

which implies $\theta' > \theta$. Similarly, we have

$$\text{area}B_2(d', \theta') = A_2 = \text{area}B_2(d, \theta) < \text{area}B_2(d', \theta)$$

which implies $\theta' < \theta$. Thus we have a contradiction and so $d = d'$. As a consequence, we have $\theta = \theta'$ and so, up to rotations and translations, $\mathcal{E}$ and $\mathcal{E}'$ are two clusters with the same endpoints which form angles of θ, $\frac{2\pi}{3} - \theta$, $\frac{2\pi}{3} + \theta$. By the previous lemma either $\mathcal{E} = \mathcal{E}'$ or $\mathcal{E}$ is the symmetric of $\mathcal{E}'$ w.r.t. the segment V_1V_2. Then, applying a rotation r of $180°$ with center in the middle point of V_1V_2, we have $r(\mathcal{E}') = \mathcal{E}$.

□

LEMMA 2.14. *The perimeter of a standard double bubble is increasing w.r.t. the larger of the two enclosed areas.*

Proof. Assume, without loss of generality, that the distance between the two vertices is 1. Then the perimeter of the cluster is

$$P(\theta) = L(\theta) + L\left(\frac{2\pi}{3} + \theta\right) + L\left(\frac{2\pi}{3} - \theta\right)$$

with $L(\theta) = \frac{\theta}{\sin\theta}$. It is easy to see that $L', L'' > 0$ on $(0, \pi)$. Moreover $L\left(\frac{2\pi}{3} + \theta\right) + L\left(\frac{2\pi}{3} - \theta\right)$ is increasing in θ. In fact

$$L''\left(\frac{2\pi}{3} + \theta\right) + L''\left(\frac{2\pi}{3} - \theta\right) > 0$$

on $(0, \pi/3)$ and so $L'\left(\frac{2\pi}{3} + \theta\right) - L'\left(\frac{2\pi}{3} - \theta\right)$ is increasing. Since

$$L'\left(\frac{2\pi}{3} + \theta\right)_{|\theta=0} - L'\left(\frac{2\pi}{3} - \theta\right)_{|\theta=0} = 0$$

then $L'\left(\frac{2\pi}{3} + \theta\right) - L'\left(\frac{2\pi}{3} - \theta\right) > 0$ on $(0, \pi/3)$.

Hence P is increasing in θ. Since enhancing θ the area of the biggest chamber rises, then P is increasing w.r.t. the area of the largest bubble.

□

In this proposition we are going to use the following notation. We denote by $P(A_1, A_2)$ the perimeter of the minimizing cluster of areas (A_1, A_2) and by $P_0(A_1, A_2)$ the perimeter of the minimizing standard double bubble enclosing (A_1, A_2). In general we write $P(\{B_1, B_2\})$ for the perimeter of the cluster $\{B_1, B_2\}$.

PROPOSITION 2.15. *The exterior of a 2-minimizing cluster is connected.*

Proof. Let $A_1 \geq A_2$ and $\{B_1, B_2\}$ a perimeter minimizer for the partitioning problem associated to A_1, A_2. Suppose by contradiction that the exterior is not connected. It is easy to see that $P(\cdot, A_2)$ has minimum in $[A_1, +\infty)$ and that $P(A, A_2) \to \infty$ as $A \to \infty$. Set $A_1' \in [A_1, \infty)$ as the value which realizes the minimum of $P(\cdot, A_2)$; moreover we take A_1' as big as possible. In particular

$$P(A_1', A_2) < P(A, A_2) \tag{2.17}$$

for every $A > A_1'$. Let $\{B_1', B_2'\}$ a perimeter-minimizer associated to (A_1', A_2). For sure $\{B_1', B_2'\}$ has exterior connected. In fact, if it wasn't so, we could construct the cluster $\{B_1'', B_2'\}$ incorporating the bounded connected components of the exterior of $\{B_1', B_2'\}$ inside B_1'. In this way, we have

$$\left|B_1''\right| > \left|B_1'\right|, \qquad P(\{B_1'', B_2'\}) \leq P(\{B_1', B_2'\})$$

which contradicts the minimality of $\{B_1', B_2'\}$.

