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NON-DIFFERENTIABILITY SETS OF TYPICAL LIPSCHITZ FUNCTIONS

Supervisors:
Prof. DAVID PREISS
Prof. ROBERTO MONTI

Candidate:
ANDREA MERLO

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Introduction

General Framework

The study of differentiability properties of Lipschitz functions has a long story. It started with H. Lebesgue who proved the almost everywhere differentiability of Lipschitz functions on the real line. In 1919 H. Rademacher understood that this almost everywhere differentiability was not just a property of the line itself, indeed he proved his famous and celbrated:

Theorem 0.1 (Rademacher, 1919)**.** *A Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable outside a Lebesgue null set.*

Such a beautiful result left a lot of questions open, which can be organized in two big families:

(i) Does a similar statement hold for infinite dimensional Banach spaces?

(ii) Does a viceversa hold for the Rademacher theorem, i.e., is it possible to give a characterization of non-differentiability sets of Lipschitz functions?

A lot of problems concerning both these macro areas of research are still open however a lot of work has been done, as well. A remarkable breaktrough in both problems occured in 1990 when D. Preiss published his paper [11], in which he proved that on Banach spaces having an equivalent norm that is differentiable away from the origin, Lipschitz functions are Fréchet differentiable on a dense subset. Moreover, in the last pages of this paper Preiss esplicitely contructed a G_δ dense in set $\mathbb{R}^n$ on which every real valued Lipschitz function has a differentiability point. This amazing and counterintuitive result made it clear that a converse for the Rademacher's Theorem was not a straightforward problem at all. From this starting point a lot of theory has been developed in the last few years in order to solve (ii), indeed in 2005 G. Alberti, M. Csörnyei and D. Preiss announced with their papers [2] and [3] a complete solution for (ii) in the case of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m > n$. In 2015 G. Alberti and A. Marchese proved in [1] that the Rademacher Theorem can be extended to finite mass Borel measures and in the same year D. Preiss and G. Speight proved in the paper [9] that for any $m \leq n$ there exists a set of universal differentiability for maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, but in this case a full characterization for non-differentiability sets seems to be not around the corner.

Main Result, scheme of its proof and some comments

The problem we investigate in this thesis could be intuitively rephrased in the following way: *"If we have a huge amount of Lipschtz functions being non-differentiable on the same set, what can we say about that set?"*

In 1995 D. Preiss and J. Tišer studyed this problem for the space of Lipschitz funtions $\mathrm{Lip}_1(1,1) := \{f : [0,1] \to \mathbb{R} : \mathrm{Lip}(f) \leq 1\}$ endowed with the uniform norm. They actually were able to give a precise meaning to the above question and a complete answer, which is contained in the main result of the paper [10]:

Theorem 0.2. *Let $E \subseteq [0,1] \subseteq \mathbb{R}$ be an analytic set. The following are equivalent:*

(i) The set S of those functions $f \in \mathrm{Lip}_1(1,1)$ which are differentiable at no point of E is residual in $\mathrm{Lip}_1(1,1)$.

(ii) E is contained in an F_σ subset of $[0,1]$ of Lebesgue measure zero.

We note that in this result the "*size*" of sets involved is described in terms of their topological properties.

The main result of this work is an extension to general dimension of domain and codomain of the (i)⇒(ii) implication of Theorem 0.2:

Theorem 0.3. *Let $E \subseteq [0,1]^n$ be an analytic set. If the set S of those functions $f \in \mathrm{Lip}_1(n,m)$ which are differentiable at no point of E is residual in $\mathrm{Lip}_1(n,m)$, then the set E is contained in an F_σ subset of $[0,1]^n$ of Lebesgue measure zero.*

The approach we use in the proof of Theorem 0.3 is similar (from the point of view of the ideas involved) to the one Preiss and Tišer used in [10]. The goal is to find a winning strategy for Player II in the Banach-Mazur game in $\mathrm{Lip}_1(n,m)$, which is introduced in Section 1.1; where Player I is dealt with the set:

$$A := \{f \in \mathrm{Lip}_1(n,m) : f \text{ is non-differentiable on } E\}.$$

Therefore, thanks to Theorem 1.3, which characterizes residual sets with Banach-Mazur game, we prove by contradiction in Theorem 2.14 that whenever the set E is *not* contained in a F_σ of null Lebesgue measure, we can construct a winning strategy for Player II, and hence the set $\mathrm{Lip}_1(n,m) \setminus A$ is residual.

To find such a strategy, in Section 1.2 we introduce the set of $\mathfrak{P}(n,m)$ functions, which are piecewse linear $\mathrm{Lip}_1(n,m)$ functions having Jacobian with maximal Hilbert Schimdt norm where the differential is defined. These functions are dense in $\mathrm{Lip}_1(n,m)$ (fact that is proved in Section 1.4) and satisfy to the following inequality (see Theorem 1.11):

$$\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 dx \leq C(f,n,m)\|f-g\|_\infty. \tag{1}$$

In the proof of this inequality is fundamental that the Hilbert-Schmidt norm of the Jacobian in constant. Moreover, in order to obtain a dense class of functions satisfying (1), we have to add the condition that the Hilbert-Schmidt norm is maximal. Therefore, using (1) and the strong L^2-estimate for the maximal operator defined in Section 1.3) we prove the following:

Proposition 0.4. *For all $0 < \epsilon < \frac{1}{2^{n+2}}$ and $f \in \mathfrak{P}(n,m)$ there exists an open neighbourhood V of f in $Lip_1(n,m)$ with $diam(V) < \epsilon$ such that for any $g \in \mathfrak{P}(n,m) \cap V$ there are an open set $G \subseteq (0,1)^n$ and a constant $D(n,m)$ having the following propertGies:*

(i) $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\epsilon$.

(ii) $\|J(f-g)(x)\|_{HS} \leq \epsilon$ *for any* $x \in G$.

(iii) $|g(y) - g(x) - (f(y) - f(x))| \leq \epsilon|x-y|$ *for any* $x \in G$ *and any* $y \in [0,1]^n$.

This proposition links the topology of $\mathrm{Lip}_1(n,m)$ to the measure of the set where we have control on the differential with a certain precision. In this way, the second player can choose a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ uniformly converging to some $f \in \mathrm{Lip}_1(n,m)$ and a sequence of open sets G_k such that $\bigcap_{k\in\mathbb{N}} G_k \cap E \neq \emptyset$. Thanks to Proposition 0.4, G_k are chosen in such a way f is differentiable at any $x \in \bigcap_{k\in\mathbb{N}} G_k \cap E$ and $Jf_k(x) \to Jf(x)$ as $k \to \infty$ (see Proposition 2.10). In this way we get a contradiction and therefore Theorem 0.3 is proved.

To conclude we would like to remark that there is a deep difference between Theorem 0.2 and Theorem 0.3. Indeed the former is an characterzation, the latter is not. Therefore one could wonder if it is possible to get a converse for Theorem 0.3, in full analogy with the one dimensional case. The answer is *not always*, since M. Doré and O.Maleva proved in [5] that for any $n > 1$ there exists a compact Lebesgue-null subset of $\mathbb{R}^n$ which contains a point of differentability for any real valued Lipschitz function. Therefore to get a converse for Theorem 0.3 we should requre some extra property of the set E.

Contents

Chapter 1

Preliminaries

This preliminary chapter is divided in four sections. In each section we provide a different tool that will be foundamental in the proof of Theorem 2.14.

In Section 1.1 we introduce and recall some basic facts about residuality and its relations with the Banach-Mazur game. In Section 1.2 we study the space of $\mathrm{Lip}_1(n, m)$ functions and its natural topology induced by nets of pointwise convergence, and we introduce the set $\mathfrak{P}(n, m)$ of piecewise affine functions. In Section 1.3 we define a maximal operator over a suitable and controlled class of cylinders and we prove that such an operator is bounded from L^p to L^p for $1 < p < \infty$. In Section 1.4 we prove that the set $\mathfrak{P}(n, m)$ is dense in $\mathrm{Lip}_1(n, m)$.

1.1 Residuality

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$:

(i) if $\mathrm{int}(\mathrm{cl}(A)) = \emptyset$, A is said to be *nowhere dense*,

(ii) if A is the countable union of nowhere dense sets, A is said to be *meagre*,

(iii) if A is the complement of a meagre set, A is said to be *residual*.

We define the $Banach - Mazur\ game$, which will be used to prove Proposition 2.10.

Definition 1.2. Let $(X, \mathcal{T})$ be a topological space. The Banach-Mazur game, is a game between two players, Player I and Player II.

Player I is dealt with an arbitrary subset $A \subseteq X$ and Player II with the set $B := X \setminus A$.

The game $\langle A, B \rangle$ is played as follows: I chooses arbitrarely an open set $U_1 \subseteq X$; then II chooses an open set $V_1 \subseteq U_1$; then I chooses an open set $U_2 \subseteq V_1$ and so on.

If the set $\left(\bigcap_{n\in\mathbb{N}} V_i\right) \cap A \neq \emptyset$ then I wins. Otherwise II wins.

The following proposition explains the connection between the Banach-Mazur game and the topology of the space on which we are playing:

Theorem 1.3. *There exists a strategy by which Player II can be sure to win if and only if B is residual in X, or equivalently if and only if A is meagre.*

Proof. The proof of this result is given in [8] only in the case of the real line. However that argoument works in the same way in a generic topological space. $\square$

1.2 Sets of $\mathrm{Lip}_1(n,m)$ and $\mathfrak{P}(n,m)$ functions

Definition 1.4. Let $\mathrm{Lip}_1(n,m)$ be the following space of Lipschitz functions:

$$\mathrm{Lip}_1(n,m) = \{f : [0,1]^n \longrightarrow \mathbb{R}^m : f \text{ is Lipschitz with } \mathrm{Lip}(f) \leq 1\},$$

where:

$$\mathrm{Lip}(f) := \sup_{\substack{x,y\in[0,1]^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}.$$

We endow $\mathrm{Lip}_1(n,m)$ with the topology $\mathcal{T}$ induced by the pointwise convergence, which means that we are taking in consideration the topology induced by nets of pointwise converging functions in $\mathrm{Lip}_1(n,m)$, see [7] for a reference.

Remark 1.5. One could be puzzled by the above definition since the topology defined by nets of pointwise convergence seems not to be the most natural one.

First of all there are many possible topologies and so one could wonder what is the most meaningful in this framework. First of all we note that the Sobolev norm and the Lipschitz-constant norm endow $\mathrm{Lip}_1(n,m)$ with a nonseparable metric space structure and in these spaces smooth functions are even not dense. Hence good candidates for meaningful topologies are pointwise convergence topology, uniform convergence topology, and local uniform convergence topology. However we first remark that Lemma 1.6 and Proposition 1.7 will provide that the topology induced by pointwise convergence on $\mathrm{Lip}_1(n,m)$ is the one induced by $\|\cdot\|_\infty$, the supremum norm, and hence for Lipschitz functions on $[0,1]^n$ topologies mentioned above are actually the same.

Lemma 1.6. *If $a,b,c \geq 0$ and $c \leq a+b$, then*

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Proof. Thanks to the fact that a,b,c are non negative:

$$\begin{aligned} & c \leq a+b \\ & \Rightarrow c \leq a+b+2ab+abc \\ & \Rightarrow c+ac+bc+abc \leq a+ac+ab+abc+b+bc+ba+abc \\ & \Rightarrow (1+a)(1+b)c \leq (1+b)(1+c)a+(1+c)(1+a)b \end{aligned}$$

Therefore dividing by $(1+a)(1+b)(1+c)$ we prove the thesis. $\square$

Proposition 1.7. *The topological space* $(\mathrm{Lip}_1(n,m),\mathcal{T})$ *is a completely metrizable space. Moreover the topology induced on* $\mathrm{Lip}_1(n,m)$ *by the pointwise convergence is equivalent to the one induced by the uniform norm* $\|\cdot\|_\infty$*, where* $\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]^n\}$.