Since $\{B_1', B_2'\}$ is a perimeter-minimizer with exterior connected then, by proposition 2.10, it is a standard double bubble and so

$$P(A_1', A_2) = P_0(A_1', A_2)$$

Now we have two possibilities: either $A_1' > A_1$ or $A_1' = A_1$. If $A_1' > A_1$, thanks to the last lemma, we have

$$P(A_1, A_2) \geq P(A_1', A_2) = P_0(A_1', A_2) > P_0(A_1, A_2) \geq P(A_1, A_2)$$

which is a contradiction. Now assume $A_1' = A_1$. As the cluster $\{B_1, B_2\}$ has disconnected exterior, define $\{B_1'', B_2\}$ incorporating the connected components of the exterior of $\{B_1, B_2\}$ inside B_1 and let $A_1'' = \left|B_1''\right| > A_1$. Then we have

$$P(A_1, A_2) \geq P(\{B_1'', B_2\}) \geq P(A_1'', A_2) > P(A_1, A_2)$$

by (2.17) and $A_1'' > A_1 = A_1'$. Also in this case we have a contradiction. Then the exterior of $\{B_1, B_2\}$ is connected.

□

Finally, we can give the proof of the main theorem of this section.

Proof of Theorem 2.8. We have just to put together the previous statements. We know that the exterior of a perimeter-minimizer is connected and so it is a standard double bubble. Moreover we know that the standard double bubble enclosing two areas A_1, A_2 is unique up to rotations and translations. This ends the proof.

□

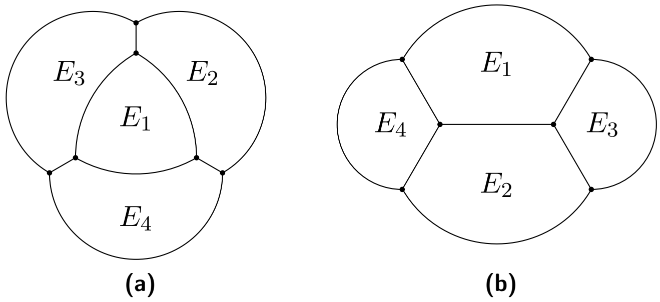


Figure 2.6: Figure **(a)** shows a cluster with the *flower* topology. Figure **(b)** instead one with the *sandwich* topology.

2.5 Quadruple planar bubble

In this final section, we are going to deal with a partitioning problem in the plane where there are four chambers. In particular we consider just the case of four equal areas. We quickly summarize what states [9] and then we are going to show the symmetric properties of these clusters. In that article it is proved that, if $\mathcal{E}$ is a 4-cluster which minimizes the perimeter and encloses the correct (equal) areas, then $\mathcal{E}$ has exactly four connected regions, two among them are quadrangular and have a common edge while the remaining two are triangular and are adjacent to both the quadrangular ones. The idea of the article is the following. First of all, it can be proved that every connected component of each chamber has at least three edges. This is true in general for an N-cluster, $N > 2$. Moreover the number of edges can be bounded in this way. If $\mathcal{E}(h)$ is a chamber with k connected components, C is one of them and M is the total number of connected components of the cluster, then C has at most $M + 1 - k$ edges. Then it is shown that a minimal cluster has at most six connected components. Since it is not possible to have exactly six connected components, because each chamber can not have three components and there are not two different regions disconnected, then we reduce to the case of five connected components. Analysing every possible configuration with five components, we see that they are not admissible. Thus we deduce that the minimal cluster has four connected components. Now there are just two possible cases: the *flower* topology and the *sandwich* topology (see Figure 2.6). The first is excluded and hence the only admissible cluster is the one described before.

The aim of this is section is to prove the following symmetry theorem.

Theorem 2.16. *Let $a > 0$. The minimizing cluster for the partitioning problem associated to the volume vector (a, a, a, a) is, up to rotations, translations and modifications by zero measure sets, composed by four regions. The regions $\mathcal{E}(1), \mathcal{E}(2)$ are quadrangular and one is the reflection of the other* *through the segment in common between them. The regions* $\mathcal{E}(3), \mathcal{E}(4)$ *are triangular and one is the mirror-image of the other through the axis of the common segment between* $\mathcal{E}(1), \mathcal{E}(2)$.