Proof. Let us prove that $(\mathrm{Lip}_1(n,m),\mathcal{T})$ is a metrizable space. Let $\{q_i\}_{i\in\mathbb{N}}$ be an enumeration of $[0,1]^n \cap \mathbb{Q}^n$ and $f,g \in \mathrm{Lip}_1(n,m)$, we define:

$$d(f,g) := \sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|}.$$

First of all we show that the function $d(\cdot,\cdot) : \mathrm{Lip}_1(n,m)\times\mathrm{Lip}_1(n,m) \longrightarrow [0,\infty[$ is well defined and it is a metric on $\mathrm{Lip}_1(n,m)$. It is well defined since for every $f,g \in \mathrm{Lip}_1(n,m)$:

$$\sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|} \leq \sum_{i\in\mathbb{N}} \frac{1}{2^i} \leq 2.$$

We verify for $d(\cdot,\cdot)$ the metric axioms:

(i) $d(f,g) \geq 0$ for any $f,g \in \mathrm{Lip}_1(n,m)$ by definition.

(ii) If $d(f,g)=0$ then $|f(q_i)-g(q_i)|=0$ for any $i\in\mathbb{N}$ and by continuity of f and g we deduce that $f(x)=g(x)$ on $[0,1]^n$, the converse is obvious.

(iii) Thanks to the fact that $|f(x)-g(x)|=|g(x)-f(x)|$ we have $d(f,g)=d(g,f)$.

(iv) $d(f,g)\leq d(f,h)+d(h,g)$ for any $f,g,h \in \mathrm{Lip}_1(n,m)$, indeed since

$$|f(q_i)-g(q_i)| \leq |f(q_i)-h(q_i)|+|g(q_i)-h(q_i)|$$

and Lemma 1.6, we have that:

$$\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|} \leq \frac{|f(q_i)-h(q_i)|}{1+|f(q_i)-h(q_i)|} + \frac{|g(q_i)-h(q_i)|}{1+|g(q_i)-h(q_i)|}.$$

Therefore, dividing by $\frac{1}{2^i}$ and summing over $i\in\mathbb{N}$ we get:

$$\sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|} \leq \sum_{i\in\mathbb{N}} \frac{1}{2^i}\left(\frac{|f(q_i)-h(q_i)|}{1+|f(q_i)-h(q_i)|} + \frac{|g(q_i)-h(q_i)|}{1+|g(q_i)-h(q_i)|}\right),$$

finally thanks to the absolute convergence we can split the right hand side obtaining the triangular inequality for d.

We prove now that $f_n(x)\to f(x)$ for any $x\in[0,1]^n$ if and only if $d(f_n,f)\to 0$. Let us suppose that $f_n(x)\to f(x)$ for any $x\in[0,1]^n$. Then for any $\epsilon>0$ there exists an $N_1 \in \mathbb{N}$ such that $|f(q_i) - f_n(q_i)| \leq \epsilon$ for any $n \geq N_1$ and every $i \in \{1, \dots, N_1\}$ Moreover there exists an $N_2 \in \mathbb{N}$ such that

$$\sum_{i>N_2} \frac{1}{2^i} \frac{|f(q_i) - f_n(q_i)|}{1 + |f(q_i) - f_n(q_i)|} \leq \sum_{i>N_2} \frac{1}{2^i} \leq \epsilon.$$

Choosing $N := \max\{N_1, N_2\}$, we have that:

$$\begin{aligned}\sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{|f(q_i) - f_n(q_i)|}{1 + |f(q_i) - f_n(q_i)|} &\leq \sum_{i=1}^{N} \frac{1}{2^i} \frac{|f(q_i) - f_n(q_i)|}{1 + |f(q_i) - f_n(q_i)|} + \sum_{i>N} \frac{1}{2^i} \frac{|f(q_i) - f_n(q_i)|}{1 + |f(q_i) - f_n(q_i)|} \\ &\leq \sum_{i=1}^{N} \frac{1}{2^i}\epsilon + \epsilon \leq 2\epsilon \ \forall n \geq N.\end{aligned}$$

Viceversa, for any $\epsilon > 0$ there exists a $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$:

$$\sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{|f(q_i) - f_n(q_i)|}{1 + |f(q_i) - f_n(q_i)|} \leq \epsilon.$$

In particular for any $i \in \mathbb{N}$, provided that $\epsilon \leq \frac{1}{2^{i+1}}$, we get that:

$$|f(q_i) - f_n(q_i)| \leq \frac{2^i\epsilon}{1 - 2^i\epsilon} \leq 2\epsilon.$$

Thus by continuity we get pointwise convergence on $[0,1]^n$.

Now we have to prove completeness. In order to do so, we take a Cauchy sequence $\{f_n\}_n \in \mathbb{N}$ in $(\mathrm{Lip}_1(n,m), d)$. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $j, k \geq N$ we have that $d(f_j, f_k) \leq \epsilon$. Therefore for any $0 < \epsilon \leq \frac{1}{2^{i+1}}$ and any $j, k \geq N$ we have that $|f_j(q_i) - f_k(q_i)| \leq 2\epsilon$. Thus $\{f_n(q_i)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for any $i \in \mathbb{N}$. Let us define:

$$f(q_i) := \lim_{n\to\infty} f_n(q_i).$$

First of all we want to prove now that f is Lipschitz on $\bigcup_{i\in\mathbb{N}}\{q_i\}$ with $\mathrm{Lip}(f) \leq 1$. Indeed fixed $i_1, i_2 \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that for any $n \geq N$

$$|f(q_{i_1}) - f_n(q_{i_2})| + |f_n(q_{i_1}) - f(q_{i_2})| \leq \epsilon,$$

thus applying triangular inequality and recalling that $f_n \in \mathrm{Lip}_1(n,m)$ we have that:

$$\begin{aligned}|f(q_{i_1}) - f(q_{i_2})| &\leq |f(q_{i_1}) - f_n(q_{i_2})| + |f_n(q_{i_1}) - f_n(q_{i_2})| + |f_n(q_{i_1}) - f(q_{i_2})| \\ &\leq \epsilon + |q_{i_1} - q_{i_2}|.\end{aligned}$$

Thus since ϵ was aribtrary, we prove that f satifies the claim. Now fix an $x \in [0,1]^n$ and let $\{r_j^1\}_{j\in\mathbb{N}}, \{r_j^2\}_{j\in\mathbb{N}} \subseteq \bigcup_{i\in\mathbb{N}}\{q_i\}$ be two sequences such that $r_j^k \to x$ as $j \to \infty$ for $k = 1, 2$. For such sequences we have that:

$$0 \le \lim_{j\to\infty} \left|f\left(r_j^1\right) - f\left(r_j^2\right)\right| \le \lim_{j\to\infty} \left|r_j^1 - r_j^2\right| = 0.$$

Therefore the function f can be well defined on the whole $[0,1]^n$ as:

$$f(x) := \lim_{j\to\infty} f(r_j) \text{ for any sequence } \{r_j\}_{j\in\mathbb{N}} \subseteq \bigcup_{i\in\mathbb{N}}\{q_i\} \text{ converging to } x.$$

We are left to show that such an f is contained in $\mathrm{Lip}_1(n,m)$, since $d(f_n, f) \to 0$ by construction. Indeed, let $x_1, x_2 \in [0,1]^n$. Then for any $\epsilon > 0$ there exist $q_{j_1}, q_{j_2} \in \bigcup_{i\in\mathbb{N}}\{q_i\}$ such that $|x_k - q_{j_k}| + |f(x_k) - f(q_{j_k})| \le \epsilon$ for $k = 1, 2$. Thus we have that:

$$\begin{aligned}
|f(x_1) - f(x_2)| &\le |f(x_1) - f(q_{j_1})| + |f(q_{j_1}) - f(q_{j_2})| + |f(q_{j_2}) - f(x_2)| \\
&\le |f(x_1) - f(q_{j_1})| + |q_{j_1} - q_{j_2}| + |f(q_{j_2}) - f(x_2)| \\
&\le |f(x_1) - f(q_{j_1})| + |q_{j_1} - x_1| + |x_1 - x_2| + |x_2 - q_{j_2}| + |f(q_{j_2}) - f(x_2)| \\
&\le 2\epsilon + |x_1 - x_2|.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have that $f \in \mathrm{Lip}_1(n,m)$, and this proves that $(\mathrm{Lip}_1(n,m), \mathcal{T})$ is a complete metric space. We have to prove the equivalence between the topology induced by the $\|\cdot\|_\infty$ norm and the one induced by pointwise convergence. Indeed:

$$d(f,g) = \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{|f(q_i) - g(q_i)|}{1 + |f(q_i) - g(q_i)|} \le \sum_{i\in\mathbb{N}} \frac{1}{2^i} \|f - g\|_\infty \le \|f - g\|_\infty.$$

For the viceversa, first of all suppose by contradiction that there exists an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there exists an $x_N \in [0,1]^n$ such that we have that $B_\epsilon(x_N) \cap \bigcup_{i=1}^N \{q_i\} = \emptyset$. Since $\{x_N\}_{N\in\mathbb{N}}$ is a sequence in $[0,1]^n$ we can find a subsequence converging to a point x, and we can assume that without loss of generality $\{x_N\}_{N\in\mathbb{N}}$ converges to x. Therefore there exists an $M \in \mathbb{N}$ such that for any $N \ge M$ we have that $|x_N - x|_\infty \le \frac{\epsilon}{4}$. Therefore for any $N \ge M$ $B_{\frac{\epsilon}{4}}(x) \subseteq B_\epsilon(x_N)$, and this would imply that $B_{\frac{\epsilon}{4}}(x) \cap \bigcup_{i=1}^N \{q_i\} = \emptyset$ for any N, which contradicts the density of $\bigcup_{i=1}^N \{q_i\} = \emptyset$ in $[0,1]^n$. Therefore for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $x \in [0,1]^n$ we have that $B_\epsilon(x) \cap \bigcup_{i=1}^N \{q_i\} \neq \emptyset$. Moreover, since $f - g$ is a 2-Lipschitz function on the unit cube we have that:

$$\begin{aligned}
|f(q_i) - g(q_i)| &\le |f(q_i) - g(q_i) - (f(q_1) - g(q_1))| + |f(q_1) - g(q_1)| \\
&\le 2|q_i - q_1| + |f(q_1) - g(q_1)| \le 2\sqrt{n} + |f(q_1) - g(q_1)|.
\end{aligned}$$

Therefore:

$$\begin{aligned}
d(f,g) &= \sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|} \geq \sum_{i=1}^{N} \frac{1}{2^i}\frac{|f(q_i)-g(q_i)|}{1+|f(q_i)-g(q_i)|}\\
&\geq \frac{1}{1+|f(q_1)-g(q_1)|+2\sqrt{n}}\sum_{i=1}^{N}\frac{1}{2^i}|f(q_i)-g(q_i)|\\
&\geq \frac{1}{1+|f(q_1)-g(q_1)|+2\sqrt{n}}\sum_{i=1}^{N}|f(q_i)-g(q_i)|\\
&\geq \frac{1}{1+|f(q_1)-g(q_1)|+2\sqrt{n}}|f(q_i)-g(q_i)|\ \forall i\in\{1,\dots N\},
\end{aligned}$$

Hence fixed an $x \in [0,1]^n$, there exists an $i \in \{1,\dots N\}$ such that $|x-q_i| \leq \epsilon$, we have that:

$$\begin{aligned}
|f(x)-g(x)| &\leq |f(q_i)-g(q_i)| + |(f(x)-g(x))-(f(q_i)-g(q_i))| \leq |f(q_i)-g(q_i)|+2|x-q_i|\\
&\leq |f(q_i)-g(q_i)|+2\epsilon \leq (1+|f(q_1)-g(q_1)|+2\sqrt{n})2^N d(f,g)+2\epsilon.
\end{aligned}$$

We choose now $\epsilon \leq d(f,g)$ so that we have

$$\|f-g\|_\infty \leq \left((1+|f(q_1)-g(q_1)|+2\sqrt{n})2^N+2\right)d(f,g).$$

Moreover by definition:

$$\frac{1}{2}\cdot\frac{|f(q_1)-g(q_1)|}{1+|f(q_1)-g(q_1)|} \leq d(f,g),$$

hence:

$$\|f-g\|_\infty \leq \left(\left(1+\frac{2d(f,g)}{1-d(f,g)}+2\sqrt{n}\right)2^N+2\right)d(f,g).$$

At last we note that inequality above proves the claim, i.e. the equivalence of the topologies induced by uniform and pointwise convergence, and we would like to remark that the constant

$$\left(1+\frac{2d(f,g)}{1-d(f,g)}+2\sqrt{n}\right)2^N+2$$

depends on $d(f,g)$ and on $N = N(\epsilon)$. Since we fixed $\epsilon \leq d(f,g)$, we have that $\epsilon = \epsilon(d(f,g))$ and hence there exists an increasing function $g : [0,1[\to [0,\infty[$ such that:

$$\|f-g\|_\infty \leq g(d(f,g))d(f,g).$$

Therefore since g is unbounded, we do not have the equivalence of metrics d and $\|\cdot\|_\infty$. $\square$

We introduce some standard notation and at last we will define the set $\mathfrak{P}(n,m)$.

Definition 1.8. We define τ to be the family of all $\Pi \subseteq \mathcal{P}([0,1]^n)$, where $\mathcal{P}([0,1]^n)$ is the power set of $[0,1]^n$, such that $\operatorname{Card}\Pi < \infty$, every element P of Π is an open n-simplex and:

$$\mathrm{cl}\left(\bigcup_{P\in\Pi} P\right) = [0,1]^n.$$

Definition 1.9 (Hilbert-Schmidt norm for matrices)**.** We define the Hilbert-Schimdt norm of a matrix $A \in M_{m\times n}(\mathbb{R})$ as:

$$\|A\|_{HS} := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}\ ,$$

where $A = (A_{ij})_{\substack{i\in\{1,\dots,m\}\\ j\in\{1,\dots,n\}}}$.

Definition 1.10 (Piecewise Affine Functions)**.** We define $\mathfrak{P}(n,m)$ as the set of $\mathrm{Lip}_1(n,m)$ functions for which there exists a $\Pi \in \tau$ such that for all $P \in \Pi$ we have that $f|_P$ is affne and the following equality holds:

$$\|Jf|_P\|_{HS}^2 = \min\{n,m\},$$

where $Jf(x)$ is the Jacobian matrix of f in $x \in [0,1]^n$. We will refer to elements of $\mathfrak{P}(n,m)$ as *piecewise affine functions* (despite the fact we are forcing a certain condition on the jacobian), and we will say that f is piecewise affine on the partion Π if for any $P \in \Pi$ we have that $f|_P$ is affine.

In order to understand why this class of Lipschitz functions is important for the solution of our problem, we prove Theorem 1.11, which will be the core of the proof of Proposition 2.8 in Chapter 2, and hence foundamental in the proof of Proposition 2.10. This theorem establishes a connection between the distance in the supremum norm of two functions with the mean behaviour of their jacobians.

Theorem 1.11. *Let $f \in \mathfrak{P}(n,m)$ be fixed, then there exists a constant $C(f,n,m)$ depending only on f, n, m such that for any $g \in \mathfrak{P}(n,m)$*

$$\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 dx \le C(f,n,m)\|f-g\|_\infty.$$

Proof. Using the definition and recalling that the set where the Jacobian of f and

g is not defined has measure zero, we get:

$$\begin{aligned}
\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 dx &= \int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n (\partial_i (f-g)_j)^2 dx \\
&= \int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n ((\partial_i f_j)^2 + (\partial_i g_j)^2 - 2\partial_i f_j \cdot \partial_i g_j) dx \\
&= \int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n (\partial_i f_j)^2 dx + \int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n (\partial_i g_j)^2 dx - 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i g_j dx \\
&= 2\min\{n,m\} - 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i g_j dx \\
&= 2\min\{n,m\} - 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i (g_j - f_j) dx - 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n (\partial_i f_j)^2 dx \\
&= 2\min\{n,m\} - 2\min\{n,m\} + 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i (f_j - g_j) dx \\
&= 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i (f_j - g_j) dx
\end{aligned}$$

Let $\{P_l\}_l$ and $\{Q_k\}_k$ be the partitions of $[0,1]^n$ on which f and g are piecewise affine. Then we have:

$$\begin{aligned}
\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 dx &= 2\int_{[0,1]^n} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i (f_j - g_j) dx \\
&= 2\sum_l \sum_k \int_{P_l \cap Q_k} \sum_{j=1}^m \sum_{i=1}^n \partial_i f_j \cdot \partial_i (f_j - g_j) dx,
\end{aligned}$$

and using the Green's formula over $P_l \cap Q_k$ on each f_j and g_j, recalling that an affine function is harmonic, we get:

$$\begin{aligned}
&\sum_l \sum_k \int_{P_l \cap Q_k} \sum_{i=1}^n \partial_i f_j \cdot \partial_i (f_j - g_j) dx = \sum_l \sum_k \int_{P_l \cap Q_k} \nabla f_j \cdot \nabla (f_j - g_j) dx \\
&= \sum_l \sum_k \int_{\partial(P_l \cap Q_k)} (f_j - g_j)(\nabla f_j \cdot \nu) d\mathcal{H}^{n-1}(x) - \sum_l \sum_k \int_{P_l \cap Q_k} (f_j - g_j) \Delta f_j dx \\
&= \sum_l \sum_k \int_{\partial(P_l \cap Q_k)} (f_j - g_j)(\nabla f_j \cdot \nu) d\mathcal{H}^{n-1}(x),
\end{aligned}$$

where ν is the outer normal to $P_l \cap Q_k$. If $P_l \cap Q_{k_1}$ and $P_l \cap Q_{k_2}$ share a common $n-1$-dimensional face S then:

$$\int_S (f_j - g_j)(\nabla f_j \cdot \nu|_{k_1}) d\mathcal{H}^{n-1}(x) = -\int_S (f_j - g_j)(\nabla f_j \cdot \nu|_{k_2}) d\mathcal{H}^{n-1}(x),$$

this means that summing over k the contribute coming from the faces of Q_k cancels:

$$\sum_l \sum_k \int_{\partial(P_l \cap Q_k)} (f_j - g_j)(\nabla f_j \cdot \nu) d\mathcal{H}^{n-1}(x) = \sum_l \int_{\partial P_l} (f_j - g_j)(\nabla f_j \cdot \mu) d\mathcal{H}^{n-1}(x),$$

where μ is the is the outer normal to P_l. So this yelds:

$$\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 = 2 \sum_l \sum_{j=1}^m \int_{\partial P_l} (f_j - g_j)(\nabla f_j \cdot \mu) d\mathcal{H}^{n-1}(x).$$

Observing that $|\nabla f_j| \leq \sqrt{\min\{n,m\}}$ and that $|f_j - g_j| \leq \|f - g\|_\infty$, we get:

$$\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 \leq 2m \cdot \sqrt{\min\{n,m\}} \cdot \left(\sum_l \mathcal{H}^{n-1}(\partial P_l) \right) \|f-g\|_\infty. \qquad \square$$

1.3 A cilindrical maximal operator

The maximal operator introduced in this section and its boundness on $L^2(\mathbb{R}^n)$ will be foundamental in the proof of Proposition 2.8.

Definition 1.12. Given $\pi \subseteq \mathbb{R}^n$ a linear subspace of dimesion $n-1$ and a point $x \in \pi$, we define the disk lying on the hyperplane $x + \pi$ of radius r and centre x as:

$$D_{\pi,r}(x) := \mathrm{cl}(B_r(x)) \cap (x + \pi).$$

Given a point $y \in \mathbb{R}^n \setminus (x+\pi)$, we define the nondegenerate cylinder with bases $D_{\pi,r}(x)$ and $D_{\pi,r}(y)$ as the convex hull of $D_{\pi,r}(x)$ and $D_{\pi,r}(y)$, and we will refer to $\frac{x+y}{2}$ as the *centre of the cylinder.*

We denote by $\mathcal{B}_i(\mathcal{C}(x,y,r,\pi))$ the set of closed balls with centre $\frac{x+y}{2}$ contained in $\mathcal{C}(x,y,r,\pi)$ and with $\mathcal{B}_e(\mathcal{C}(x,y,r,\pi))$ the set of closed balls with centre $\frac{x+y}{2}$ that contain $\mathcal{C}(x,y,r,\pi)$. Thus we define:

$$\rho(\mathcal{C}(x,y,r,\pi)) := \max_{(B_1,B_2) \in \mathcal{B}_i(\mathcal{C}(x,y,r,\pi)) \times \mathcal{B}_e(\mathcal{C}(x,y,r,\pi))} \frac{\mathrm{diam}(B_1)}{\mathrm{diam}(B_2)}.$$

Definition 1.13. Let us fix $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{x\}$ and introduce the hyperplane:

$$\pi(x,y,r) := x + \frac{r}{2} \cdot \frac{x-y}{|x-y|} + (x-y)^\perp.$$

In order to simplify the notation we define:

$$\mathcal{C}_{x,y,r} := \mathcal{C}\left(x + \frac{r}{2} \cdot \frac{x-y}{|x-y|}, y + \frac{r}{2} \cdot \frac{y-x}{|y-x|}, r, \pi(x,y,r) \right).$$

Remark 1.14. The cylinders $\mathcal{C}_{x,y,r}$ are only right circular cylinders with centre $\frac{x+y}{2}$, basis with radius r and height $(1+r)|x-y|$.

Definition 1.15. Fixed $\gamma > 0$ and $R > 0$, we define $\mathscr{C}(\gamma, R)$ as the set of cylinders $\mathcal{C}_{x,y,r}$ with $x \in \mathbb{R}^n$, $y \in \mathbb{Q}^n \setminus \{x\}$ and $r > 0$ for whch $\rho\left(\mathcal{C}_{x,y,r}\right) \geq \gamma$ and $\operatorname{diam}(\mathcal{C}_{x,y,r}) \leq R$.

Definition 1.16 (Cilindrical maximal operator)**.** Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, $\gamma > 0$ and $R > 0$, we define the maximal operator on the set of cylinders $\mathscr{C}(\gamma, R)$:

$$M_{\mathscr{C}(\gamma,R)} f(x) := \sup_{\substack{x \in \operatorname{int}(C) \\ C \in \mathscr{C}(\gamma,R)}} \fint_C |f(\xi)| d\xi.$$

Proposition 1.17. *For any $\gamma > 0$, $R > 0$ and any $f \in L^1_{loc}(\mathbb{R}^n)$, the function $x \mapsto M_{\mathscr{C}(\gamma,R)} f(x)$ is lower semicontinous.*

Proof. We will prove that for any α, the set $\left\{M_{\mathscr{C}(\gamma,R)} f > \alpha\right\}$ is open. If $x \in \left\{M_{\mathscr{C}(\gamma,R)} f > \alpha\right\}$ then

$$M_{\mathscr{C}(\gamma,R)} f(x) > \alpha.$$

Therefore we can choose $\epsilon > 0$ such that $M_{\mathscr{C}(\gamma,R)} f(x) - \epsilon > \alpha$. Moreover by definition, for any $\epsilon > 0$ there exists a cylinder $C_\epsilon \in \mathscr{C}(\gamma, R)$ such that $x \in \operatorname{int}(C_\epsilon)$ and

$$\fint_{C_\epsilon} |f(\xi)| d\xi > M_{\mathscr{C}(\gamma,R)} f - \epsilon.$$

Hence:

$$\fint_{C_\epsilon} |f(\xi)| d\xi > M_{\mathscr{C}(\gamma,R)} f - \epsilon > \alpha.$$

Since $x \in \operatorname{int}(C_\epsilon)$, then there is a ball centred in x such that $B \subseteq C_\epsilon$, and hence, for any $z \in B$, we have that $z \in C_\epsilon$. Thus

$$M_{\mathscr{C}(\gamma,R)} f(z) > \alpha \ \forall z \in B,$$

and so this proves the result. $\square$

Definition 1.18. Let $\beta > 0$,$\gamma > 0$, $R > 0$, and $C \in \mathscr{C}(\gamma, R)$. We define the dilated cylinder βC as:

$$\beta C := \left\{z \in \mathbb{R}^n : \exists \lambda \in [0, \beta], y \in C \text{ s.t. } z = G(C) + \lambda(y - G(C))\right\},$$

where $G(C)$ is the centre of the cylinder C of Definition 1.12.

Lemma 1.19 (A covering lemma for cylinders)**.** *For any $\gamma > 0$ and $R > 0$, there exists a countable subfamily $\mathscr{G} \subseteq \mathscr{C}(\gamma, R)$ of pairwise disjoint cylinders such that:*

$$\bigcup_{C \in \mathscr{C}(\gamma,R)} C \subseteq \bigcup_{C \in \mathscr{G}} \left(\frac{32}{\gamma}\right)^2 C,$$

where the set $\left(\frac{32}{\gamma}\right)^2 C$ is the dilated cylinder defined in Definition 1.18.