We give these two preliminary statements. We say that a cluster $\mathcal{E}$ is *stationary* if the boundary of $\mathcal{E}$ is composed of arcs (circular arcs or line segments) which meet in threes forming $120°$ angles and if, at each endpoint of an arc, the sum of the signed curvature is zero. The following lemmas deal with the stationarity of a cluster under some particular constructions.

LEMMA 2.17. *Stationarity is preserved under isometries, homotheties and circle inversion.*

LEMMA 2.18. *Let* T *be a triangular region of a stationary cluster* $\mathcal{E}$*. Consider the three arcs not edges of* T *that have a certain vertex of* T *as endpoint. If these three arcs are prolonged inside* T*, they meet in a single point* P *inside* T *with three* $120°$ *angles. The cluster obtained in this way is also stationary.*

Now we are able to start the proof of the theorem. Let p_0, p_1, p_2 the vertices of the triangular region $\mathcal{E}(3)$ and p_3, p_4, p_5 the ones of the other triangular region $\mathcal{E}(4)$ (see Figure 2.7). We remove the chambers $\mathcal{E}(3), \mathcal{E}(4)$ and by lemma 2.18 we know that, extending the remaining arcs up to they meet, we get a standard double bubble. Let us denote with $\mathcal{E}' = \{\mathcal{E}'(1), \mathcal{E}'(2)\}$ this new cluster. From now we are going to identify the plane with $\mathbb{C}$. Without loss of generality (up to rotations, translations and rescaling) assume that the vertices of $\mathcal{E}'$ are the points $(0,0), (1,0)$. Since $\mathcal{E}'$ is a standard double bubble, either $\mathcal{E}'(1)$ or $\mathcal{E}'(2)$ is convex. We can suppose that $\mathcal{E}'(2)$ is convex and that $\mathcal{E}'(1)$ is contained in $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Let $\theta \in [0, \pi/3)$ be the angle formed by the arc separating $\mathcal{E}'(1), \mathcal{E}'(2)$ and the line segment between the two vertices of the cluster. We want to prove that $\theta = 0$. Define $\mathcal{F}$ as the cluster obtained from $\mathcal{E}$ through the circle inversion

$$R(\omega) = \frac{\omega}{|\omega|^2}$$

By this we mean that $\mathcal{F}(i) = R(\mathcal{E}(i)), i = 1, \ldots, 4$. Moreover define the points $q_i = R(p_i), i = 0, \ldots, 5$.

By lemma 2.17 we know that $\mathcal{F}$ is also stationary. We notice that, since $0 \in \mathcal{E}(4)$, then $\mathcal{F}(4)$ is unbounded, while the exterior of $\mathcal{E}$ is mapped into $\mathcal{F}(0)$ which is bounded. Moreover, we observe that the arcs $\widehat{p_1p_4}, \widehat{p_0p_3}, \widehat{p_2p_5}$, which are contained in the edges of the cluster $\mathcal{E}'$, become three line segments whose extensions meet in the point 1 with $120°$ angles. In fact since the arcs of $\mathcal{E}'$ join the points 0 and 1, they are mapped into three half lines starting at the point 1. They form $120°$ angles because R is a conformal function. In particular, since the arc $\widehat{p_0p_3}$ forms an angle θ with the real axis, then also the segment q_0q_3 forms an angle θ with the same axis.