Proof. Let $C \in \mathscr{C}(\gamma, R)$. By definition there are two balls $B_i(C) \in \mathcal{B}_i(C)$ and $B_e(C) \in \mathcal{B}_e(C)$ such that $\frac{\operatorname{diam}(B_i(C))}{\operatorname{diam}(B_e(C))} \geq \frac{\gamma}{2}$. Observe now that the balls $B_e(C)$ and $B_i(C)$ have the same centre, and we know by construction that $\operatorname{diam}(B_i(C)) \geq \frac{\gamma}{2}\operatorname{diam}(B_e(C))$. Hence $B_e(C) \subseteq \frac{2}{\gamma}B_i(C)$, and therefore $B_e(C) \subseteq 3 \cdot \frac{2}{\gamma}B_i(C)$. This implies that

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} C \subseteq \bigcup_{C\in\mathscr{C}(\gamma,R)} B_e(C) \subseteq \bigcup_{C\in\mathscr{C}(\gamma,R)} 3 \cdot \frac{2}{\gamma}B_i(C).$$

By the Vitali covering lemma for the set $\{B_i(C)\}_{C\in\mathscr{C}(\gamma,R)}$, there exists a countable subset $\mathscr{F}$ in $\mathscr{C}(\gamma, R)$, such that:

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} B_i(C) \subseteq \bigcup_{C\in\mathscr{F}} 5B_i(C).$$

This implies that:

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} 3 \cdot \frac{2}{\gamma}B_i(C) \subseteq \bigcup_{C\in\mathscr{F}} 5 \cdot 3 \cdot \frac{2}{\gamma}B_i(C) \subseteq \bigcup_{C\in\mathscr{F}} \frac{30}{\gamma}C,$$

and therefore summing up:

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} C \subseteq \bigcup_{C\in\mathscr{F}} \frac{30}{\gamma}C.$$

Moreover by contruction $C \subseteq B_e(C)$, and hence we deduce that:

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} C \subseteq \bigcup_{C\in\mathscr{F}} \frac{30}{\gamma}C \subseteq \bigcup_{C\in\mathscr{F}} \frac{30}{\gamma}B_e(C).$$

Applying the Vitali covering lemma to the set of balls $\{B_e(C)\}_{C\in\mathscr{F}}$, we get a subfamily $\mathscr{G}$ of $\mathscr{F}$ such that $\{B_e(C)\}_{C\in\mathscr{G}}$ are pairwise disjoint and moreover:

$$\bigcup_{C\in\mathscr{C}(\gamma,R)} B_e(C) \subseteq \bigcup_{C\in\mathscr{F}} 5B_e(C).$$

Thus

$$\begin{aligned}\bigcup_{C\in\mathscr{F}} \frac{30}{\gamma}B_e(C) &\subseteq \bigcup_{C\in\mathscr{G}} 5 \cdot \frac{30}{\gamma}B_e(C) \subseteq \bigcup_{C\in\mathscr{G}} 5 \cdot \frac{30}{\gamma} \cdot \left(3 \cdot \frac{2}{\gamma}B_i(C)\right) \\ &= \bigcup_{C\in\mathscr{G}} \left(\frac{30}{\gamma}\right)^2 B_i(C) \subseteq \bigcup_{C\in\mathscr{G}} \left(\frac{30}{\gamma}\right)^2 C \subseteq \bigcup_{C\in\mathscr{G}} \left(\frac{32}{\gamma}\right)^2 C.\end{aligned}$$

$\square$

Proposition 1.20 (Weak estimate for the conical maximal operator)**.** *Let $f \in L^1(\mathbb{R}^n)$, then the following estimate holds true:*

$$\mathcal{L}^n\left(\left\{M_{\mathscr{C}(\gamma,R)}f > t\right\}\right) \leq \frac{1}{t}\cdot\left(\frac{32}{\gamma}\right)^2 \int_{\mathbb{R}^n} |f(z)|dz.$$

Proof. Let us define the set:

$$\mathfrak{C}_t := \left\{C \in \mathscr{C}(\gamma,R):\ \exists\, x \in \left\{M_{\mathscr{C}(\gamma,R)}f > t\right\} \text{ s.t. } x \in \mathrm{int}(C) \text{ and } \fint_C |f(z)|dz > t\right\},$$

Recalling Lemma 1.19, we get a countable subset $\mathscr{G} \subseteq \mathfrak{C}_t$ of disjoint cylinders such that

$$\left\{M_{\mathscr{C}(\gamma,R)}f > t\right\} \subseteq \bigcup_{C\in\mathfrak{C}_t} C \subseteq \bigcup_{C\in\mathscr{G}} \left(\frac{32}{\gamma}\right)^2 C,$$

thus:

$$\begin{aligned}\mathcal{L}^n\left(\left\{M_{\mathscr{C}(\gamma,R)}f > t\right\}\right) &\leq \mathcal{L}^n\left(\bigcup_{C\in\mathscr{G}}\left(\frac{32}{\gamma}\right)^2 C\right) \leq \sum_{C\in\mathscr{G}} \mathcal{L}^n\left(\left(\frac{32}{\gamma}\right)^2 C\right)\\ &= \left(\frac{32}{\gamma}\right)^{2n} \sum_{C\in\mathscr{G}} \mathcal{L}^n(C) \leq \left(\frac{32}{\gamma}\right)^{2n} \sum_{C\in\mathscr{G}} \frac{1}{t}\int_C |f(z)|dz.\end{aligned}$$

Since the cylinders $C \in \mathscr{G}$ are pairwise disjoint, then:

$$\begin{aligned}\mathcal{L}^n\left(\left\{M_{\mathscr{C}(\gamma,R)}f > t\right\}\right) &\leq \left(\frac{32}{\gamma}\right)^{2n} \sum_{C\in\mathscr{G}} \frac{1}{t}\int_C |f(z)|dz\\ &= \left(\frac{32}{\gamma}\right)^{2n}\cdot\frac{1}{t}\int_{\bigcup_{C\in\mathscr{G}} C} |f(z)|dz \leq \left(\frac{32}{\gamma}\right)^{2n}\cdot\frac{1}{t}\int_{\mathbb{R}^n}|f(z)|dz.\end{aligned}$$

□

Proposition 1.21. *Let $1 < p < \infty$. If $f \in L^p(\mathbb{R}^n)$, then*

$$\|M_{\mathscr{C}(\gamma,R)}f\|_p \leq N(n,\gamma,p)\|f\|_p,$$

where $N(n,\gamma,p) := 2\left(\frac{32}{\gamma}\right)^{\frac{2n}{p}}\left(\frac{p}{p-1}\right)^{\frac{1}{p}}$, in particoular if $f \in L^2(\mathbb{R}^n)$, then:

$$\|M_{\mathscr{C}(\gamma,R)}f\|_2 \leq \sqrt{2}\left(\frac{32}{\gamma}\right)^n \|f\|_2.$$

Proof. We have:

$$\begin{aligned}
\int_{\mathbb{R}^n} |M_{\mathscr{C}(\gamma,R)} f(z)|^p dz &= p \int_{\mathbb{R}^n} \int_0^{M_{\mathscr{C}(\gamma,R)} f(z)} t^{p-1} dt dz \\
&= p \int_{\mathbb{R}^n} \int_0^{+\infty} \chi_{[0, M_{\mathscr{C}(\gamma,R)} f(z))}(t) t^{p-1} dt dz \\
&= p \int_{\mathbb{R}^n} \int_0^{+\infty} \chi_{\{M_{\mathscr{C}(\gamma,R)} f(z) > t\}}(z) t^{p-1} dt dz \\
&= p \int_0^{+\infty} \left(\int_{\mathbb{R}^n} \chi_{\{M_{\mathscr{C}(\gamma,R)} f(z) > t\}}(z) dz \right) t^{p-1} dt \\
&= p \int_0^{+\infty} \mathcal{L}^n(\{M_{\mathscr{C}(\gamma,R)} f(z) > t\}) t^{p-1} dt.
\end{aligned}$$

Since $|f(z)| \le |f(z)\chi_{\{|f| > \frac{t}{2}\}}(z)| + \frac{t}{2}$. Thus:

$$⨍_C |f(z)| \le ⨍_C |f(z)| \chi_{\{|f| > \frac{t}{2}\}}(z) dz + \frac{t}{2}$$

for any cylinder $C \in \mathscr{C}(\gamma, R)$. Therefore we obtain:

$$M_{\mathscr{C}(\gamma,R)} f(x) \le M_{\mathscr{C}(\gamma,R)} \left(f(z) \chi_{\{|f| > \frac{t}{2}\}} \right) + \frac{t}{2}.$$

This yelds:

$$\begin{aligned}
\mathcal{L}^n \left(\left\{ M_{\mathscr{C}(\gamma,R)} f > t \right\} \right) &\le \mathcal{L}^n \left(\left\{ M_{\mathscr{C}(\gamma,R)} \left(f \chi_{\{|f| > \frac{t}{2}\}} \right) + \frac{t}{2} > t \right\} \right) \\
&= \mathcal{L}^n \left(\left\{ M_{\mathscr{C}(\gamma,R)} \left(f \chi_{\{|f| > \frac{t}{2}\}} \right) > \frac{t}{2} \right\} \right).
\end{aligned}$$

Applying now Proposition 1.20 to the function $f\chi_{\{|f|>\frac{t}{2}\}}$ we have that:

$$\begin{aligned}
\int_{\mathbb{R}^n} |M_{\mathscr{C}(\gamma,R)}f(z)|^p dz &= p\int_0^{+\infty} \mathcal{L}^n(\{M_{\mathscr{C}(\gamma,R)}f(z) > t\})t^{p-1}dt \\
&\le p\int_0^{+\infty} \mathcal{L}^n\left(\left\{M_{\mathscr{C}(\gamma,R)}f\chi_{\{|f|>\frac{t}{2}\}} > \frac{t}{2}\right\}\right) t^{p-1}dt \\
&\le p\int_0^{+\infty} \left(\frac{2}{t}\cdot\left(\frac{32}{\gamma}\right)^{2n}\int_{\mathbb{R}^n}|f(z)\chi_{\{|f|>\frac{t}{2}\}}(z)|dz\right) t^{p-1}dt \\
&= 2p\left(\frac{32}{\gamma}\right)^{2n}\int_0^{+\infty}\left(\int_{\mathbb{R}^n}|f(z)\chi_{\{|f|>\frac{t}{2}\}}(z)|dz\right)t^{p-2}dt \\
&= 2p\left(\frac{32}{\gamma}\right)^{2n}\int_{\mathbb{R}^n}\left(\int_0^{+\infty}\chi_{[0,2|f(z)|]}(t)t^{p-2}dt\right)|f(z)|dz \\
&= 2p\left(\frac{32}{\gamma}\right)^{2n}\int_{\mathbb{R}^n}\left(\int_0^{2|f(z)|}t^{p-2}dt\right)|f(z)|dz \\
&= 2\left(\frac{32}{\gamma}\right)^{2n}\cdot\frac{p}{p-1}\int_{\mathbb{R}^n}(2|f(z)|)^{p-1}|f(z)|dz \\
&= 2^p\left(\frac{32}{\gamma}\right)^{2n}\cdot\frac{p}{p-1}\int_{\mathbb{R}^n}|f(z)|^p dz.
\end{aligned}$$

Hence, we have that:

$$\|M_{\mathscr{C}(\gamma,R)}f\|_p \le 2\left(\frac{32}{\gamma}\right)^{\frac{2n}{p}}\cdot\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\|f\|_p.$$

□

1.4 Density of $\mathfrak{P}(n,m)$ in $\mathrm{Lip}_1(n,m)$

First of all, we introduce some notation and facts contained in [4], in order to prove that the set $\mathfrak{P}(n,m)$ is dense in $\mathrm{Lip}_1(n,m)$. Brehm introduced the concept of piecewise congruent mappings in [4]. Using our notation his definition can be rephrased as:

Definition 1.22 (Piecewise congruent mappings)**.** Let $n \le m$. A mapping $f : [0,1]^n \longrightarrow \mathbb{R}^m$ is called a piecewise congruent piecewise mapping if it is continuous and there exists a $\Pi \in \tau$ such that the restriction $f|_P$ for any $P \in \Pi$ is an affine isometric mapping.