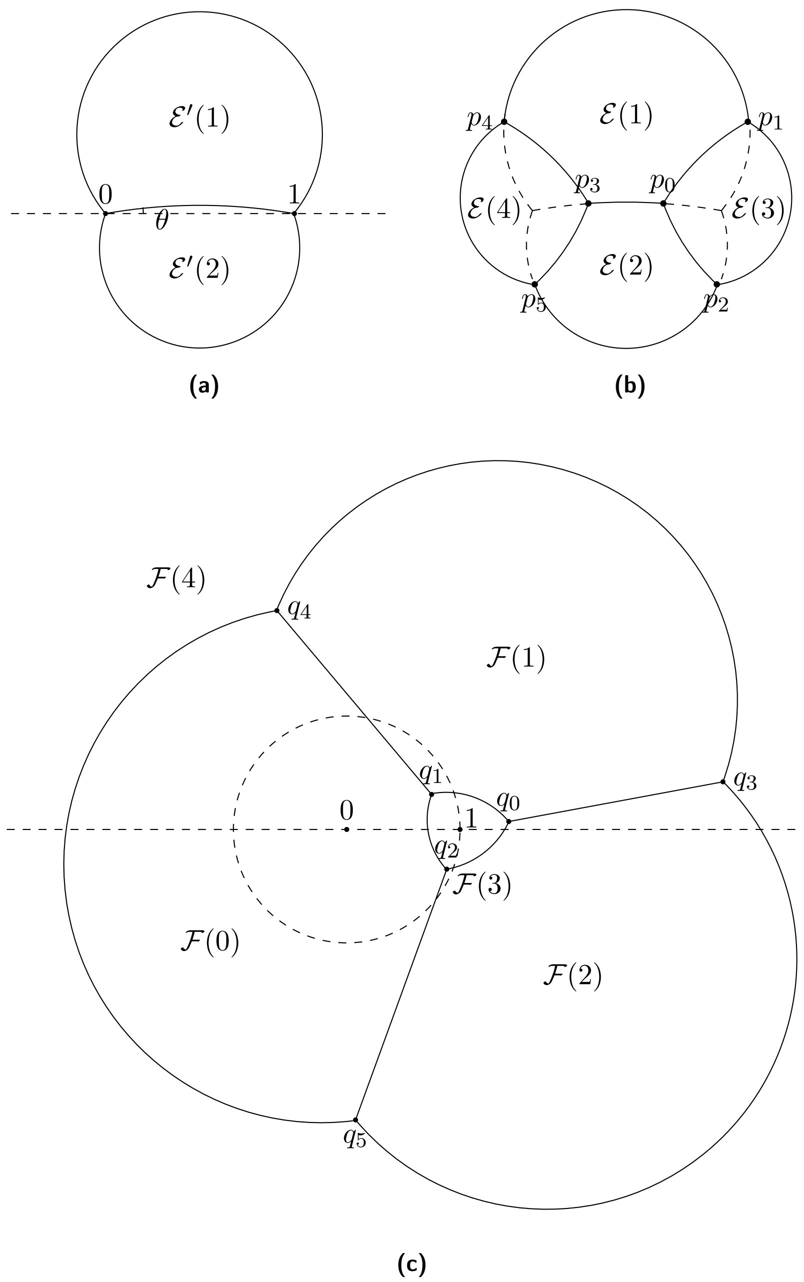


Figure 2.7: In figure **(b)** we see the original cluster $\mathcal{E}$, while in figure **(c)** the reflected one $\mathcal{F}$.

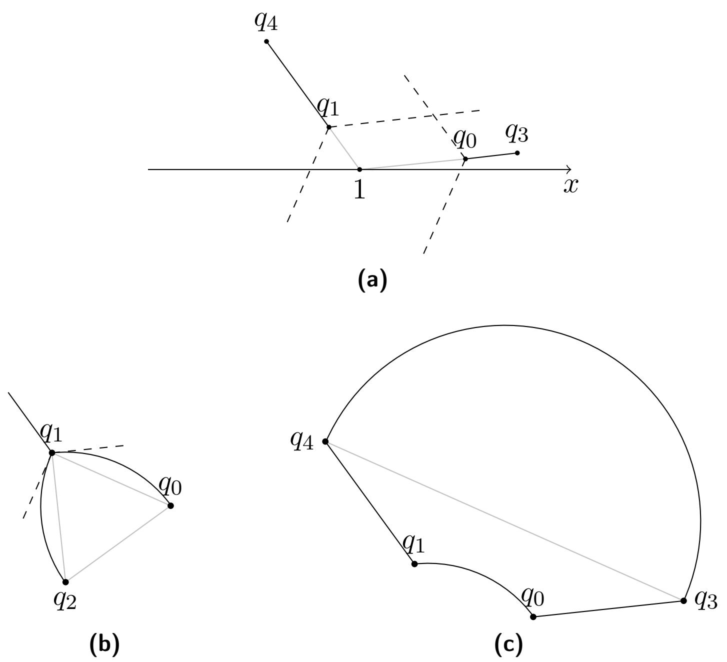


Figure 2.8: In figure **(a)** we see that it is not possible to have an arc joining q_0 and q_1 and tangent to the dashed lines.