Let us prove the following:

Proposition 1.23. *Let $n \le m$. If f is a piecewise congruent mapping then it is contained in $\mathfrak{P}(n,m)$.*

Proof. Let Π be a partition on which f is a congruent piecewise mapping and fix $x, y \in [0,1]^n$ and consider the closed segment $[x,y]$. There exist an $N \in \mathbb{N}$ and $\{x_i\}_{i\in\{1,\dots,N\}} \subseteq [x,y]$ such that for every $P \in \Pi$ either $\mathrm{Card}(\mathrm{cl}(P) \cap [x,y]) < \infty$ or there exists an $i \in \{1,\dots,N-1\}$ such that $\mathrm{cl}(P \cap [x,y]) = [x_i, x_{i+1}]$. Then:

$$|f(x)-f(y)| = \left|\sum_{i=1}^{k} f(x_i) - f(x_{i+1})\right| \le \sum_{i=1}^{k} |f(x_i) - f(x_{i+1})| = \sum_{i=1}^{k} |x_i - x_{i+1}| = |x-y|,$$

and thus f is in $\mathrm{Lip}_1(n,m)$. Now for any $P \in \Pi$, we can write $f|_P$ as $f|_P(x) := Ax+b$ where $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Therefore for such an affine mapping we have that:

$$1 = |e_j - 0|^2 = |Ae_j + b - A0 - b|^2 = |Ae_j|^2 = \sum_{i=1}^{m} A_{i,j}^2 \ ,$$

and thus

$$\min\{m,n\} = n = \sum_{j=1}^{n} 1 = \sum_{j=1}^{n}\sum_{i=1}^{m} A_{i,j}^2 = \|A\|_{HS}^2.$$

This implies that if f is a congruent mapping, then it is contained in $\mathfrak{P}(n,m)$. $\square$

Brehm in [4] proves the following:

Theorem 1.24. *Let $n \le m$ and $M \subseteq \mathbb{R}^n$, with $\mathrm{Card}(M) < \infty$. Then for any distance-reducing mapping $f : M \longrightarrow \mathbb{R}^m$, there is an extension to a piecewise congruent mapping $\overline{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.*

Hence we can use the Theorem 1.24 in order to prove the following:

Proposition 1.25. *The set $\mathfrak{P}(n,m)$ is dense in $\mathrm{Lip}_1(n,m)$.*

Proof. Let us consider the case $n \le m$ first. Consider the set:

$$M(k) := \left\{ x_I := \left(\frac{I_1}{2^k}, \dots, \frac{I_n}{2^k}\right) : I = (I_1, \dots, I_n) \in \{0, \dots, 2^k\}^n \right\}.$$

Given a function $f \in \mathrm{Lip}_1(n,m)$ we have that $f|_{M(k)}$ is distance reducing, and thanks to Theorem 1.24 and the fact that $\mathrm{Card}(M(k)) = 2^{(k+1)n}$ we can find a function $\overline{f}_k : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $\overline{f}_k(x) = f(x)$ if $x \in M(k)$. Moreover, thanks to Proposition 1.23 we have that $g_k := \overline{f}_k|_{[0,1]^n}$ is in $\mathfrak{P}(n,m)$. We split now $[0,1]^n$ and we have the following fact:

$$\|g_k - f\|_\infty = \max_{I\in\{0,\dots,2^k-1\}^n} \left\| (g-f)|_{\prod_{i=1}^n \left[\frac{I_i}{2^k}, \frac{I_i+1}{2^k}\right]} \right\|_\infty .$$

Moreover since $g_k - f$ is a 2-Lipschitz function and $g_k(x_I) - f(x_I) = 0$, we have the following estimate:

$$\left\| (g_k - f)|_{\prod_{i=1}^n \left[\frac{I_i}{2^k}, \frac{I_i+1}{2^k}\right]} \right\|_\infty \le 2\cdot\mathrm{diam}\left(\prod_{i=1}^{n}\left[\frac{I_i}{2^k}, \frac{I_i+1}{2^k}\right]\right) = \frac{\sqrt{n}}{2^{k-1}} \ \forall I \in \{0,\dots,2^k-1\}^n.$$

Therefore:

$$\|g_k - f\|_\infty \leq \frac{\sqrt{n}}{2^{k-1}}.$$

Thus if $n \leq m$, for any $f \in \mathrm{Lip}_1(n,m)$ and any $\epsilon > 0$ we can find a function in $g \in \mathfrak{P}(n,m)$ such that $\|g - f\|_\infty \leq \epsilon$.

If $n > m$ we can define $F : [0,1]^n \longrightarrow \mathbb{R}^n$ to be $F(x) := (f_1(x), \ldots, f_m(x), 0, \ldots, 0)$ and hence for any $\epsilon > 0$ we can find $G \in \mathfrak{P}(n,n)$ such that $\|G - F\|_\infty \leq \epsilon$. Let us define now the function $g : [0,1]^n \longrightarrow \mathbb{R}^m$ as:

$$g(x) := (G_1(x), \ldots, G_m(x)),$$

we have that:

$$\begin{aligned}|g(x) - g(y)| &= \sqrt{\sum_{i=1}^{m} |g_i(x) - g_i(y)|^2} = \sqrt{\sum_{i=1}^{m} |G_i(x) - G_i(y)|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} |G_i(x) - G_i(y)|^2} = |x - y|.\end{aligned}$$

In order to show that $g \in \mathfrak{P}(n,m)$ we have to prove that $\|Jg\|_{HS}^2 = m$, since g is trivially piecewise affine. To do this we only have to note that $JG(x) \in O(n)$ for $x \in [0,1]^n$ where the differential exists:

$$JG(x) := \begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \\ \vdots \\ r_n^T \end{pmatrix},$$

where $r_i \in \mathbb{R}^n$ and $r_i \cdot r_j = \delta_{i,j}$ for any $i, j \in \{1, \ldots, n\}$. Therefore we have that:

$$Jg(x) = \begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix},$$

and $\|Jg\|_{HS}^2 = \sum_{i=1}^m |r_i|^2 = \sum_{i=1}^m 1 = m$. This concludes the proof that $\mathfrak{P}(n,m)$ is dense in $\mathrm{Lip}_1(n,m)$ for any $n, m \in \mathbb{N}$. $\square$

Chapter 2

Residuality implies F_σ

This chapter is divided in three sections. Section 2.1 is the prelude of Section 2.2, since in the former there are some technical details which will be used in the latter. Lemma 2.3, Proposition 2.2 are foundamental in the proof of Proposition 2.8 since they will be used in order to introduce a family of controlled cylinders on which the maximal operator of Section 1.3 will be built. Moreover, Lemma 2.7 togheter with results of Section 2.1, will be used in order to get the estimate (2.2). Section 2.3 containtains the proof of the main result Theorem 2.14. In this last section the Banach-Mazur game (see Section 1.1) and the density of $\mathfrak{P}(n,m)$ (see Section 1.4) functions will come into play.

2.1 Construction of a cylinder family

Definition 2.1. Let $0 < \epsilon < \frac{1}{2^{n+2}}$ and $x \in \mathbb{R}^n$. For any $y \in [0,1]^n \setminus \{x\}$ we define:

$$r(\epsilon, x, y) := \frac{\epsilon}{32}|x-y|.$$

In order to have a manageable notation, we slightly modify the one introduced in Definition 1.12 and in Definition 1.13. We let:

(i)
$$D_\epsilon(x,y) := D_{\pi(x,y,r(\epsilon,x,y)),r(\epsilon,x,y)}\left(x + \frac{r(\epsilon,x,y)}{2} \cdot \frac{x-y}{|x-y|}\right),$$

(ii)
$$\mathcal{C}_\epsilon(x,y) = \mathcal{C}(x,y,r(\epsilon,x,y)).$$

Proposition 2.2. *Fix $0 < \epsilon < \frac{1}{2^{n+2}}$ and $x \in [0,1]^n$. Then for any $y \in [0,1]^n \setminus \{x\}$ there holds:*

$$r(\epsilon,x,y) \le \min\left\{\frac{1}{2}, \frac{\epsilon\sqrt{n}}{32}, \frac{\epsilon|x-y|}{32}\right\}.$$

Proof. Since $0 < \epsilon < \frac{1}{2^n}$ and $|x-y| \leq \mathrm{diam}([0,1]^n) = \sqrt{n}$, we have:

$$r(\epsilon, x, y) = \frac{\epsilon}{32}|x-y| \leq \frac{\epsilon}{32}\sqrt{n} \leq \frac{\frac{1}{2^n}}{2^5}\sqrt{n} \leq \frac{1}{2^{n+5}}\sqrt{n} \leq \frac{1}{2}.$$

□

Lemma 2.3 (A geometric property for cylinders of the type $\mathcal{C}_\epsilon(x,y)$)**.** *Fix $0 < \epsilon < \frac{1}{2^{n+2}}$, and consider a cylinder $\mathcal{C}_\epsilon(x,y)$ as in Definition 2.1, then:*

$$\rho(\mathcal{C}_\epsilon(x,y)) \geq \frac{\epsilon}{128}$$

uniformily in $x \in [0,1]^n$ and $y \in [0,1]^n \setminus \{x\}$, where ρ is as in Definition 1.12.

Proof. First of all, recall that $\mathcal{C}_\epsilon(x,y)$ is a right circular cylinder with centre $\frac{x+y}{2}$, basis congruent to the disk $D_\epsilon(x,y)$ with radious $\frac{\epsilon}{32}|x-y|$, and height $\left(1+\frac{\epsilon}{32}\right)|x-y|$ as observed in Remark 1.14, recall that in this case $r = \frac{\epsilon}{32}|x-y|$. Thus we have $B_{\frac{\epsilon}{64}|x-y|}(\frac{x+y}{2}) \subseteq C_\epsilon(x,y)$. On the other hand $B_{(1+\epsilon)|x-y|}\left(\frac{x+y}{2}\right) \supseteq \mathcal{C}_\epsilon(x,y)$ and hence:

$$\rho(\mathcal{C}_\epsilon(x,y)) \geq \frac{\mathrm{diam}\left(B_{\frac{\epsilon}{64}|x-y|}(\frac{x+y}{2})\right)}{\mathrm{diam}\left(B_{(1+\epsilon)|x-y|}\left(\frac{x+y}{2}\right)\right)} = \frac{\frac{\epsilon}{64}|x-y|}{(1+\epsilon)|x-y|} = \frac{\frac{\epsilon}{64}}{(1+\epsilon)} \geq \frac{\epsilon}{128}.$$

□

2.2 Functions in $\mathfrak{P}(n,m)$ close in $\|\cdot\|_\infty$ norm have close differentials

This section is devoted to the proof of the core statement of this thesis. What we are going to prove is that if two functions in $\mathfrak{P}(n,m)$ are close in the supremum norm, then there exists an open set with large measure where Jacobians are close in the Hilbert-Schmidt norm.

Lemma 2.4. *For any $0 < l < \frac{1}{2^{n+2}}$ we have:*

$$\mathcal{L}^n([-2l, 2l+1]^n \setminus [0,1]^n) \leq 2^{n+2}l.$$

Proof. This is an easy computation. □

Definition 2.5. For $f \in \mathrm{Lip}_1(n,m)$ we define the function F in the following way:

$$F(x) := \begin{cases} f(x) & \text{if } x \in [0,1]^n \\ 0 & \text{if } x \notin [-\mathrm{Lip}(f)\|f\|_\infty, 1 + \mathrm{Lip}(f)\|f\|_\infty]^n \end{cases},$$

Then we define $E(f) : \mathbb{R}^n \to \mathbb{R}^m$ as the exension of F to the whole $\mathbb{R}^n$ provided by the Kirszbraun's theorem. Moreover we define $E(f)_j : \mathbb{R}^n \to \mathbb{R}$ to be the j-th component of $E(f)$.

Remark 2.6. We note that $\mathrm{Lip}(E(f)) = \mathrm{Lip}(f)$ by the Kirszbraun's theorem, and therefore $\mathrm{Lip}(E(f))_j \leq \mathrm{Lip}(f)$.