LEMMA 2.19. *The points q_0, q_1, q_2 are at the same distance $r = r(\theta) > 0$ from the point 1. Similarly, the points q_3, q_4, q_5 have the same distance $R = R(\theta) > r$ from the point 1. It follows that*

$$q_i = 1 + re^{i(\theta+2i\pi/3)}, \quad q_{i+3} = 1 + Re^{i(\theta+2i\pi/3)}, \quad i = 0, 1, 2$$

Proof. This is an immediate consequence of the fact that q_1 is an endpoint of two circular arcs starting from q_0 and q_2 which form 120° angles in q_1. In fact, if the distances of q_0 and q_1 from 1 are different, then it is not be possible to have an arc between q_1 and q_0 which forms 120° angles with the two line segments, as it can be seen in Figure 2.8. In a similar way, it can be proved the statement for the points q_3, q_4, q_5. □

Since the angles formed by the arcs meeting in q_0, q_1, q_2 are 120°, it follows that the arcs $\widehat{q_0q_1}, \widehat{q_1q_2}, \widehat{q_2q_0}$ are centered, respectively, in q_2, q_0, q_1. In fact the perpendicular line to the segment q_0q_1 is also tangent to the arc $\widehat{q_1q_2}$ in q_1 and so q_0q_1 is the radius of the arc $\widehat{q_1q_2}$ (see Figure 2.8 (b)).

Similarly, the arcs $\widehat{q_3q_4}, \widehat{q_4q_5}, \widehat{q_5q_3}$ are half circles. Indeed, the angle between the segment q_1q_4 and q_3q_4 is 30° and so, by the 120° rule, we deduce that the arc $\widehat{q_3q_4}$ is an half circle (see Figure 2.8 (c)).

Thanks to these two statements, we get the following corollary.

COROLLARY 2.20. *The cluster $\mathcal{F}$ is symmetric w.r.t. the line q_0q_3.*

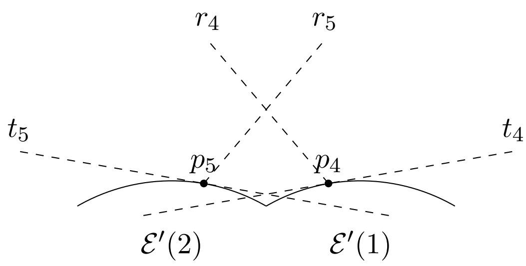


Figure 2.9

The following lemma together with the last corollary will provide the symmetry of $\mathcal{E}$ w.r.t. the line $\{z : \mathrm{Re}z = \frac{1}{2}\}$.

LEMMA 2.21. *The radii $r(\theta), R(\theta)$ are uniquely determined by the conditions $\big|\mathcal{E}(3)\big| = a, \big|\mathcal{E}(4)\big| = a$.*

Proof. The set $\mathcal{F}(3)$ is (strictly) increasing (w.r.t. the inclusion) in r. Then, since $\mathcal{E}(3) = R(\mathcal{F}(3))$, we get that $\big|\mathcal{E}(3)\big|$ is strictly increasing in r. Thus there exists an unique radius r such that $\big|\mathcal{E}(3)\big| = a$. With an analogous argument we deduce that R is uniquely determined by $\big|\mathcal{E}(4)\big| = a$ (in this case $\mathcal{F}(4)$ is strictly decreasing in R). $\square$

COROLLARY 2.22. *The cluster $\mathcal{E}$ is symmetric w.r.t. the line $s = \{z \in \mathbb{C} : \mathrm{Re}z = \frac{1}{2}\}$.*