Lemma 2.7. *Let $0 < \epsilon < \frac{1}{2^{n+2}}$ and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a 2-Lipschitz function. Then for all points $x, y \in \mathbb{R}^n$ such that:*

$$|F(y) - F(x)| \geq \epsilon|x - y|,$$

for any $0 < r < \frac{\epsilon}{16}|x-y|$, $z \in B_r(x)$ and $w \in B_r(y)$ we have:

$$|F(w) - F(z)| \geq \frac{\epsilon}{2}|z - w|.$$

Proof. Thanks to the condition on r we have that $\frac{\epsilon}{16}|x-y| > r$, and thus $\frac{\epsilon}{2}|x-y| > 8r \geq 6r$. Hence,

$$\begin{aligned}
\frac{\epsilon}{2}|z-w| &\leq \frac{\epsilon}{2}\left(|z-x| + |x-y| + |y-w|\right) \leq \frac{\epsilon}{2}\left(|x-y| + 2r\right) \\
&\leq \frac{\epsilon}{2}\left(|x-y| + 2r\right) + \frac{\epsilon}{2}|x-y| - 6r = \epsilon\,|x-y| + \epsilon r - 6r \leq \epsilon\,|x-y| - 4r.
\end{aligned}$$

Recall that $|z-x| \leq r$ and $|y-w| \leq r$. This implies that:

$$\begin{aligned}
\epsilon|x-y| - 4r &\leq \epsilon\,|x-y| - 2\,|x-z| - 2\,|y-w| \leq |F(x) - F(y)| - 2\,|x-z| - 2\,|y-w| \\
&\leq |F(x) - F(y)| - \left(|F(x) - F(z)| + |F(w) - F(y)|\right) \\
&\leq |F(x) - F(y)| - |F(x) - F(z) + F(w) - F(y)| \\
&\leq |F(x) - F(y) - [F(x) - F(z) + F(w) - F(y)]| = |F(w) - F(z)|.
\end{aligned}$$

□

Proposition 2.8. *For all $0 < \epsilon < \frac{1}{2^{n+2}}$ and $f \in \mathfrak{P}(n,m)$ there exists an open neighbourhood V of f in $Lip_1(n,m)$ with $\mathrm{diam}(V) < \epsilon$ such that for any $g \in \mathfrak{P}(n,m) \cap V$ there are an open set $G \subseteq (0,1)^n$ and a constant $D(n,m)$ having the following properties:*

(i) $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\epsilon$.

(ii) $\|J(f-g)(x)\|_{HS} \leq \epsilon$ *for any* $x \in G$.

(iii) $|g(y) - g(x) - (f(y) - f(x))| \leq \epsilon|x-y|$ *for any* $x \in G$ *and any* $y \in [0,1]^n$.

Proof. Let $\{P_l\}_l = \Pi \in \tau$ be a partition on which f is piecewise affine and define V as:

$$V := \left\{ g \in \mathrm{Lip}_1(n,m) : \|f - g\|_\infty < \frac{\epsilon^{2n+3}}{m \cdot 64^{4n} \cdot \sqrt{\min\{n,m\}} \cdot (1 + \sum_l \mathcal{H}^{n-1}(\partial P_l))} \right\}.$$

We will show that such a neighbourhood of f satisfies the claim of Proposition 2.8. Let $g \in V \cap \mathfrak{P}(n,m)$. By Theorem 1.11 and the definition of V we have:

$$\int_{[0,1]^n} \|J(f-g)\|_{HS}^2 dx \leq m \cdot \sqrt{\min\{n,m\}} \cdot \left(\sum_l \mathcal{H}^{n-1}(\partial P_l) \right) \|f-g\|_\infty < \frac{\epsilon^{2n+3}}{64^{4n}}. \tag{2.1}$$

Let us introduce now three sets:

(i) $F_0 := \{x \in [0,1]^n : f \text{ is non-differentiable}\} \cup \{x \in [0,1]^n : g \text{ is non-differentiable}\}$.

(ii) $F_1 := \{x \in [0,1]^n \setminus F_0 : \|J(f-g)\|_{HS} \geq \epsilon\}$.

(iii) $F_2 := \{x \in [0,1]^n \setminus (F_0 \cup F_1) : \exists\, y \in [0,1]^n \setminus \{x\} \text{ s.t. } |g(y) - g(x) - (f(y) - f(x))| \geq \epsilon |x-y|\}$.

Suppose the following claim holds:

Claim: $F_0 \cup F_1 \cup F_2$ is a closed set with $\mathcal{L}^n(F_0 \cup F_1 \cup F_2) \leq 64(n+m)^2 \epsilon$.

Then the open set we are looking for is $G := (F_0 \cup F_1 \cup F_2)^c$. Indeed, G is open since it is the complement of a closed set, and with

$$D(n,m) := 64(n+m)^2,$$

we have $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\epsilon$. Moreover by definition of F_1 and F_2, conditions (ii), (iii) in the statement of the proposition are satisfied by G. We are therefore left to prove the claim.

First Step: $F_0 \cup F_1 \cup F_2$ is a closed set.

Since f and g are Lipschitz, the set were they are non-differentiable has measure zero by Rademacher's Theorem, so $\mathcal{L}^n(F_0) = 0$. Moreover F_0 is closed since f, g are piecewise affine by hypothesis. Analogously $F_0 \cup F_1$ is a closed set, since $F_0 \cup F_1$ is a union of finitely many close portions of hyperplanes and close symplexes (recall that we are dealing with piecewise affine functions, and thus the differential is piecewise constant). Finally, $F_0 \cup F_1 \cup F_2$ is closed since it is the union of two closed sets, $F_0 \cup F_1$ and:

$$\{x \in [0,1]^n : \exists\, y \in [0,1]^n \setminus \{x\} \text{ s.t. } |g(y) - g(x) - (f(y) - f(x))| \geq \epsilon |x-y|\}.$$

Second Step: estimate of the measure of F_1.

We estimate the measure of F_1 using Chebyshev's inequality and (2.1):

$$\begin{aligned}\mathcal{L}^n(\{\|J(f-g)\|_{HS}\chi_{[0,1]^n\setminus F_0} \geq \epsilon\}) &\leq \frac{1}{\epsilon^2}\int_{[0,1]^n\setminus F_0}\|J(f-g)\|_{HS}^2 dx\\ &\leq \frac{1}{\epsilon^2}\int_{[0,1]^n}\|J(f-g)\|_{HS}^2 dx\\ &< \frac{1}{\epsilon^2}\cdot\frac{\epsilon^{2n+3}}{64^{4n}} = \frac{\epsilon^{2n+1}}{64^{4n}}.\end{aligned}$$

Third Step: estimate of the measure of F_2.

We introduce the following function defined on $\mathbb{R}^n$:

$$\Psi_\epsilon(x) := \sup_{y\in(\mathbb{Q}^n\cap[0,1]^n)\setminus\{x\}}\sum_{j=1}^m \fint_{\mathcal{C}_\epsilon(x,y)}\left|\left\langle\frac{y-x}{|y-x|}, \nabla(E(f-g)_j)(z)\right\rangle\right| dz.$$

First of all, we have to show that this function is well defined and measurable. Let

$$\begin{aligned}I_{\epsilon,y,j}(x) :&= \fint_{\mathcal{C}_\epsilon(x,y)}\left|\left\langle\frac{y-x}{|y-x|}, \nabla(E(f-g)_j)(z)\right\rangle\right| dz\\ &\leq \fint_{\mathcal{C}_\epsilon(x,y)}|\nabla(E(f-g)_j)(z)|\, dz\\ &\leq \fint_{\mathcal{C}_\epsilon(x,y)} 2dz = 2\end{aligned}$$

for any $y \in (\mathbb{Q}^n\cap[0,1]^n)\setminus\{x\}$. The integral in the above computation is well defined since we know that $\nabla(E(f-g)_j)$ exists almost everywehere by Rademacher's theorem. The first inequality comes from Cauchy-Schwartz inequality and the last line from the fact that $\mathrm{Lip}(E(f-g)) \leq 2$. Therefore $0 \leq I_{\epsilon,y,j}(x) \leq 2$ and hence $\Psi_\epsilon(x) = \sup_{y\in(\mathbb{Q}^n\cap[0,1]^n)\setminus\{x\}}\sum_{j=1}^m I_{\epsilon,y,j}(x)$ exists is finite and we deduce that $0 \leq \Psi_\epsilon(x) \leq 2m$ for any $x \in \mathbb{R}^n$. Thus $\Psi_\epsilon(x)$ is well defined. It is measurable because it is a pointwise supremum of countable many measurable functions.

We will use this function in order to bound the measure of F_2. Indeed suppose the two following inequalities hold:

$$\int_{\mathbb{R}^n}\Psi_\epsilon(x)^2 dx \geq \frac{\epsilon^2}{16m}\mathcal{L}^n(F_2), \tag{2.2}$$

$$\int_{\mathbb{R}^n}\Psi_\epsilon(x)^2 \leq 2(n+m)\epsilon^3. \tag{2.3}$$

Using (2.2) and (2.3) we can estimate the measure of the set F_2:

$$\mathcal{L}^n(F_2) \leq 32m(m+n)\epsilon.$$

In order to get such an estimate we are left to prove inequalities (2.2) and (2.3).

First of all, let us prove inequality (2.2). Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{x\}$. By Rademacher's theorem we know that the set $\mathcal{B}$ where the differential of $E(f-g)_j$ does not exists has measure 0 and that on $\mathcal{B}^c$ it holds that:

$$\left\langle \frac{y-x}{|y-x|}, \nabla(E(f-g)_j)(z) \right\rangle = \partial_{\frac{y-x}{|y-x|}} E(f-g)_j(z).$$

We can express any $z \in \mathcal{C}_\epsilon(x,y)$ as $z = \xi + t\frac{y-x}{|y-x|}$ where $\xi \in D_\epsilon(x,y)$ and $t \in \left[0, \left(1+\frac{\epsilon}{32}\right)|x-y|\right]$, since as we already noticed in the proof of Proposition 2.3, the height of the cylinder $\mathcal{C}_\epsilon(x,y)$ is $\left(1+\frac{\epsilon}{32}\right)|x-y|$. Thus by Fubini-Tonelli Theorem we have that for $\mathcal{H}^{n-1}$-a.e. $\xi \in D_\epsilon(x,y)$ it holds that:

$$\partial_{\frac{y-x}{|y-x|}} E(f-g)_j \left(\xi + t\frac{y-x}{|y-x|}\right) = \frac{d}{ds} E(f-g)_j \left(\xi + s\frac{y-x}{|y-x|}\right)\Big|_{s=t}$$

for a.e. $t \in \left[0, \left(1+\frac{\epsilon}{16}\right)|x-y|\right]$. Therefore:

$$\begin{aligned}
I_{\epsilon,y,j}(x) :&= \fint_{\mathcal{C}_\epsilon(x,y)} \left| \left\langle \frac{y-x}{|y-x|}, \nabla(E(f-g)_j)(z) \right\rangle \right| dz \\
&= \fint_{\mathcal{C}_\epsilon(x,y)} \left| \partial_{\frac{y-x}{|y-x|}} E(f-g)_j(z) \right| dz \\
&= \fint_{D_\epsilon(x,y)\times\left[0,\left(1+\frac{\epsilon}{32}\right)|x-y|\right]} \left| \partial_{\frac{y-x}{|y-x|}} E(f-g)_j \left(\xi + t\frac{y-x}{|y-x|}\right) \right| dt \otimes d\mathcal{H}^{n-1}(\xi) \\
&= \fint_{D_\epsilon(x,y)} \left(\fint_0^{\left(1+\frac{\epsilon}{32}\right)|y-x|} \left| \partial_{\frac{y-x}{|y-x|}} E(f-g)_j \left(\xi + t\frac{y-x}{|y-x|}\right) \right| dt \right) d\mathcal{H}^{n-1}(\xi) \\
&= \fint_{D_\epsilon(x,y)} \left(\fint_0^{\left(1+\frac{\epsilon}{32}\right)|y-x|} \left| \frac{d}{ds} E(f-g)_j \left(\xi + s\frac{y-x}{|y-x|}\right)\Big|_{s=t} \right| dt \right) d\mathcal{H}^{n-1}(\xi).
\end{aligned}$$