Proof. We know that the chambers $\mathcal{E}(3), \mathcal{E}(4)$ are uniquely determined by r, R and so by θ. Consider the cluster $\mathcal{E}''$ defined as the symmetric of $\mathcal{E}$ w.r.t. s. Removing the triangular region of $\mathcal{E}''$ and prolonging the remaining edges, one of them forms an angle θ in 0 and 1. As a consequence the two triangular regions $\mathcal{E}(3)'', \mathcal{E}(4)''$ satisfy $\{\mathcal{E}(3), \mathcal{E}(4)\} = \{\mathcal{E}(3)'', \mathcal{E}(4)''\}$. Hence $\mathcal{E}(3), \mathcal{E}(4)$ are symmetric w.r.t. s and so it follows that $\mathcal{E}$ is symmetric w.r.t. s. $\square$

Thus the first symmetry property is proved. Now we are going to prove that $\theta = 0$. The idea is the following. If we assume by contradiction that $\theta > 0$, we would have $\big|\mathcal{E}(1)\big| > \big|\mathcal{E}(2)\big|$ because $\mathcal{F}(1)$ is closer to 0 rather than $\mathcal{F}(2)$, in a certain sense. The sets $\mathcal{F}(1), \mathcal{F}(2)$ can be written as

$$\mathcal{F}(1) = \left\{1 + \rho e^{i(\theta+\psi)} : \psi \in \left[0, \frac{2}{3}\pi\right], \rho \in [r_1(\psi,\theta), r_2(\psi,\theta)]\right\}$$

$$\mathcal{F}(2) = \left\{1 + \rho e^{i(\theta-\psi)} : \psi \in \left[0, \frac{2}{3}\pi\right], \rho \in [r_1(\psi,\theta), r_2(\psi,\theta)]\right\}$$

The Jacobian determinant of R is $JR(z) = \frac{1}{|z|^2}$. Then by the area formula we get

$$\begin{aligned} \left|\mathcal{E}(1)\right| - \left|\mathcal{E}(2)\right| &= \int_{\mathcal{F}(1)} \frac{1}{|x+iy|^2} \mathrm{d}x\mathrm{d}y - \int_{\mathcal{F}(2)} \frac{1}{|x+iy|^2} \mathrm{d}x\mathrm{d}y \\ &= \int_0^{\frac{2}{3}\pi} \int_{r_1(\psi,\theta)}^{r_2(\psi,\theta)} \left[\frac{1}{\left|1+\rho e^{i(\theta+\psi)}\right|^2} - \frac{1}{\left|1+\rho e^{i(\theta-\psi)}\right|^2} \right] \rho \,\mathrm{d}\rho \mathrm{d}\psi \end{aligned} \tag{2.18}$$

Since it holds $\left|1+\rho e^{i\alpha}\right|^2 = 1+\rho^2+2\rho\cos\alpha$ and by the addition formula $\cos(\theta+\psi) < \cos(\theta-\psi)$ whenever $\psi \in (0, \frac{2}{3}\pi], \theta \in (0, \frac{\pi}{3})$, it follows that

$$\left|1+\rho e^{i(\theta+\psi)}\right|^2 < \left|1+\rho e^{i(\theta-\psi)}\right|^2 \tag{2.19}$$

Then by (2.18) and (2.19), we deduce that $\left|\mathcal{E}(1)\right| > \left|\mathcal{E}(2)\right|$ which is a contradiction. Thus $\theta = 0$. Then we conclude that the real axis is a symmetry axis for the cluster $\mathcal{E}$. In fact $\mathcal{E}'$ is a standard double bubble with equal areas and it is formed by a straight line and two arcs with the same radius. As a consequences the points p_4, p_5 are symmetric because, otherwise, there could not be a circular arc between p_4, p_5 that satisfies the 120° rule. Indeed a necessary condition is that the tangent lines to the arc in p_4, p_5 (the lines r_4, r_5 the Figure 2.9) and the axis of p_4p_5 meet in a single point. Moreover, by the 120° rule, this is equivalent to have the tangent lines to the arcs of $\mathcal{E}'(1), \mathcal{E}'(2)$ in p_4, p_5 (the lines t_4, t_5) and the axis of p_4p_5 meeting in a single point. We see that, if p_4, p_5 are not symmetric, this is not true (see Figure 2.9).

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