Choose now $x \in F_2$, then there exists $y \in [0,1]^n \setminus \{x\}$ such that $|g(y)-g(x)-(f(y)-f(x))| \geq \epsilon|x-y|$. Hence, by density of $\mathbb{Q}^n \cap [0,1]^n$ in $[0,1]^n$ and continuity of $f-g$, there exists $\tilde{y} \in \mathbb{Q}^n \cap [0,1]^n \setminus \{x\}$ such that $|(f-g)(\tilde{y}) - (f-g)(x))| \geq \frac{\epsilon}{2}|x-\tilde{y}|$ i.e., $|E(f-g)(\tilde{y}) - E(f-g)(x))| \geq \frac{\epsilon}{2}|x-\tilde{y}|$. Therefore there exists a coordinate function $E(f-g)_k$ such that $|E(f-g)_k(\tilde{y}) - E(f-g)_k(x))| \geq \frac{\epsilon}{2\sqrt{m}}|x-\tilde{y}|$. Hence:

$$\begin{aligned}
\Psi_\epsilon(x) &= \sum_{j=1}^m \fint_{D_\epsilon(x,\tilde{y})} \left(\fint_0^{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|} \left| \frac{d}{ds} E(f-g)_j \left(\xi + s\frac{\tilde{y}-x}{|\tilde{y}-x|}\right)\Big|_{s=t} \right| dt \right) d\mathcal{H}^{n-1}(\xi) \\
&\geq \sum_{j=1}^m \fint_{D_\epsilon(x,\tilde{y})} \left| \fint_0^{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|} \frac{d}{ds} E(f-g)_j \left(\xi + s\frac{\tilde{y}-x}{|\tilde{y}-x|}\right)\Big|_{s=t} dt \right| d\mathcal{H}^{n-1}(\xi) \\
&= \sum_{j=1}^m \fint_{D_\epsilon(x,\tilde{y})} \left| \frac{E(f-g)_j\left(\xi + \left(1+\frac{\epsilon}{32}\right)(\tilde{y}-x)\right) - E(f-g)_j(\xi)}{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|} \right| d\mathcal{H}^{n-1}(\xi).
\end{aligned}$$

Since $\xi+\left(1+\frac{\epsilon}{32}\right)(\tilde{y}-x)\in D_\epsilon(\tilde{y},x)\subseteq B_{\frac{\epsilon}{16}|x-\tilde{y}|}(\tilde{y})$ for any $\xi\in D_\epsilon(x,\tilde{y})\subseteq B_{\frac{\epsilon}{16}|x-\tilde{y}|}(x)$, we can apply Lemma 2.7, obtaining:

$$\left|\frac{E(f-g)_k\left(\xi+\left(1+\frac{\epsilon}{32}\right)(\tilde{y}-x)\right)-E(f-g)_k(\xi)}{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|}\right|\geq\frac{1}{2}\cdot\frac{\epsilon}{2\sqrt{m}}|x-\tilde{y}|,$$

and hence:

$$\begin{aligned}
\sum_{j=1}^{m}&\fint_{D_\epsilon(x,\tilde{y})}\left|\frac{E(f-g)_j\left(\xi+\left(1+\frac{\epsilon}{32}\right)(\tilde{y}-x)\right)-E(f-g)_j(\xi)}{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|}\right|d\mathcal{H}^{n-1}(\xi)\\
&\geq\fint_{D_\epsilon(x,\tilde{y})}\left|\frac{E(f-g)_k\left(\xi+\left(1+\frac{\epsilon}{32}\right)(\tilde{y}-x)\right)-E(f-g)_k(\xi)}{\left(1+\frac{\epsilon}{32}\right)|\tilde{y}-x|}\right|d\mathcal{H}^{n-1}(\xi)\\
&\geq\fint_{D_\epsilon(x,\tilde{y})}\frac{\epsilon}{4\sqrt{m}}d\mathcal{H}^{n-1}(z)=\frac{\epsilon}{4\sqrt{m}}.
\end{aligned}$$

So $\Psi_\epsilon(x)\geq\frac{\epsilon}{4\sqrt{m}}$ if $x\in F_2$ and we can evaluate the L^2-norm of Ψ_ϵ from below:

$$\int_{\mathbb{R}^n}\Psi_\epsilon(x)^2dx\geq\int_{F_2}\Psi_\epsilon(x)^2dx\geq\frac{\epsilon^2}{16m}\mathcal{L}^n(F_2).$$

Next, let us prove inequality (2.3). First of all we note that for any $x\in[0,1]^n$ and any $y\in[0,1]^n\setminus\{x\}$, thanks to Lemma 2.3 and $|x-y|\leq\sqrt{n}$, we have that $\mathcal{C}_\epsilon(x,y)\in\mathscr{C}\left(\frac{\epsilon}{128},2\sqrt{n}\right)$, where the set $\mathscr{C}(\gamma(\epsilon))$ is defined in Definition 1.15. Thus:

$$\begin{aligned}
\Psi_\epsilon(x)&=\sup_{y\in(\mathbb{Q}^n\cap[0,1]^n)\setminus\{x\}}\sum_{j=1}^{m}\fint_{\mathcal{C}_\epsilon(x,y)}\left|\left\langle\frac{y-x}{|y-x|},\nabla(E(f-g)_j)(z)\right\rangle\right|\\
&\leq\sup_{\substack{x\in int(C)\\ C\in\mathscr{C}\left(\frac{\epsilon}{128},2\sqrt{n}\right)}}\fint_C\sum_{j=1}^{m}\left|\left\langle\frac{y-x}{|y-x|},\nabla(E(f-g)_j)(z)\right\rangle\right|dz\\
&\leq\sup_{\substack{x\in int(C)\\ C\in\mathscr{C}\left(\frac{\epsilon}{128},2\sqrt{n}\right)}}\fint_C\sum_{j=1}^{m}|\nabla(E(f-g)_j)(z)|\,dz
\end{aligned}$$

Recalling the definition of the cylindrical maximal function $M_{\mathscr{C}\left(\frac{\epsilon}{128},2\sqrt{n}\right)}$ in Definition 1.16 we have:

$$\Psi_\epsilon(x)\leq M_{\mathscr{C}\left(\frac{\epsilon}{128},2\sqrt{n}\right)}\left(\sum_{j=1}^{m}|\nabla(E(f-g)_j)|\right)(x).$$

We can now apply Proposition 1.21 obtaining:

$$\int_{\mathbb{R}^n} \Psi_\epsilon(x)^2 dx \le \int_{\mathbb{R}^n} \left(M_{\mathscr{C}\left(\frac{\epsilon}{128}, 2\sqrt{n}\right)} \left(\sum_{j=1}^m |\nabla(E(f-g)_j)| \right)(x) \right)^2 dx$$
$$= 2\left(\frac{64^2}{\epsilon}\right)^{2n} \int_{\mathbb{R}^n} \left(\sum_{j=1}^m |\nabla(E(f-g)_j)(z)| \right)^2 dz.$$

Using the inequality between the aritmetic mean and the quadratic mean we get:

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^m |\nabla(E(f-g)_j)(z)| \right)^2 dz = \int_{\mathbb{R}^n} m^2 \cdot \left(\frac{\sum_{j=1}^m |\nabla(E(f-g)_j)(z)|}{m} \right)^2 dz$$
$$\le \int_{\mathbb{R}^n} m^2 \cdot \frac{\sum_{j=1}^m |\nabla(E(f-g)_j)(z)|^2}{m} dz = m \int_{\mathbb{R}^n} \sum_{j=1}^m |\nabla(E(f-g)_j)(z)|^2 \, dz$$
$$= m \int_{\mathbb{R}^n} \|J(E(f-g))(z)\|_{HS}^2 dz.$$

Therefore summing up we get the following estimate:

$$\int_{\mathbb{R}^n} \Psi_\epsilon(x)^2 dx \le 2m \left(\frac{64^2}{\epsilon}\right)^{2n} \int_{\mathbb{R}^n} \|J(E(f-g))(z)\|_{HS}^2 dz$$
$$= 2m \left(\frac{64^2}{\epsilon}\right)^{2n} \left(\int_{[0,1]^n} \|J(E(f-g))(z)\|_{HS}^2 dz + \int_{([0,1]^n)^c} \|J(E(f-g))(z)\|_{HS}^2 dz \right).$$

In order to bound the first integral in the last line, we recall Definition 2.5 and we apply (2.1) obtaining:

$$\int_{[0,1]^n} \|J(E(f-g))(z)\|_{HS}^2 dz = \int_{[0,1]^n} \|J(f-g)(z)\|_{HS}^2 dz \le \frac{\epsilon^{2n+3}}{64^{4n}}.$$

In order to estimate the second one, we recall that $E(f-g) = 0$ outside $[-2\|f-g\|_\infty, 1 + 2\|f-g\|_\infty]^n$, by Defintion 2.5. Hence we have:

$$\int_{([0,1]^n)^c} \|J(E(f-g))(z)\|_{HS}^2 dz = \int_K \|J(E(f-g))(z)\|_{HS}^2 dz,$$

Where we let $K := [-2\|f-g\|_\infty, 1+2\|f-g\|_\infty]^n \setminus ([0,1]^n)$ Moreover since $\mathrm{Lip}(E(f-g)) = 2$, then $\sup_{|v|=1} |J(E(f-g))(z)(v)| \le 2$ for every $z \in \mathbb{R}^n$ where the differential exists. Provided that $\{e_k\}_{k=1,\dots,n}$ is the standard orthonormal basis of $\mathbb{R}^n$, we get the following estimate (recall that outside $[0,1]^n$ we extended by Kirzbraun's theorem and hence we do not have any control on the Hilbert-Schmidt norm of the Jacobian besides the one given by the Lipschitz constant):

$$\|J(E(f-g))(z)\|_{HS}^2 = \sum_{k=1}^n |J(E(f-g))(z)(e_k)|^2 \le \sum_{k=1}^n 4 = 4n,$$

and hence, thanks to Lemma 2.4 we have:

$$\begin{aligned}
&\int_K \|J(E(f-g))(z)\|_{HS}^2 dz \leq \mathcal{L}^n(K)\cdot 4n \\
&\leq 2^{n+2}\cdot 2\|f-g\|_\infty \cdot 4n = 2^{n+5}n\|f-g\|_\infty \\
&\leq 2^{n+5}n\frac{\epsilon^{2n+3}}{m\cdot 64^{4n}\cdot\sqrt{\min\{n,m\}}\cdot(1+\sum_l \mathcal{H}^{n-1}(\partial P_l))} \\
&\leq \frac{n\epsilon^{2n+3}}{m\cdot 64^{4n}\cdot\sqrt{\min\{n,m\}}}.
\end{aligned}$$

Therefore can write now:

$$\begin{aligned}
&\int_{\mathbb{R}^n}\Psi_\epsilon(x)^2 dx \leq 2m\left(\frac{64^2}{\epsilon}\right)^{2n}\left(\int_{[0,1]^n}\|J(E(f-g))(z)\|_{HS}^2 dz + \int_{([0,1]^n)^c}\|J(E(f-g))(z)\|_{HS}^2 dz\right) \\
&\leq 2m\left(\frac{64^2}{\epsilon}\right)^{2n}\left(\frac{\epsilon^{2n+3}}{64^{4n}}+\frac{n\epsilon^{2n+3}}{m\sqrt{\min\{n,m\}}64^{2n}}\right) = \left(2m+\frac{2n}{\sqrt{\min\{n,m\}}}\right)\epsilon^3 \\
&\leq 2(n+m)\epsilon^3.
\end{aligned}$$

Fourth Step: Estimate of the measure of $F_0\cup F_1\cup F_2$.

We can now estimate the measure of the closed set $F_0\cup F_1\cup F_2$. Indeed, we have:

$$\begin{aligned}
\mathcal{L}^n(F_0\cup F_1\cup F_2) &\leq \mathcal{L}^n(F_0)+\mathcal{L}^n(F_1)+\mathcal{L}^n(F_2) \\
&< \frac{\epsilon^{2n+1}}{64^{4n}} + 32m(n+m)\epsilon \leq 64(n+m)^2\epsilon.
\end{aligned}$$

□

2.3 Proof of the main result

In this section, we prove the main result of the thesis. The core argument is contained in the following proposition.

Definition 2.9. Let $F\subseteq\mathbb{R}^n$ be a Borel set. F has *each portion of positive measure* if for any open set $U\subseteq\mathbb{R}^n$ such that $U\cap F\neq\emptyset$ then $\mathcal{L}^n(U\cap F)>0$.

Proposition 2.10. *Let $F\subseteq[0,1]^n$ be a closed, nonempty subset with any partition of positive measure on the unit cube and let $E\subseteq[0,1]^n$ be such that $E\cap F$ is residual in F. If $S\subseteq\mathrm{Lip}_1(n,m)$ is the set of functions which are differentiable at least in a point of $E\cap F$ then S is residual in* $\mathrm{Lip}_1(n,m)$.

Proof. Let $E_1 \supseteq E_2 \supseteq \dots$ be relatively dense open subsets of F such that $\bigcap_{k=1}^{\infty} E_k \subseteq E$. To prove that S is residual we will build a winning strategy for the Player II in the corresponding Banach-Mazur game in which in addition to the nonempty open subsets $V_k \subseteq \mathrm{Lip}_1(n,m)$, we will make the second player choose functions in $V_k \cap \mathfrak{P}(n,m)$ and nonempty relatively open subsets $M_k \subseteq F$ in such a way that:

(i)
$$\mathrm{diam}(V_k) \leq \frac{1}{2^{(k-1)\cdot(2n+3)}}.$$

(ii) For every $g \in V_k \cap \mathfrak{P}(n,m)$ there is an open set $G \subseteq (0,1)^n$ such that:

(a) $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\mathcal{L}^n(M_k \cap E_k)$.

(b) $\|J(f_k - g)(x)\|_{HS} \leq \frac{1}{2^k}$ for any $x \in G$.

(c) $|g(y) - g(x) - (f_k(y) - f_k(x))| \leq \frac{1}{2^k}|x-y|$ for any $x \in G$ and any $y \in [0,1]^n$.

(iii) For any $x \in M_k$ $f_1, \dots, f_k$ are differentiable at x and if $k \geq 2$ the following holds:
$$\|J(f_k - f_{k-1})(x)\|_{HS} \leq \frac{1}{2^{k-1}}.$$

(iv) $\overline{M_k} \subseteq M_{k-1} \cap E_{k-1}$ if $k \geq 2$.

(v) $|f_k(y) - f_k(x) - (f_{k-1}(y) - f_{k-1}(x))| \leq \frac{1}{2^{k-1}}|x-y|$ for any $x \in M_k$ and any $y \in [0,1]^n$.

The required strategy for Player II can be described as follows.

The construction of the answer of Player II to the first move U_1 of Player I starts by picking an arbitrary $f_1 \in U \cap \mathfrak{P}(n,m)$. It exists thanks to the fact that $\mathfrak{P}(n,m)$ is dense in $\mathrm{Lip}_1(n,m)$. The set M_1 is defined as the set of points where f_1 is differentiable, and thus this choice satisfies (iii). Moreover since $F \setminus M_1$ has measure zero, then M_1 is a nonempty open set in $[0,1]^n$ (because $f_1 \in \mathfrak{P}(n,m)$). Since E_1 is a dense and relatively open subset of F, then the set $M_1 \cap E_1$ is a nonempty relatively open subset of F and therefore it has a positive measure. Thus, applying Proposition 2.8 to f_1 and $\epsilon := \min\left\{\frac{1}{2^{n+6}}, \mathcal{L}^n(M_1 \cap E_1)\right\}$ gives an open neighbourhood V_1 of f_1 such that (ii) holds, indeed we get for every $g \in V_1$ that:

(a) $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\epsilon \leq D(n,m)\mathcal{L}^n(M_1 \cap E_1)$.

(b) $\|J(f_1 - g)(x)\|_{HS} \leq \epsilon \leq \frac{1}{2^{n+6}} \leq \frac{1}{2}$ for any $x \in G$.

(c) $|g(y) - g(x) - (f_1(y) - f_1(x))| \leq \epsilon|x-y| \leq \frac{1}{2^{n+6}}|x-y| \leq \frac{1}{2}|x-y|$ for any $x \in G$ and any $y \in [0,1]^n$.

Moreover (i) holds for such a V_1, indeed:

$$\mathrm{diam}(V_1) \leq \frac{\epsilon^{2n+3}}{m(32\cdot 64)^{2n}\sqrt{\min\{n,m\}}\,(1 + \sum_l \mathcal{H}^{n-1}(\partial P_l))} \leq \epsilon \leq \frac{1}{2^{n+6}}.$$

Moreover (iv) and (v) are trivially satisfied.

Now, let $k \geq 2$ and

$$\mathrm{Lip}_1(n,m) \supseteq U_1 \supseteq V_1 \supseteq \ldots \supseteq U_{k-1} \supseteq V_{k-1},$$

functions $f_1, \ldots, f_{k-1}$ and nonempty relatively open subsets $M_1, \ldots, M_{k-1}$ of F verifying the required conditions have been already defined. Let $U_k \subseteq V_{k-1}$ be the arbitrary k-th move of Player I. Then Player II chooses an arbitrary function $f_k \in U_k \cap \mathfrak{P}(n,m)$ and uses (ii) of the $(k-1)$-step: for every $g \in V_{k-1} \cap \mathfrak{P}(n,m)$ there exists an open set $G \subseteq (0,1)^n$ such that:

(α) $\mathcal{L}^n([0,1]^n \setminus G) < D(n,m)\mathcal{L}^n(M_{k-1} \cap E_{k-1})$.

(β) $\|J(g - f_{k-1})(x)\|_{HS} \leq \frac{1}{2^{k-1}}$ for every $x \in G$.

(γ) $|g(y) - g(x) - (f(y) - f(x))| \leq \frac{1}{2^{k-1}}|x - y|$ for any $x \in G$ and any $y \in [0,1]^n$.

Since the set $G \cap M_{k-1} \cap E_{k-1}$ is relatively open in F and since (α) implies that it is nonempty, then there is a nonempty relatively open subset M_k of F such that $\overline{M_k} \subseteq G \cap M_{k-1} \cap E_{k-1}$. Because of ($\alpha$), ($\beta$), ($\gamma$) the choice of f_k and M_k verifies (iii), (iv) and (v). The remaining part of the construction is similar to the case $k = 1$: since E_k is a dense relatively open subset of F, it has positive measure. Hence Proposition 2.8 with $f := f_k$ and $\epsilon := \min\left\{\frac{1}{2^k}, \frac{1}{2^{n+6}}, \mathcal{L}^n(M_k \cap E_k)\right\}$ gives a neighbourhood V_k of f_k such that (i) and (ii) hold true. It remains to show that any function $f \in \bigcap_{k=1}^{\infty} V_k$ is differentiable at some point of $E \cap F$. Because of (i) a function f such that $f_k \to f$ uniformily. In view of (iv), we have that $\emptyset \neq \bigcap_{k=1}^{\infty} M_k \subseteq E$, because $M_k \subset\subset M_{k-1}$ and compact sets have the property of finite intersection. Thus it suffices to show that f is differentiable at every $x \in \bigcap_{k=1}^{\infty} M_k$. Thanks to the way we choosed functions f_k, we have that $\{Jf_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $M_{m\times n}(\mathbb{R})$ endowed with the $\|\cdot\|_{HS}$ norm, indeed without loss of generality for $i < j$ we have:

$$\begin{aligned}\|Jf_i(x) - Jf_j(x)\|_{HS} &\leq \sum_{k=i}^{n-1} \|Jf_k(x) - Jf_{k+1}(x)\|_{HS} \leq \sum_{k=i}^{j-1} \frac{1}{2^k} \\ &= \sum_{k=0}^{j-i-1} \frac{1}{2^{i+k}} = \frac{1}{2^i} \sum_{k=0}^{j-i-1} \frac{1}{2^k} \leq \frac{1}{2^i} \cdot 2 = \frac{1}{2^i},\end{aligned}$$

for any $x \in \bigcap_{k=1}^{\infty} M_k$. Thus for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n, i > N$ we have that $\|Jf_i(x) - Jf_j(x)\|_{HS} < \epsilon$ for any $x \in \bigcap_{k=1}^{\infty} M_k$. Therefore the limit of the sequence $Jf_k(x)$ exists. Let $J(x)$ be such a limit. Let us prove that $J(x)$ is the differential of f in $x \in \bigcap_{k=1}^{\infty} M_k$. In order to do this, let us fix $i \in \mathbb{N}$:

$$\begin{aligned}&\left|\frac{f(y) - f(x) - J(x)(y-x)}{|x-y|}\right| \leq \left|\frac{f(y) - f(x) - Jf_i(x)(y-x)}{|x-y|}\right| + \left|\frac{(Jf_i(x) - J(x))(y-x)}{|x-y|}\right| \\ &\leq \left|\frac{f_i(y) - f_i(x) - Jf_i(x)(y-x)}{|x-y|}\right| + \sum_{k=i}^{\infty}\left|\frac{f_{k+1}(y) - f_{k+1}(x)}{|x-y|} - \frac{f_k(y) - f_k(x)}{|x-y|}\right| + \|Jf_i(x) - J(x)\|,\end{aligned}$$

where $\|\cdot\|$ is the operator norm. Now recalling that all norms are equivalent in finite dimensional vector spaces, and property (v) on points of $\bigcap_{k=1}^{\infty} M_k$ we have that:

$$\begin{aligned}
&\left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| + \sum_{k=i}^{\infty}\left|\frac{f_{k+1}(y)-f_{k+1}(x)}{|x-y|} - \frac{f_k(y)-f_k(x)}{|x-y|}\right| \\
&\qquad\qquad\qquad\qquad + \|Jf_i(x)-J(x)\| \\
&\le \left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| + \sum_{k=i}^{\infty}\frac{1}{2^k} + \sqrt{m}\,\|Jf_i(x)-J(x)\|_{HS} \\
&= \left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| + \frac{1}{2^{i-1}} + \sqrt{m}\,\|Jf_i(x)-J(x)\|_{HS}\,.
\end{aligned}$$

But thanks to the definition of $J(x)$:

$$\|Jf_i(x)-J(x)\|_{HS} \le \sum_{k=i}^{\infty}\|Jf_{k+1}(x)-Jf_k(x)\|_{HS} \le \sum_{k=i}^{\infty}\frac{1}{2^k} = \frac{1}{2^i}\cdot 2 \le \frac{1}{2^{i-1}},$$

hence we deduce that:

$$\begin{aligned}
&\left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| + \frac{1}{2^{i-1}} + \sqrt{m}\,\|Jf_i(x)-J(x)\|_{HS} \\
&\le \left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| + \frac{1+\sqrt{m}}{2^{i-1}}.
\end{aligned}$$

Therefore, taking the lim sup for $y \to x$,

$$\begin{aligned}
\limsup_{y\to x}\left|\frac{f(y)-f(x)-J(x)(y-x)}{|x-y|}\right| &\le \frac{1+\sqrt{m}}{2^{i-1}} + \limsup_{y\to x}\left|\frac{f_i(y)-f_i(x)-Jf_i(x)(y-x)}{|x-y|}\right| \\
&= \frac{1+\sqrt{m}}{2^{i-1}},
\end{aligned}$$

for any $i \in \mathbb{N}$, hence:

$$\limsup_{y\to x}\left|\frac{f(y)-f(x)-J(x)(y-x)}{|x-y|}\right| = 0,$$

and thus f is differentiable in $x \in \bigcap_{k=1}^{\infty} M_k$ and its differential is precisely $J(x)$. □

We recall the definition of analytic set.

Definition 2.11 (Analytic sets)**.** A set in a complete metric space is said to be *analytic* if it is a continuous image of a complete metric space.

Remark 2.12. Recall that every Borel set is analytic (see [6]).

The following theorem relates analytic sets to closed sets of zero measure.

Theorem 2.13 (Solecki)**.** *Any analytic set $E \subseteq \mathbb{R}^n$ is either covered by a countable union of closed sets of zero measure, or there exists a closed set F such that $E \cap F$ contains a G_δ set dense in F. Moreover F has each portion of positive measure.*

Proof. It is a straightforward application of the main result of [12]. □

Theorem 2.14. *Let $E \subseteq [0,1]^n$ be an analytic set. If the set S of those functions $f \in \mathrm{Lip}_1(n,m)$ which are differentiable at no point of E is residual in $\mathrm{Lip}_1(n,m)$, then the set E is contained in an F_σ subset of $[0,1]^n$ of Lebesgue measure zero.*

Proof. Suppose it is false. Then E cannot be covered by any F_σ set of measure zero. Using Theorem 2.13 we find a closed nonempty set $F \subseteq [0,1]^n$ with every portion of positive measure such that $E \cap F$ is residual in F. Applying Proposition 2.10 we get a contradiction. □

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